



Relations and Functions

Question 1

The number of functions $f : \{1, 2, 3, 4\} \rightarrow \{a, b, c\}$, which are not onto, is :

JEE Main 2026 (Online) 4th April Morning Shift

Options:

A. 48

B. 45

C. 51

D. 35

Answer: B

Solution:

Total number of functions from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$ is

$$3^4 = 81$$

Now we find the number of onto functions.

For a function to be onto, all the three elements a, b, c must appear as images of elements of $\{1, 2, 3, 4\}$. Using inclusion-exclusion principle:

Total functions: $3^4 = 81$ Let

A = functions missing a

B = functions missing b

C = functions missing c

If one element is missing, then the function goes into only 2 elements.

So,

$$|A| = |B| = |C| = 2^4 = 16$$

Hence,

$$|A| + |B| + |C| = 3 \times 16 = 48$$

If two elements are missing, then all 4 elements of domain must go to only one element of codomain. So,

$$|A \cap B| = |B \cap C| = |C \cap A| = 1^4 = 1$$

Thus,
 $|A \cap B| + |B \cap C| + |C \cap A| = 3$

If all three are missing, that is impossible, so

$$|A \cap B \cap C| = 0$$

Therefore, number of functions which are not onto is $|A \cup B \cup C| = 48 -$

$3 = 45$ So the correct answer is 45

Question2

Let f and g be functions satisfying $f(x + y) = f(x)f(y)$, $f(1) = 7$ and $g(x + y) = g(xy)$, $g(1) = 1$, for all $x, y \in \mathbb{N}$. If $\sum_{x=1}^n \left(\frac{f(x)}{g(x)} \right) = 19607$, then n is equal to : JEE Main 2026 (Online) 22nd January Evening Shift

Options:

A. 6

B. 7

C. 4

D. 5

Answer: D

Solution:

$$f(x+y) = f(x) \cdot f(y) \Rightarrow f(x) = a^x$$
$$(\because f(1) = 7 \Rightarrow a^1 = 7)$$

$$\text{So } f(x) = 7^x$$

Now

$$g(x+y) = g(xy) \text{ (put } y = 1) \Rightarrow g(x+1) = g(x)$$

$$\text{So } g(1) = g(2) = g(3) = \dots = g(n) = 1$$

$$\text{Given } \sum_{x=1}^n \frac{f(x)}{g(x)} = 19607$$

$$\sum_{x=1}^n \frac{7^x}{1} = 19607 \Rightarrow 7 \left(\frac{7^n - 1}{7 - 1} \right) = 19607$$

$$7^n - 1 = \frac{6}{7} \times 19607$$

$$7^n = 16807 \Rightarrow n = 5$$

Question3

Let the domain of the function

$$f(x) = \log_3 \log_5 (7 - \log_2 (x^2 - 10x + 85)) + \sin^{-1} \left(\left| \frac{3x-7}{17-x} \right| \right) \text{ be } (\alpha, \beta]$$

. Then $\alpha + \beta$ is equal to :

JEE Main 2026 (Online) 22nd January Evening Shift

Options:

A. 12

B. 8

C. 10

D.

Answer: D

Solution:

$$\text{let } x^2 - 10x + 85 = \lambda$$

$$\therefore \text{Domain for first term } \lambda > 0 \dots\dots\dots (1)$$

$$\& 7 - \log_2 \lambda > 0 \Rightarrow \lambda < 2^7 \dots\dots\dots (2)$$

$$\& \log_5 (7 - \log_2 \lambda) > 0 \Rightarrow \lambda < 2^6 \dots\dots\dots (3)$$

$$\therefore \text{from (1), (2) \& (3)}$$

$$0 < \lambda < 2^6$$

$$0 < x^2 - 10x + 85 < 64 \Rightarrow x \in (3, 7) \dots\dots\dots (A)$$

$$\& \text{domain for second term } -1 \leq \frac{3x-7}{x-17} \leq 1 \Rightarrow x \in [-5, 6] \dots\dots\dots$$

From (A) \& (B), domain of function will be (3, 6]

$$\Rightarrow \alpha = 3, \beta = 6 \Rightarrow \alpha + \beta = 9$$

Question4

Let $f(x) = [x]^2 - [x + 3] - 3, x \in \mathbf{R}$, where $[.]$ is the greatest integer funtion. Then

JEE Main 2026 (Online) 22nd January Evening Shift

Options:

A.

$$f(x) = 0 \text{ for finitely many values of } x$$

B.

$$f(x) < 0 \text{ only for } x \in [-1, 3)$$

C.

$$\int_0^2 f(x) dx = -6$$

D.

$$f(x) > 0 \text{ only for } x \in [4, \infty)$$

Answer: B

Solution:

$$(3) \Rightarrow f(x) = 0 \Rightarrow [x] = 3 \text{ or } [x] = -2 \\ \Rightarrow x \in [3, 4) \cup [-2, -1) \Rightarrow f(x) = 0 \text{ has infinite solutions}$$

$$f(x) = [x]^2 - [x + 3] - 3 \\ = [x^2] - [x] - 6 = ([x] - 3)([x] + 2)$$

$$(2) \Rightarrow \int_0^2 f(x) dx = \int_0^2 ([x]^2 - [x] - 6) dx \\ \int_0^2 f(x) dx = \int_0^2 ([x]^2 - [x] - 6) dx = \int_0^2 ([x]^2 - [x]) dx - 12 = -12$$

$$(4) \Rightarrow f(x) < 0 \Rightarrow ([x] - 3)([x] + 2) < 0 \Rightarrow -2 < [x] < 3 \Rightarrow -1 \leq [x] < 3$$

$$(1) \Rightarrow f(x) > 0 \Rightarrow [x] < -2 \text{ or } [x] > 3 \Rightarrow x < -2 \text{ or } x \geq 4 \Rightarrow x \in (-\infty, -2) \cup [4, \infty)$$

Question 5

Let f be a function such that $3f(x) + 2f\left(\frac{m}{19x}\right) = 5x, x \neq 0$, where

$m = \sum_{i=1}^9 (i)^2$. Then $f(5) - f(2)$ is equal to

JEE Main 2026 (Online) 24th January Evening Shift

Options:

A.

36

B.

9

C.

-9

D.

18

Answer: D

Solution:

$$\text{Given } m = \sum_{i=1}^9 (i)^2 = 1^2 + 2^2 + \dots + 9^2$$

$$\text{We know, } 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{so, } m = \frac{9 \times 10 \times 19}{6} = 3 \times 5 \times 19 = 285$$

Given equation

$$3f(x) + 2f\left(\frac{m}{19x}\right) = 5x, x \neq 0$$

$$\text{put } m = 285, 3f(x) + 2f\left(\frac{285}{19x}\right) = 5x$$

$$3f(x) + 2f\left(\frac{15}{x}\right) = 5x \quad (1)$$

$$\text{put } x = \frac{15}{x}$$

$$3f\left(\frac{15}{x}\right) + 2f\left(\frac{15}{15/x}\right) = 5 \times \frac{15}{x}$$

$$3f\left(\frac{15}{x}\right) + 2f(x) = \frac{75}{x} \quad (2)$$

$$\text{from (1) } f\left(\frac{15}{x}\right) = \frac{5x - 3f(x)}{2} \text{ substitute in equation (2)}$$

$$3 \left[\frac{5x - 3f(x)}{2} \right] + 2f(x) = \frac{75}{x}$$

$$15x - 9f(x) + 4f(x) = \frac{150}{x}$$

$$5f(x) = 15x - \frac{150}{x}$$

$$f(x) = 3x - \frac{30}{x}$$

to find $f(5) - f(2)$

$$f(5) - f(2) = \left(3(5) - \frac{30}{5} \right) - \left(3 \times 2 - \frac{30}{2} \right)$$

$$= (15 - 6) - (6 - 15)$$

$$= 9 - (-9) = 18$$

Question 6

If $g(x) = 3x^2 + 2x - 3$, $f(0) = -3$ and $4g(f(x)) = 3x^2 - 32x + 72$, then $f(g(2))$ is equal to:

JEE Main 2026 (Online) 28th January Morning Shift

Options:

A.

$$\frac{7}{2}$$

B.

$$-\frac{25}{6}$$

C.

$$\frac{25}{6}$$

D.

$$-\frac{7}{2}$$

Answer: A

Solution:

Given

$$g(x) = 3x^2 + 2x - 3, f(0) = -3, \text{ and } 4g(f(x)) = 3x^2 - 32x + 72$$

Since $g(x)$ is a quadratic function (degree 2) and the composite function $4g(f(x))$ is also a quadratic expression ($3x^2 - 32x + 72$), $f(x)$ must be a linear function.

- Let $f(x) = ax + b$.
- Using $f(0) = -3$: $a(0) + b = -3 \implies b = -3$.
- So, $f(x) = ax - 3$.

First, find $g(f(x))$ by substituting $f(x)$ into $g(x)$:

$$g(f(x)) = 3(f(x))^2 + 2f(x) - 3$$

Substitute this into the given equation $4g(f(x)) = 3x^2 - 32x + 72$:

$$4[3(ax - 3)^2 + 2(ax - 3) - 3] = 3x^2 - 32x + 72$$

Solve for the Coefficient 'a'

Expand the left side of the equation:

- $4[3(a^2x^2 - 6ax + 9) + 2ax - 6 - 3] = 3x^2 - 32x + 72$
- $4[3a^2x^2 - 18ax + 27 + 2ax - 9] = 3x^2 - 32x + 72$
- $4[3a^2x^2 - 16ax + 18] = 3x^2 - 32x + 72$
- $12a^2x^2 - 64ax + 72 = 3x^2 - 32x + 72$

Compare the coefficients:

- For $x^2 : 12a^2 = 3 \implies a^2 = \frac{1}{4} \implies a = \pm \frac{1}{2}$

- For $x : -64a = -32 \implies a = \frac{32}{64} = \frac{1}{2}$ Thus, $f(x) = \frac{1}{2}x - 3$. Calculate $f(g(2))$

Find $g(2)$

$$g(2) = 3(2)^2 + 2(2) - 3 = 12 + 4 - 3 = 13$$

Step B: Find $f(g(2))$ which is $f(13)$

$$f(13) = \frac{1}{2}(13) - 3 = 6.5 - 3 = 3.5 = \frac{7}{2}$$

Question 7

The sum of all the elements in the range of

$$f(x) = \text{Sgn}(\sin x) + \text{Sgn}(\cos x) + \text{Sgn}(\tan x) + \text{Sgn}(\cot x),$$

$x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$, where

$$\text{Sgn}(t) = \begin{cases} 1, & \text{if } t > 0 \\ -1, & \text{if } t < 0 \end{cases}$$

is :

JEE Main 2026 (Online) 28th January Evening Shift

Options:

A.

4

B.

0

C.

2

D.

-2

Answer: C

Solution:

$$f(x) = \text{Sgn}(\sin x) + \text{Sgn}(\cos x) + \text{Sgn}(\tan x) + \text{Sgn}(\cot x)$$

Since the trig functions change signs depending on which quadrant x is in, we analyze the value of $f(x)$ quadrant by quadrant.

Given that $x \neq \frac{n\pi}{2}$, we don't have to worry about any of these terms being zero. Each Sgn term will simply be either 1 (if positive) or -1 (if negative).

Quadrant	Interval	$\sin x$	$\cos x$	$\tan x$	$\cot x$	Sum of Sgn values	$f(x)$
I	$0 < x < \frac{\pi}{2}$	+	+	+	+	$1 + 1 + 1 + 1$	4
II	$\frac{\pi}{2} < x < \pi$	+	-	-	-	$1 - 1 - 1 - 1$	-2
III	$\pi < x < \frac{3\pi}{2}$	-	-	+	+	$-1 - 1 + 1 + 1$	0
IV	$\frac{3\pi}{2} < x < 2\pi$	-	+	-	-	$-1 + 1 - 1 - 1$	-2

Range of $f(x)$:

From table, Range = $\{-2, 0, 4\}$

Sum of all elements in this range = $4 + (-2) + 0 = 2$

Question8

Given below are two statements :

Statement I : The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{1+|x|}$ is one-one.

Statement II : The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2+4x-30}{x^2-8x+18}$ is many-one.

In the light of the above statements, choose the correct answer from the options given below :

JEE Main 2026 (Online) 28th January Evening Shift

Options:

A.

Statement I is true but Statement II is false

B.

Both Statement I and Statement II are false

C.

Both Statement I and Statement II are true

D.

Statement I is false but Statement II is true

Answer: C

Solution:

Statement I: $f(x) = \frac{x}{1+|x|}$ is one-one

To check if a function is one-one (injective), we can check its monotonicity (whether it's always increasing or always decreasing).

1. For $x \geq 0$: $f(x) = \frac{x}{1+x}$. The derivative is $f'(x) = \frac{1}{(1+x)^2}$, which is always positive.

2. For $x < 0$: $f(x) = \frac{x}{1-x}$. The derivative is $f'(x) = \frac{1}{(1-x)^2}$, which is also always positive.

Since the derivative is positive for all x in the domain, the function is strictly increasing. A strictly monotonic function is always one-one.

Statement I is True.

Statement II: $f(x) = \frac{x^2+4x-30}{x^2-8x+18}$ is many-one

Discriminant of Denominator: For $x^2 - 8x + 18$, $D = (-8)^2 - 4(1)(18) = 64 - 72 = -8$. Since $D < 0$, the denominator is always positive and the function is continuous for all $x \in \mathbb{R}$.

Behavior at Infinity: As $x \rightarrow \infty$ and $x \rightarrow -\infty$, the value of $f(x)$ approaches 1 (the ratio of the leading coefficients).

Turning Points: Since the function starts near $y = 1$ at $-\infty$, moves through various values, and returns toward $y = 1$ at $+\infty$, it must have at least one local maximum or minimum. This means a horizontal line will intersect the graph at more than one point.

Also, let

$$y = \frac{x^2+4x-30}{x^2-8x+18}$$

Cross multiply:

$$y(x^2 - 8x + 18) = x^2 + 4x - 30$$

Rearrange:

$$(y - 1)x^2 + (-8y - 4)x + (18y + 30) = 0$$

This is a quadratic in x .

For many values of y , this quadratic has two distinct real solutions. Function is MANY-ONE

Question9

The number of functions $f : \{1, 2, 3, 4\} \rightarrow \{a, b, c\}$, which are not onto, is :

JEE Main 2026 (Online) 4th April Morning Shift

Options:

A.

48

B.

45

C.

51

D.

35

Answer: B

Solution:

Total number of functions from $\{1, 2, 3, 4\}$ to $\{a, b, c\}$ is

$$3^4 = 81$$

Now we find the number of onto functions.

For a function to be onto, all the three elements a, b, c must appear as images of elements of $\{1, 2, 3, 4\}$.

Using inclusion-exclusion principle:

- Total functions: $3^4 = 81$

Let

- A = functions missing a
- B = functions missing b
- C = functions missing c

If one element is missing, then the function goes into only 2 elements.

So,

$$|A| = |B| = |C| = 2^4 = 16$$

Hence,

$$|A| + |B| + |C| = 3 \times 16 = 48$$

If two elements are missing, then all 4 elements of domain must go to only one element of codomain.

So,

$$|A \cap B| = |B \cap C| = |C \cap A| = 1^4 = 1$$

Thus,

$$|A \cap B| + |B \cap C| + |C \cap A| = 3$$

If all three are missing, that is impossible, so

$$|A \cap B \cap C| = 0$$

Therefore, number of functions which are not onto is

$$|A \cup B \cup C| = 48 - 3 = 45$$

So the correct answer

is

45

Hence, **Option B** is correct.

Question 10

Let $[\cdot]$ denote the greatest integer function. If the domain of the function

$f(x) = \cos^{-1} \left(\frac{4x+2[x]}{3} \right)$ is $[\alpha, \beta]$, then $12(\alpha + \beta)$ is equal to :

JEE Main 2026 (Online) 4th April Morning Shift

Options:

A.

6

B.

8

C.

9

D.

4

Answer: A

Solution:

For the function

$$f(x) = \cos^{-1} \left(\frac{4x+2[x]}{3} \right),$$

the expression inside $\cos^{-1}$ must lie between -1 and 1 .

So, for domain:

$$-1 \leq \frac{4x+2[x]}{3} \leq 1$$

Multiplying by 3:

$$-3 \leq 4x + 2[x] \leq 3$$

Now let

$$[x] = n$$

where n is an integer and

$$n \leq x < n + 1.$$

Then the inequality becomes

$$-3 \leq 4x + 2n \leq 3.$$

Subtracting $2n$:

$$-3 - 2n \leq 4x \leq 3 - 2n$$

Dividing by 4:

$$\frac{-3-2n}{4} \leq x \leq \frac{3-2n}{4}$$

But also,

$$n \leq x < n + 1.$$

So we find possible integer values of n for which these intervals overlap.

Case 1: $n = -1$

Then

$$[x] = -1, \quad -1 \leq x < 0$$

and

$$\frac{-3-2(-1)}{4} \leq x \leq \frac{3-2(-1)}{4}$$

$$\frac{-1}{4} \leq x \leq \frac{5}{4}$$

Intersection with $[-1, 0)$ gives

$$-\frac{1}{4} \leq x < 0.$$

Case 2: $n = 0$

Then

$$0 \leq x < 1$$

and

$$\frac{-3}{4} \leq x \leq \frac{3}{4}$$

Intersection gives

$$0 \leq x \leq \frac{3}{4}.$$

Case 3: $n = -2$

Then

$$-2 \leq x < -1$$

and

$$\frac{-3-2(-2)}{4} \leq x \leq \frac{3-2(-2)}{4}$$

$$\frac{1}{4} \leq x \leq \frac{7}{4}$$

No intersection.

Case 4: $n = 1$

Then

$$1 \leq x < 2$$

and

$$\frac{-5}{4} \leq x \leq \frac{1}{4}$$

No intersection.

So the domain is

$$\left[-\frac{1}{4}, 0\right) \cup \left[0, \frac{3}{4}\right] = \left[-\frac{1}{4}, \frac{3}{4}\right].$$

Thus,

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{4}.$$

Therefore,

$$\alpha + \beta = \frac{1}{2}$$

and

$$12(\alpha + \beta) = 12 \cdot \frac{1}{2} = 6.$$

Hence the correct answer is

6

So, **Option A** is correct.

Question 11

Let for some $\alpha \in \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying

$f(x + y) = f(x) + 2y^2 + y + \alpha xy$ for all $x, y \in \mathbb{R}$. If $f(0) = -1$ and

$f(1) = 2$, then the value of $\sum_{n=1}^5 (\alpha + f(n))$ is :

JEE Main 2026 (Online) 4th April Evening Shift

Options:

A.

110

B.

140

C.

150

D.

170

Answer: B

Solution:

Given

$$f(x+y) = f(x) + 2y^2 + y + \alpha xy \quad \text{for all } x, y \in \mathbb{R}$$

with

$$f(0) = -1, \quad f(1) = 2.$$

We have to find

$$\sum_{n=1}^5 (\alpha + f(n)).$$

Step 1: Use the given relation to find α

Put $x = 0$ in

$$f(x+y) = f(x) + 2y^2 + y + \alpha xy.$$

Then

$$f(y) = f(0) + 2y^2 + y + \alpha \cdot 0 \cdot y$$

so

$$f(y) = -1 + 2y^2 + y.$$

Thus, for every real y ,

$$f(y) = 2y^2 + y - 1.$$

Now use $f(1) = 2$:

$$f(1) = 2(1)^2 + 1 - 1 = 2.$$

This is already satisfied, so now we must use the original equation to determine α .

Substitute

$$f(x+y) = 2(x+y)^2 + (x+y) - 1$$

and

$$f(x) = 2x^2 + x - 1$$

into the given equation:

$$2(x+y)^2 + (x+y) - 1 = 2x^2 + x - 1 + 2y^2 + y + \alpha xy.$$

Expand the left side:

$$2x^2 + 4xy + 2y^2 + x + y - 1.$$

Right side is:

$$2x^2 + x - 1 + 2y^2 + y + \alpha xy.$$

Comparing both sides,

$$4xy = \alpha xy.$$

Hence,

$$\alpha = 4.$$

Step 2: Find $f(n)$ for $n = 1$ to 5

Since

$$f(n) = 2n^2 + n - 1,$$

we get:

$$f(1) = 2(1)^2 + 1 - 1 = 2$$

$$f(2) = 2(2)^2 + 2 - 1 = 8 + 2 - 1 = 9$$

$$f(3) = 2(3)^2 + 3 - 1 = 18 + 3 - 1 = 20$$

$$f(4) = 2(4)^2 + 4 - 1 = 32 + 4 - 1 = 35$$

$$f(5) = 2(5)^2 + 5 - 1 = 50 + 5 - 1 = 54$$

Step 3: Compute the required sum

We need

$$\sum_{n=1}^5 (\alpha + f(n)).$$

Since $\alpha = 4$,

$$\alpha + f(1) = 4 + 2 = 6$$

$$\alpha + f(2) = 4 + 9 = 13$$

$$\alpha + f(3) = 4 + 20 = 24$$

$$\alpha + f(4) = 4 + 35 = 39$$

$$\alpha + f(5) = 4 + 54 = 58$$

Therefore,

$$6 + 13 + 24 + 39 + 58 = 140.$$

So the required value is

$$\boxed{140}.$$

Hence, the correct option is **B**.

Question 12

For the function $f: [1, \infty) \rightarrow [1, \infty)$ defined by $f(x) = (x-1)^4 + 1$, among the two statements:

(I) The set $S = \{x \in [1, \infty) : f(x) = f^{-1}(x)\}$ contains exactly two elements, and

(II) The set $S = \{x \in [1, \infty) : f(x) = f^{-1}(x+1)\}$ is an empty set,

JEE Main 2026 (Online) 4th April Evening Shift

Options:

A.

only (I) is TRUE

B.

only (II) is TRUE

C.

both (I) and (II) are TRUE

D.

neither (I) nor (II) is TRUE

Answer: A

Solution:

For

$$f: [1, \infty) \rightarrow [1, \infty), \quad f(x) = (x-1)^4 + 1,$$

we first find the inverse function.

Since for $x \geq 1$, the function $(x-1)^4 + 1$ is strictly increasing, f is one-one and onto from $[1, \infty)$ to $[1, \infty)$.

Let

$$y = (x-1)^4 + 1.$$

Then

$$y-1 = (x-1)^4.$$

Since $x \geq 1$, we take the positive fourth root:

$$x - 1 = (y - 1)^{1/4}.$$

Hence

$$f^{-1}(y) = 1 + (y - 1)^{1/4}.$$

So

$$f^{-1}(x) = 1 + (x - 1)^{1/4}.$$

Now check the two statements one by one.

Statement (I)

We need to solve

$$f(x) = f^{-1}(x).$$

That is,

$$(x - 1)^4 + 1 = 1 + (x - 1)^{1/4}.$$

So

$$(x - 1)^4 = (x - 1)^{1/4}.$$

Put

$$t = x - 1 \quad \text{where } t \geq 0.$$

Then the equation becomes

$$t^4 = t^{1/4}.$$

Now solve:

$$t^4 - t^{1/4} = 0.$$

This gives

$$t^{1/4} (t^{15/4} - 1) = 0.$$

Hence

$$t = 0 \quad \text{or} \quad t^{15/4} = 1.$$

Since $t \geq 0$, from $t^{15/4} = 1$ we get

$$t = 1.$$

Therefore

$$x - 1 = 0 \Rightarrow x = 1,$$

or

$$x - 1 = 1 \Rightarrow x = 2.$$

So the set

$$S = \{x \in [1, \infty) : f(x) = f^{-1}(x)\}$$

contains exactly two elements:

$$S = \{1, 2\}.$$

Thus, Statement (I) is TRUE.

Statement (II)

We need to solve

$$f(x) = f^{-1}(x + 1).$$

Now

$$f(x) = (x - 1)^4 + 1,$$

and

$$f^{-1}(x + 1) = 1 + ((x + 1) - 1)^{1/4} = 1 + x^{1/4}.$$

So the equation becomes

$$(x - 1)^4 + 1 = 1 + x^{1/4}.$$

Hence

$$(x - 1)^4 = x^{1/4}.$$

Again put

$$t = x^{1/4}, \quad t \geq 1 \text{ since } x \geq 1.$$

Then

$$x = t^4.$$

So the equation becomes

$$(t^4 - 1)^4 = t.$$

Instead of solving this directly, test simple values.

For $x = 1$:

$$f(1) = (1 - 1)^4 + 1 = 1,$$

and

$$f^{-1}(1 + 1) = f^{-1}(2) = 1 + (2 - 1)^{1/4} = 2.$$

So $x = 1$ is not a solution.

For $x = 2$:

$$f(2) = (2 - 1)^4 + 1 = 2,$$

and

$$f^{-1}(2 + 1) = f^{-1}(3) = 1 + (3 - 1)^{1/4} = 1 + 2^{1/4} \neq 2.$$

So $x = 2$ is not a solution.

Now define

$$g(x) = (x - 1)^4 - x^{1/4}, \quad x \geq 1.$$

We need to check whether $g(x) = 0$ has any solution.

Observe:

$$g(1) = 0 - 1 = -1 < 0,$$

and

$$g(2) = 1 - 2^{1/4} < 0,$$

while

$$g(3) = 16 - 3^{1/4} > 0.$$

Since g is continuous, there is at least one root in $(2, 3)$.

So the set

$$S = \{x \in [1, \infty) : f(x) = f^{-1}(x + 1)\}$$

is not empty.

Thus, Statement (II) is FALSE.

Therefore, the correct option is

Option A: only (I) is TRUE

Question 13

Let $[\cdot]$ denote the greatest integer function. If the domain of the function $f(x) = \sin^{-1}\left(\frac{x + [x]}{3}\right)$ is $[\alpha, \beta)$, then $\alpha^2 + \beta^2$ is equal to:

JEE Main 2026 (Online) 6th April Morning Shift

Options:

A.

2

B.

5

C.

10

D.

13

Answer: B

Solution:

For the function

$$f(x) = \sin^{-1}\left(\frac{x+[x]}{3}\right),$$

the expression inside $\sin^{-1}$ must lie between -1 and 1 .

So, we need

$$-1 \leq \frac{x+[x]}{3} \leq 1$$

Multiplying throughout by 3,

$$-3 \leq x + [x] \leq 3$$

Now let

$$[x] = n$$

Then by definition of greatest integer function,

$$n \leq x < n + 1$$

and so

$$x + [x] = x + n$$

Hence the condition becomes

$$-3 \leq x + n \leq 3$$

But since $x \in [n, n + 1)$, let us check possible integer values of n .

Case 1: $n = -2$

Then

$$x \in [-2, -1)$$

and

$$x + [x] = x - 2$$

Since $x \in [-2, -1)$,

$$x - 2 \in [-4, -3)$$

This does not satisfy $-3 \leq x + [x]$ except at $x = -1$, but $-1 \notin [-2, -1)$.

So no value from this interval is allowed.

Case 2: $n = -1$

Then

$$x \in [-1, 0)$$

and

$$x + [x] = x - 1$$

So

$$x - 1 \in [-2, -1)$$

This lies completely within $[-3, 3]$.

Hence all $x \in [-1, 0)$ are allowed.

Case 3: $n = 0$

Then

$$x \in [0, 1)$$

and

$$x + [x] = x$$

So

$$x \in [0, 1)$$

This also lies within $[-3, 3]$.

Hence all $x \in [0, 1)$ are allowed.

Case 4: $n = 1$

Then

$$x \in [1, 2)$$

and

$$x + [x] = x + 1$$

So

$$x + 1 \in [2, 3)$$

This lies within $[-3, 3]$.

Hence all $x \in [1, 2)$ are allowed.

Case 5: $n = 2$

Then

$$x \in [2, 3)$$

and

$$x + [x] = x + 2$$

So

$$x + 2 \in [4, 5)$$

This exceeds 3, so no value is allowed.

Now let us also check the boundary points:

- At $x = -\frac{3}{2}$, actually $[x] = -2$, so this does not help.
- Better to solve more carefully near the ends.

We want the full set of x satisfying

$$-3 \leq x + [x] \leq 3$$

Let $x \in [n, n+1)$, so $[x] = n$.

Then

$$-3 \leq x + n \leq 3$$

which gives

$$-3 - n \leq x \leq 3 - n$$

Together with $n \leq x < n+1$, we need intersection of

$$[n, n+1) \cap [-3-n, 3-n].$$

Now check relevant integers:

For $n = -2$

$$x \in [-2, -1) \cap [-1, 5] = \{-1\}$$

but -1 is not included in $[-2, -1)$, so no solution.

For $n = -1$

$$x \in [-1, 0) \cap [-2, 4] = [-1, 0)$$

For $n = 0$

$$x \in [0, 1) \cap [-3, 3] = [0, 1)$$

For $n = 1$

$$x \in [1, 2) \cap [-4, 2] = [1, 2)$$

For $n = 2$

$$x \in [2, 3) \cap [-5, 1] = \emptyset$$

Thus domain is

$$[-1, 0) \cup [0, 1) \cup [1, 2) = [-1, 2)$$

So,

$$\alpha = -1, \quad \beta = 2$$

Therefore,

$$\alpha^2 + \beta^2 = (-1)^2 + 2^2 = 1 + 4 = 5$$

Hence the correct answer is

5

So, **Option B** is correct.

Question 14

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \frac{2x^2 - 3x + 2}{3x^2 + x + 3}$. Then f is :

JEE Main 2026 (Online) 6th April Evening Shift

Options:

A.

both one-one and onto

B.

one-one but not onto

C.

onto but not one-one

D.

neither one-one nor onto

Answer: D

Solution:

To determine the nature of the given function, let us carefully analyze it for both the one-one (injective) and onto (surjective) properties. The function is given by:

$$f(x) = \frac{2x^2 - 3x + 2}{3x^2 + x + 3}$$

where the domain is the set of real numbers $\mathbf{R}$ and the codomain is also $\mathbf{R}$.

Checking if the function is one-one

A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is said to be one-one if every element in the domain has a distinct image in the codomain. That is, if $f(x_1) = f(x_2)$, then we must have $x_1 = x_2$.

Let us check if we can find two different real values of x that give the same value for $f(x)$. Let us set $f(x) = 1$ and solve for x :

$$\frac{2x^2 - 3x + 2}{3x^2 + x + 3} = 1$$

Cross-multiplying, we obtain:

$$2x^2 - 3x + 2 = 3x^2 + x + 3$$

Bringing all terms to one side, we get:

$$x^2 + 4x + 1 = 0$$

Using the quadratic formula to find the roots:

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$$

Since we found two distinct real values, $x_1 = -2 + \sqrt{3}$ and $x_2 = -2 - \sqrt{3}$, such that $f(x_1) = f(x_2) = 1$, the function is not one-one.

Checking if the function is onto

A function is onto if its range is equal to its codomain. Here, the codomain is $\mathbf{R}$. To find the range, let us set $f(x) = y$:

$$y = \frac{2x^2 - 3x + 2}{3x^2 + x + 3}$$

Cross-multiplying gives us:

$$y(3x^2 + x + 3) = 2x^2 - 3x + 2$$

$$3yx^2 + yx + 3y = 2x^2 - 3x + 2$$

Grouping the terms by the powers of x , we form a quadratic equation in x :

$$(3y - 2)x^2 + (y + 3)x + (3y - 2) = 0$$

Since x is a real number, the discriminant (D) of this quadratic equation must be greater than or equal to zero ($D \geq 0$).

$$D = (y + 3)^2 - 4(3y - 2)(3y - 2) \geq 0$$

$$(y + 3)^2 - 4(3y - 2)^2 \geq 0$$

This is in the form of $a^2 - b^2 \geq 0$, which can be factored as $(a - b)(a + b) \geq 0$. Here, $a = y + 3$ and $b = 2(3y - 2) = 6y - 4$:

$$[(y + 3) - (6y - 4)][(y + 3) + (6y - 4)] \geq 0$$

$$(-5y + 7)(7y - 1) \geq 0$$

To solve this inequality, we can multiply by -1 (which reverses the inequality sign):

$$(5y - 7)(7y - 1) \leq 0$$

The critical points are $y = \frac{1}{7}$ and $y = \frac{7}{5}$. Based on the inequality, y must lie between these two values:

$$\frac{1}{7} \leq y \leq \frac{7}{5}$$

This means the range of the function is the closed interval $[\frac{1}{7}, \frac{7}{5}]$.

Because the range is a small subset of the real numbers and not equal to the entire codomain $\mathbf{R}$, the function is not onto.

Conclusion:

Since $f(x)$ is neither one-one nor onto, the correct statement is given by Option D.

Option D

Question 15

If the domain of the function

$f(x) = \sqrt{\log_{(0.6)} \left(\left| \frac{2x-5}{x^2-4} \right| \right)}$ is $(-\infty, a] \cup \{b\} \cup [c, d) \cup (e, \infty)$, then the value of $a + b + c + d + e$ is _____.

JEE Main 2026 (Online) 2nd April Morning Shift

Answer: 4

Solution:

For domain $\log_{0.6} \left| \frac{2x-5}{x^2-4} \right| \geq 0$

$$\frac{2x-5}{x^2-4} \leq 1 \quad \& \quad x \neq \frac{5}{2} \quad \dots\dots (1)$$

$$-1 \leq \frac{2x-5}{x^2-4} \leq 1$$

$$\frac{2x-5}{x^2-4} + 1 \geq 0$$

$$\frac{x^2+2x-9}{x^2-4} \geq 0$$

$$\frac{(x+1)^2-10}{(x-2)(x+2)} \geq 0$$



$$x \in \left(-\infty, -1 - \sqrt{10}\right] \cup (-2, 2) \cup \left[-1 + \sqrt{10}, \infty\right) \dots\dots (2)$$

$$\frac{2x-5}{x^2-4} - 1 \leq 0$$

$$\frac{2x-5-x^2+4}{x^2-4} \leq 0$$

$$\frac{x^2-2x+1}{x^2-4} \geq 0$$

$$\frac{(x-1)^2}{(x-2)(x+2)} \geq 0$$

$$x \in (-\infty, -2) \cup (2, \infty) \cup \{1\} \dots\dots (3)$$

$$(1) \cap (2) \cap (3)$$

$$x \in \left(-\infty, -1 - \sqrt{10}\right] \cup \{1\} \cup \left[-1 + \sqrt{10}, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$$

$$a + b + c + d + e = -2 + 1 + 5 = 4$$

Question16

Let $A = \{1, 2, 3, 4, 5, 6\}$. The number of one-one functions $f : A \rightarrow A$ such that $f(1) \geq 3$, $f(3) \leq 4$ and $f(2) + f(3) = 5$, is _____.

JEE Main 2026 (Online) 5th April Morning Shift

Answer: 72

Solution:

We need the number of one-one functions $f : A \rightarrow A$, where

$$A = \{1, 2, 3, 4, 5, 6\}$$

and the conditions are:

1. $f(1) \geq 3$
2. $f(3) \leq 4$
3. $f(2) + f(3) = 5$

Since f is one-one and domain and codomain are the same finite set, such a function is a permutation of A .

Now let us count carefully.

From

$$f(2) + f(3) = 5$$

and since values are from $A = \{1, 2, 3, 4, 5, 6\}$, the possible pairs are:

$$(f(2), f(3)) = (1, 4), (2, 3), (3, 2), (4, 1)$$

Also $f(3) \leq 4$, which is already satisfied by all these cases.

Now we use the condition that f is one-one, so all function values must be distinct.

We also have:

$$f(1) \geq 3 \implies f(1) \in \{3, 4, 5, 6\}$$

But $f(1)$ must be different from both $f(2)$ and $f(3)$.

Case 1: $(f(2), f(3)) = (1, 4)$

Then $f(1) \in \{3, 4, 5, 6\}$, but it cannot be 4 because function is one-one.

So,

$$f(1) \in \{3, 5, 6\}$$

Number of choices for $f(1) = 3$.

After fixing $f(1), f(2), f(3)$, the remaining 3 elements of the domain can be mapped to the remaining 3 elements of the codomain in

$$3! = 6$$

ways.

So total in this case:

$$3 \times 6 = 18$$

Case 2: $(f(2), f(3)) = (2, 3)$

Then $f(1) \in \{3, 4, 5, 6\}$, but it cannot be 3.

So,

$$f(1) \in \{4, 5, 6\}$$

Number of choices for $f(1) = 3$.

Remaining mappings can be done in

$$3! = 6$$

ways.

Total in this case:

$$3 \times 6 = 18$$

Case 3: $(f(2), f(3)) = (3, 2)$

Then $f(1) \in \{3, 4, 5, 6\}$, but it cannot be 3.

So again number of choices for $f(1) = 3$.

Remaining mappings:

$$3! = 6$$

Total:

$$3 \times 6 = 18$$

Case 4: $(f(2), f(3)) = (4, 1)$

Then $f(1) \in \{3, 4, 5, 6\}$, but it cannot be 4.

So number of choices for $f(1) = 3$.

Remaining mappings:

$$3! = 6$$

Total:

$$3 \times 6 = 18$$

Total number of functions

$$18 + 18 + 18 + 18 = 72$$

Hence, the required number of one-one functions is

$$\boxed{72}$$

Question 17

Let f be a polynomial function such that

$$\log_2(f(x)) = \left(\log_2\left(2 + \frac{2}{3} + \frac{2}{9} + \dots + \infty\right)\right) \cdot \log_3\left(1 + \frac{f(x)}{f(1/x)}\right), x > 0$$

and $f(6) = 37$. Then $\sum_{n=1}^{10} f(n)$ is equal to _____.

JEE Main 2026 (Online) 8th April Evening Shift

Answer: 395

Solution:

Statement of Work :

1. 1. $f(x)$ is a polynomial function

2. 2. A logarithmic equation involving function

$$f : \log_2[f(x)] = \left[\log_2 \left(2 + \frac{2}{3} + \frac{2}{9} + \dots \infty \right) \right] \cdot \log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\}; x > 0 \text{ and } f(6) = 37$$

3. 3. To evaluate the sum : $\sum_{n=1}^{10} f(n)$

$$\begin{aligned} \log_2[f(x)] &= \left[\log_2 \left(2 + \frac{2}{3} + \frac{2}{9} + \dots \infty \right) \right] \cdot \left[\log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\} \right] \\ &= \left[\log_2 2 \left\{ 1 + \frac{1}{3} + \left(\frac{1}{3} \right)^2 + \dots \infty \right\} \right] \cdot \left[\log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\} \right] \\ &= \left[\log_2 2 \left(\frac{1}{1 - \frac{1}{3}} \right) \right] \cdot \left[\log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\} \right] \\ &= [\log_2 3] \cdot \left[\log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\} \right] \end{aligned}$$

$$\frac{\log_2[f(x)]}{\log_2 3} = \left[\log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\} \right]$$

$$\Rightarrow \log_3[f(x)] = \log_3 \left\{ 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)} \right\}$$

$$\Rightarrow f(x) = 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}$$

$$\Rightarrow f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \dots (i)$$

The function $f(x)$ is a polynomial function stated in SOW where $f(6) = 37 = (6)^2 + 1 \dots (ii)$

Equation (ii) on logical analysis reduces the polynomial $f(x)$ to the possible form below :

$$f(x) = x^2 + 1 \dots (iii) \leftarrow \text{Cubic powers of } x \text{ is not possible where } f(6) > 216$$

Now let us analyze the polynomial function $f(x)$ from preconditions set in SOW :

$$f(x) \cdot f\left(\frac{1}{x}\right) = \left(1 + x^2\right) \cdot \left(1 + \frac{1}{x^2}\right) = 1 + \frac{1}{x^2} + x^2 + 1 = \left(1 + x^2\right) + \left(1 + \frac{1}{x^2}\right) = f(x) + f\left(\frac{1}{x}\right) \dots (iv)$$

Equation (ii), (iii) and (iv) mathematically certify the polynomial $f(x) = 1 + x^2$:

$$f(x) = 1 + x^2$$

$$\begin{aligned}\Rightarrow s_{10} &= \sum_{n=1}^{10} f(n) = \sum_{n=1}^{10} (1 + n^2) = \sum_{n=1}^{10} (1) + \sum_{n=1}^{10} (n^2) \\ &= 10 + \frac{(10) \cdot (10+1) \cdot \{(2) \cdot (10)+1\}}{6} = 10 + 385 = 395\end{aligned}$$

Question18

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16\}$. Then the number of many-one functions $f : A \rightarrow B$ such that $1 \in f(A)$ is equal to :

JEE Main 2025 (Online) 22nd January Evening Shift

Options:

A. 151

B. 139

C. 163

D. 127

Answer: A

Solution:

Step 1: Total Functions with $1 \in f(A)$

Any function $f : A \rightarrow B$ where $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16\}$ is defined by choosing one of the four elements of B for each element of A . Thus, the total number of functions is

$$4^4 = 256.$$

To count those functions where 1 appears at least once in the set $f(A)$, we can use the complementary counting method: subtract the functions that never use 1. If 1 is excluded, each element of A has only 3 choices (namely, $\{4, 9, 16\}$), so the number of such functions is

$$3^4 = 81.$$

Thus, the number of functions such that $1 \in f(A)$ is

$$256 - 81 = 175.$$

Step 2: Counting Many-One Functions

In this context, "many-one functions" are understood to be non-injective functions. Since an injective (one-to-one) function from A to B must be a permutation (because both sets have 4 elements), the number of one-to-one functions is

$$4! = 24.$$

It is important to note that every injective function $f : A \rightarrow B$ has $f(A) = B$ (a full permutation) which automatically means $1 \in f(A)$.

Thus, the number of many-one (non-injective) functions $f : A \rightarrow B$ with $1 \in f(A)$ is found by subtracting the one-to-one functions from the total functions that include 1:

$$175 - 24 = 151.$$

151

This detailed explanation shows that the number of many-one functions $f : A \rightarrow B$ such that $1 \in f(A)$ is indeed 151.

Question 19

Let $f(x) = \log_e x$ and $g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$. Then the domain of $f \circ g$ is

JEE Main 2025 (Online) 23rd January Morning Shift

Options:

A. $(0, \infty)$

B. $[1, \infty)$

C. $\mathbb{R}$

D. $[0, \infty)$

Answer: C

Solution:

$$f(x) = \ln x$$

$$g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$$

$$D_g \in \mathbb{R}$$

$$D_f \in (0, \infty)$$

$$\text{For } D_{f \circ g} \Rightarrow g(x) > 0$$

$$\frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1} > 0$$

$$\Rightarrow x^4 - 2x^3 + 3x^2 - 2x + 2 > 0$$

Clearly $x < 0$ satisfies which are included in option (1) only.

Question 20

Let $f(x) = \frac{2^{x+2} + 16}{2^{2x+1} + 2^{x+4} + 32}$. Then the value of $8\left(f\left(\frac{1}{15}\right) + f\left(\frac{2}{15}\right) + \dots + f\left(\frac{59}{15}\right)\right)$ is equal to

JEE Main 2025 (Online) 24th January Morning Shift

Options:

- A. 108
- B. 92
- C. 118
- D. 102

Answer: C

Solution:

$$f(x) = \frac{42^x + 16}{2 \cdot 2^{2x} + 16 \cdot 2^x + 32}$$

$$f(x) = \frac{2(2^x + 4)}{2^{2x} + 8 \cdot 2^x + 16}$$

$$f(x) = \frac{2}{2^x + 4}$$

$$f(4-x) = \frac{2^x}{2(2^x + 4)}$$

$$f(x) + f(4-x) = \frac{1}{2}$$

$$\text{So, } f\left(\frac{1}{15}\right) + f\left(\frac{59}{15}\right) = \frac{1}{2}$$

$$\begin{aligned}\text{Similarly } &= f\left(\frac{29}{15}\right) + f\left(\frac{31}{15}\right) = \frac{1}{2} \\ f\left(\frac{30}{15}\right) &= f(2) = \frac{2}{2^2 + 4} = \frac{2}{8} = \frac{1}{4} \\ \Rightarrow &8\left(29 \times \frac{1}{2} + \frac{1}{4}\right)\end{aligned}$$

Ans. 118

Question 21

The function $f: (-\infty, \infty) \rightarrow (-\infty, 1)$, defined by $f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$ is :

JEE Main 2025 (Online) 24th January Evening Shift

Options:

- A. One-one but not onto
- B. Onto but not one-one
- C. Both one-one and onto
- D. Neither one-one nor onto

Answer: A

Solution:

$$\begin{aligned}f(x) &= \frac{2^{2x} - 1}{2^{2x} + 1} \\ &= 1 - \frac{2}{2^{2x} + 1} \\ f'(x) &= \frac{2}{(2^{2x} + 1)^2} \cdot 2 \cdot 2^{2x} \cdot \ln 2 \text{ i.e always } +ve\end{aligned}$$

so $f(x)$ is $\uparrow$ function

$$\therefore f(-\infty) = -1$$

$$f(\infty) = 1$$

$$\therefore f(x) \in (-1, 1) \neq \text{co-domain}$$

so function is one-one but not onto

Question 22

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = (2 + 3a)x^2 + \left(\frac{a+2}{a-1}\right)x + b, a \neq 1$. If

$f(x + y) = f(x) + f(y) + 1 - \frac{2}{7}xy$, then the value of $28 \sum_{i=1}^5 |f(i)|$ is

JEE Main 2025 (Online) 28th January Morning Shift

Options:

A. 735

B. 675

C. 715

D. 545

Answer: B

Solution:

$$f(x) = (3a + 2)x^2 + \left(\frac{a+2}{a-1}\right)x + b$$

$$f\left(x + \frac{1}{2}\right) = f(x) + f(y) + 1 - \frac{2}{7}xy \quad \dots (1)$$

$$\text{In (1) Put } x = y = 0 \Rightarrow f(0) = 2f(0) + 1 \Rightarrow f(0) = -1$$

$$\text{So, } f(0) = 0 + 0 + b = -1 \Rightarrow b = -1$$

$$\text{In (1) Put } y = -x \Rightarrow f(0) = f(x) + f(-x) + 1 + \frac{2}{7}x^2$$

$$-1 = 2(3a + 2)x^2 + 2b + 1 + \frac{2}{7}x^2$$

$$-1 = \left(2(3a + 2) + \frac{2}{7}\right)x^2 + 1 - 2$$

$$\Rightarrow 6a + 4 + \frac{2}{7} = 0$$

$$a = -\frac{5}{7}$$

$$\text{So } f(x) = -\frac{1}{7}x^2 - \frac{3}{4}x - 1$$

$$\Rightarrow |f(x)| = \frac{1}{28} |4x^2 + 21x + 28|$$

Now, $28 \sum_{i=1}^5 |f(i)| = 28(|f(1)| + |f(2)| + \dots + |f(5)|)$

$$28 \cdot \frac{1}{28} \cdot 675 = 675$$

Question 23

If $f(x) = \frac{2^x}{2^x + \sqrt{2}}$, $x \in \mathbb{R}$, then $\sum_{k=1}^{81} f\left(\frac{k}{82}\right)$ is equal to

JEE Main 2025 (Online) 28th January Morning Shift

Options:

A. 82

B. $81\sqrt{2}$

C. 41

D. $\frac{81}{2}$

Answer: D

Solution:

$$f(x) = \frac{2^x}{2^x + \sqrt{2}}$$

$$\begin{aligned} f(x) + f(1-x) &= \frac{2^x}{2^x + \sqrt{2}} + \frac{2^{1-x}}{2^{1-x} + \sqrt{2}} \\ &= \frac{2^x}{2^x + \sqrt{2}} + \frac{2}{2 + \sqrt{2}2^x} = \frac{2^x + \sqrt{2}}{2^x + \sqrt{2}} = 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } \sum_{k=1}^{81} f\left(\frac{k}{82}\right) &= f\left(\frac{1}{82}\right) + f\left(\frac{2}{82}\right) + \dots + f\left(\frac{81}{82}\right) \\ &= f\left(\frac{1}{82}\right) + f\left(\frac{2}{82}\right) + \dots + f\left(1 - \frac{2}{82}\right) + f\left(1 - \frac{1}{82}\right) \\ &= \left[f\left(\frac{1}{82}\right) + f\left(1 - \frac{1}{82}\right)\right] + \left[f\left(\frac{2}{82}\right) + f\left(1 - \frac{2}{82}\right)\right] + \dots + 40 \text{ cases} + f\left(\frac{41}{82}\right) \end{aligned}$$

$$= (1 + 1 + \dots 40 \text{ times}) + \frac{2^{1/2}}{2^{1/2} + 2^{1/2}}$$

$$40 + \frac{1}{2} = \frac{81}{2}$$

Question 24

Let $f : [0, 3] \rightarrow A$ be defined by $f(x) = 2x^3 - 15x^2 + 36x + 7$ and $g : [0, \infty) \rightarrow B$ be defined by $g(x) = \frac{x^{2025}}{x^{2025} + 1}$. If both the functions are onto and $S = \{x \in \mathbb{Z}; x \in A \text{ or } x \in B\}$, then $n(S)$ is equal to :

JEE Main 2025 (Online) 28th January Evening Shift

Options:

A.

29

B.

31

C.

30

D.

36

Answer: C

Solution:

as $f(x)$ is onto hence A is range of $f(x)$

$$\begin{aligned} \text{now } f'(x) &= 6x^2 - 30x + 36 \\ &= 6(x-2)(x-3) \end{aligned}$$

$$f(2) = 16 - 60 + 72 + 7 = 35$$

$$f(3) = 54 - 135 + 108 + 7 = 34$$

$$f(0) = 7$$

$$\text{hence range} \in [7, 35] = A$$

also for range of $g(x)$

$$g(x) = 1 - \frac{1}{x^{2025} + 1} \in [0, 1) = B$$

$$s = \{0, 7, 8, \dots, 35\} \text{ hence } n(s) = 30$$

Question 25

If the domain of the function $\log_5(18x - x^2 - 77)$ is (α, β) and the domain of the function $\log_{(x-1)}\left(\frac{2x^2+3x-2}{x^2-3x-4}\right)$ is (γ, δ) , then $\alpha^2 + \beta^2 + \gamma^2$ is equal to:

JEE Main 2025 (Online) 29th January Evening Shift

Options:

A.

186

B.

179

C.

195

D.

174

Answer: A

Solution:

$$f_1(x) = \log_5(18x - x^2 - 77)$$

$$\therefore 18x - x^2 - 77 > 0$$

$$x^2 - 18x + 77 < 0$$

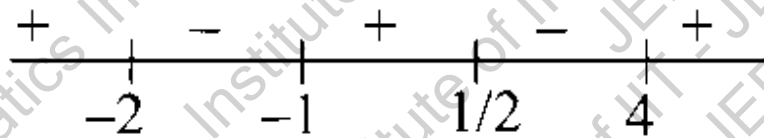
$$x \in (7, 11) \alpha = 7, \beta = 11$$

$$f_2(x) = \log_{(x-1)}\left(\frac{2x^2+3x-2}{x^2-3x-4}\right)$$

$$\therefore x - 1 > 0, x - 1 \neq 1, \frac{2x^2+3x-2}{x^2-3x-4} > 0$$

$$x > 1, x \neq 2, \frac{(2x-1)(x+2)}{(x-4)(x+1)} > 0$$

$$x > 1, x \neq 2,$$



$$\begin{aligned}\therefore x &\in (4, \infty) \\ \therefore \gamma &= 4 \\ \therefore \alpha^2 + \beta^2 + \gamma^2 &= 49 + 121 + 16 \\ &= 186\end{aligned}$$

Question 26

If the domain of the function $f(x) = \frac{1}{\sqrt{10+3x-x^2}} + \frac{1}{\sqrt{x+|x|}}$ is (a, b) , then $(1+a)^2 + b^2$ is equal to :

JEE Main 2025 (Online) 2nd April Evening Shift

Options:

- A. 29
- B. 30
- C. 25
- D. 26

Answer: D

Solution:

$$x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\Rightarrow \frac{1}{\sqrt{x+|x|}}, \text{ domain is } x > 0, \text{ as } 2x \neq 0$$

Similarly,

$$\frac{1}{\sqrt{3x+10-x^2}} \text{ is defined when } 3x+10-x^2 > 0$$

$$\Rightarrow x^2 - 3x - 10 < 0$$

$$(x-5)(x+2) < 0$$

$$\Rightarrow x \in (-2, 5)$$

$$\Rightarrow \text{Domain will be } (0, \infty) \cap (-2, 5) = (0, 5)$$

$$\Rightarrow (1+a)^2 + b^2 = 1 + 25 = 26$$

Question 27

If the domain of the function is

$f(x) = \log_e \left(\frac{2x-3}{5+4x} \right) + \sin^{-1} \left(\frac{4+3x}{2-x} \right)$ is $[\alpha, \beta)$, then $\alpha^2 + 4\beta$ equal to

Options:

A. 4

B. 3

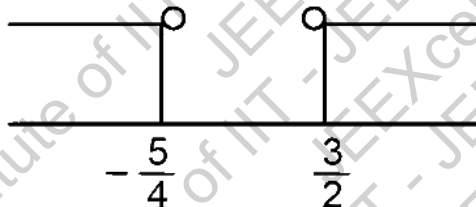
C. 7

D. 5

Answer: A

Solution:

$$\frac{2x-3}{4x+5} > 0$$



$$\therefore x \in \left(-\infty, -\frac{5}{4} \right) \cup \left(\frac{3}{2}, \infty \right) \dots\dots\dots (i)$$

$$-1 \leq \frac{3x+4}{2-x} \leq 1$$

$$\frac{3x+4}{2-x} \leq 1$$

$$\Rightarrow \frac{3x+4}{2-x} - 1 \leq 0$$

$$\Rightarrow \frac{3x+4-2+x}{x-2} \geq 0$$

$$\Rightarrow \frac{4x+2}{x-2} \geq 0$$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{2} \right] \cup (2, \infty) \dots\dots\dots (ii)$$

$$\begin{aligned}\frac{3x+4}{2-x} &\geq -1 \\ \Rightarrow \frac{3x+4}{2-x} + 1 &\geq 0 \\ \Rightarrow \frac{3x+4+2-x}{2-x} &\geq 0 \\ \Rightarrow \frac{2x+6}{x-2} &\leq 0 \\ \therefore x \in [-3, 2) \dots\dots\dots (iii)\end{aligned}$$

Taking intersection of (i), (ii) and (iii)

$$\begin{aligned}x &\in \left[-3, -\frac{5}{4}\right) \\ \alpha = -3, \beta &= -\frac{5}{4} \\ \alpha^2 + 4\beta &= 4\end{aligned}$$

Question 28

If the domain of the function $f(x) = \log_7(1 - \log_4(x^2 - 9x + 18))$ is $(\alpha, \beta) \cup (\gamma, \delta)$, then $\alpha + \beta + \gamma + \delta$ is equal to

JEE Main 2025 (Online) 3rd April Evening Shift

Options:

- A. 17
- B. 15
- C. 16
- D. 18

Answer: D

Solution:

$$1 - \log_4(x^2 - 9x + 18) > 0$$

$$\log_4(x^2 - 9x + 18) < 1$$

$$x^2 - 9x + 18 < 4$$

$$x^2 - 9x + 14 < 0$$

$$x \in (2, 7)$$

$$x^2 - 9x + 18 > 0$$

$$x \in (-\infty, 3) \cup (6, \infty)$$



$$x \in (2, 3) \cup (6, 7)$$

$$\alpha + \beta + \gamma + \delta = 18$$

Question 29

Let f be a function such that $f(x) + 3f\left(\frac{24}{x}\right) = 4x, x \neq 0$. Then $f(3) + f(8)$ is equal to

JEE Main 2025 (Online) 3rd April Evening Shift

Options:

A. 13

B. 11

C. 10

D. 12

Answer: B

Solution:

$$f(x) + 3f\left(\frac{24}{x}\right) = 4x, x \neq 0 \dots (1)$$

replace x by $\frac{24}{x}$

$$f\left(\frac{24}{x}\right) + 3f\left(\frac{24}{24}\right) = 4\left(\frac{24}{x}\right) = \frac{96}{x} \quad \dots (2)$$

$$3 \times (2) - (1)$$

$$\Rightarrow 8f(x) = \frac{96.3}{x} - 4x \Rightarrow f(x) = \frac{36}{x} - \frac{x}{2}$$

$$f(3) + f(8) = \left(12 - \frac{3}{2}\right) + \left(\frac{36}{8} - 4\right)$$

$$= 8 + \frac{36}{8} - \frac{12}{8} = 11$$

Question 30

Let $f, g : (1, \infty) \rightarrow \mathbb{R}$ be defined as $f(x) = \frac{2x+3}{5x+2}$ and $g(x) = \frac{2-3x}{1-x}$. If the range of the function fog: $[2, 4] \rightarrow \mathbb{R}$ is $[\alpha, \beta]$, then $\frac{1}{\beta-\alpha}$ is equal to

JEE Main 2025 (Online) 4th April Morning Shift

Options:

- A. 56
- B. 2
- C. 29
- D. 68

Answer: A

Solution:

$$g(2) = 4, g(4) = \frac{10}{3}$$

f is decreasing in $\left(\frac{10}{3}, 4\right)$

$$\begin{aligned}\therefore \alpha &= f(4) = \frac{1}{2} \\ \beta &= f\left(\frac{10}{3}\right) = \frac{29}{56} \\ \frac{1}{\beta - \alpha} &= \frac{1}{\frac{29}{56} - \frac{1}{2}} = 56\end{aligned}$$

Question 31

Let the domains of the functions

$f(x) = \log_4 \log_3 \log_7 (8 - \log_2 (x^2 + 4x + 5))$ and $g(x) = \sin^{-1} \left(\frac{7x+10}{x-2} \right)$ be (α, β) and $[\gamma, \delta]$, respectively. Then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is equal to :

JEE Main 2025 (Online) 4th April Evening Shift

Options:

- A. 15
- B. 13
- C. 16
- D. 14

Answer: A

Solution:

$$f(x) = \log_4 (\log_3 (\log_7 (8 - \log_2 (x^2 + 4x + 5))))$$

$$\log_3 (\log_1 (8 - \log_2 (x^2 + 4x + 5))) > 0$$

$$\log_7 (8 - \log_2 (x^2 + 4x + 5)) > 1$$

$$8 - \log_2 (x^2 + 4x + 5) > 7$$

$$-\log_2 (x^2 + 4x + 5) > -1$$

$$\log_2(x^2 + 4x + 5) < 1$$

$$x^2 + 4x + 5 < 2$$

$$x^2 + 4x + 3 < 0$$

$$\Rightarrow (x+3)(x+1) < 0 \quad \dots (1)$$

$$\log_7(8 - \log_2(x^2 + 4x + 5)) > 0$$

$$8 - \log_2(x^2 + 4x + 5) > 1$$

$$\log_2(x^2 + 4x + 5) < 9$$

$$x^2 + 4x + 5 < 2^9$$

$$x^2 + 4x + 5 < 512$$

$$\Rightarrow x^2 + 4x - 507 < 0$$

$$\Rightarrow x = -4 \pm \sqrt{16 + 2028}$$

$$x = \frac{-4 \pm \sqrt{2044}}{2} \quad \dots (2)$$

$$\Rightarrow \left(x - \left(\frac{-4 + \sqrt{2044}}{2} \right) \right) \left(x - \left(\frac{-4 - \sqrt{2044}}{2} \right) \right) < 0$$

$$x^2 + 4x + 5 > 0$$

$$D > 0$$

$$x \in R$$

$$\text{Also, } 8 - \log_2(x^2 + 4x + 5) > 0$$

$$\log_2(x^2 + 4x + 5) < 8$$

$$x^2 + 4x + 5 < 256$$

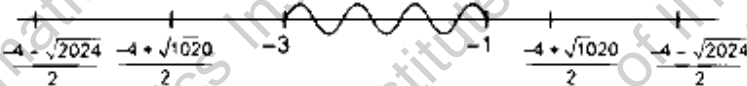
$$\Rightarrow x^2 + 4x - 251 < 0$$

$$\Rightarrow x = -4 \pm \sqrt{16 + 1004}$$

$$\Rightarrow x = \frac{-4 \pm \sqrt{1020}}{2}$$

$$\Rightarrow \left(x - \left(\frac{-4 + \sqrt{1020}}{2} \right) \right) \left(x - \left(\frac{-4 - \sqrt{1020}}{2} \right) \right) < 0$$

$\therefore$ Intersection of (1), (2) and (3)



$$\therefore x \in (-3, -1)$$

$$-1 \leq \frac{7x + 10}{x - 2} \leq 1$$

$$\Rightarrow x \in [-2, -1]$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (-3)^2 + (-1)^2 + (-2)^2 + (-1)^2$$

$$= 9 + 1 + 4 + 1$$

$$= 15$$

Question32

If the range of the function $f(x) = \frac{5-x}{x^2-3x+2}$, $x \neq 1, 2$, is $(-\infty, \alpha] \cup [\beta, \infty)$, then $\alpha^2 + \beta^2$ is equal to :

JEE Main 2025 (Online) 7th April Evening Shift

Options:

A.

188

B.

192

C.

190

D.

194

Answer: D

Solution:

$$y = \frac{5-x}{x^2-3x+2}$$

$$yx^2 - 3xy + 2y + x - 5 = 0$$

$$yz^2 + (-3y+1)x + (2y-5) = 0$$

Case I : If $y = 0$ (Accepted)

$$\Rightarrow x = 5$$

Case II : If $y \neq 0$

$$D \geq 0$$

$$(-3y+1)^2 - 4(y)(2y-5) \geq 0$$

$$9y^2 + 1 - 6y - 8y^2 + 20y \geq 0$$

$$y^2 + 14y + 1 \geq 0$$

$$(y+7)^2 - 48 \geq 0$$

$$|y+7| \geq 4\sqrt{3}$$

$$\Rightarrow y+7 \geq 4\sqrt{3} \text{ or } y+7 \leq -4\sqrt{3}$$

$$\Rightarrow y \geq 4\sqrt{3} - 7 \text{ or } y \leq -4\sqrt{3} - 7$$

From Case I and Case II

$$y \in (-\infty, -4\sqrt{3} - 7] \cup [4\sqrt{3} - 7, \infty)$$

$$\text{So } \alpha = -4\sqrt{3} - 7$$

$$\beta = 4\sqrt{3} - 7$$

$$\begin{aligned} \Rightarrow a^2 + b^2 &= (-4\sqrt{3} - 7)^2 + (4\sqrt{3} - 7)^2 \\ &= 2(48 + 49) \\ &= 194 \end{aligned}$$

Question 33

Let the domain of the function $f(x) = \cos^{-1}\left(\frac{4x+5}{3x-7}\right)$ be $[\alpha, \beta]$ and the domain of $g(x) = \log_2(2 - 6\log_{27}(2x+5))$ be (γ, δ) .

Then $|7(\alpha + \beta) + 4(\gamma + \delta)|$ is equal to _____.

JEE Main 2025 (Online) 8th April Evening Shift

Answer: 96

Solution:

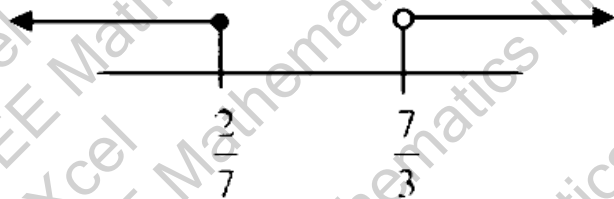
$$f(x) = \cos^{-1}\left(\frac{4x+5}{3x-7}\right)$$

$$\Rightarrow -1 \leq \left(\frac{4x+5}{3x-7} \right) \leq 1$$

$$\left(\frac{4x+5}{3x-7} \right) \geq -1$$

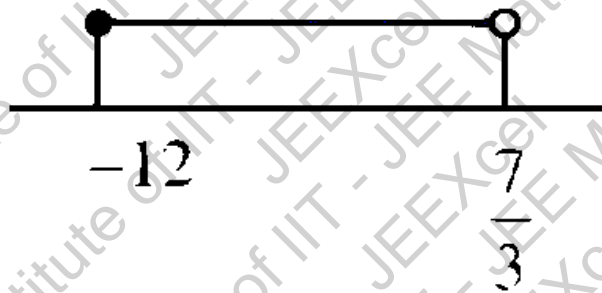
$$\frac{4x+5+3x-7}{3x-7} \geq 0$$

$$\Rightarrow \frac{7x-2}{3x-7} \geq 0$$



$$x \in \left(-\infty, \frac{2}{7} \right] \cup \left(\frac{7}{3}, \infty \right)$$

$$\& \frac{4x+5}{3x-7} \leq 1 \Rightarrow \frac{x+12}{3x-7} \leq 0$$



$\therefore$ Domain of $f(x)$ is

$$\left[-12, \frac{2}{7} \right] \alpha = -12, \beta = \frac{2}{7}$$

$$g(x) = \log_2 (2 - 6 \log_{27} (2x + 5))$$

Domain

$$2 - 6 \log_{27} (2x + 5) > 0$$

$$\Rightarrow 6 \log_{27} (2x + 5) < 2$$

$$\Rightarrow \log_{27} (2x + 5) < \frac{1}{3}$$

$$\Rightarrow 2x + 5 < 3$$

$$\Rightarrow x < -1$$

$$\& 2x + 5 > 0 \Rightarrow x > -\frac{5}{2}$$

$$\text{Domain is } x \in \left(-\frac{5}{2}, -1 \right)$$

$$\gamma = -\frac{5}{2}, \delta = -1$$

$$|7(\alpha + \beta) + 4(\gamma + \delta)| = \left| 7\left(-12 + \frac{2}{7}\right) + 4\left(-\frac{5}{2} - 1\right) \right|$$

$$|-82 - 14| = 96$$

Question34

The function $f : \mathbb{N} - \{1\} \rightarrow \mathbb{N}$; defined by $f(n) =$ the highest prime factor of n , is :

[27-Jan-2024 Shift 1]

Options:

A.
both one-one and onto

B.
one-one only

C.
onto only

D.
neither one-one nor onto

Answer: D

Solution:

$$f: \mathbb{N} - \{1\} \rightarrow \mathbb{N}$$

$f(n) =$ The highest prime factor of n .

$$f(2) = 2$$

$$f(4) = 2$$

$\Rightarrow$ many one

4 is not image of any element

$\Rightarrow$ into

Hence many one and into

Neither one-one nor onto.

Question35

Let $f: \mathbb{R} - \left\{ \frac{-1}{2} \right\} \rightarrow \mathbb{R}$ and $g: \mathbb{R} - \left\{ \frac{-5}{2} \right\} \rightarrow \mathbb{R}$ be defined as $f(x) = \frac{2x+3}{2x+1}$ and $g(x) = \frac{|x|+1}{2x+5}$. Then the domain of the function $f \circ g$ is :

[27-Jan-2024 Shift 2]

Options:

A.

$$\mathbb{R} - \left\{ -\frac{5}{2} \right\}$$

B.

$$\mathbb{R}$$

C.

$$\mathbb{R} - \left\{ -\frac{7}{4} \right\}$$

D.

$$\mathbb{R} - \left\{ -\frac{5}{2}, -\frac{7}{4} \right\}$$

Answer: A

Solution:

$$f(x) = \frac{2x+3}{2x+1}; x \neq -\frac{1}{2}$$

$$g(x) = \frac{|x|+1}{2x+5}; x \neq -\frac{5}{2}$$

Domain of $f(g(x))$

$$f(g(x)) = \frac{2g(x)+3}{2g(x)+1}$$

$$x \neq -\frac{5}{2} \text{ and } \frac{|x|+1}{2x+5} \neq -\frac{1}{2}$$

$$x \in \mathbb{R} - \left\{ -\frac{5}{2} \right\} \text{ and } x \in \mathbb{R}$$

$$\therefore \text{Domain will be } \mathbb{R} - \left\{ -\frac{5}{2} \right\}$$

Question 36

Consider the function $f : [1/2, 1] \rightarrow \mathbb{R}$ defined by $f(x) = 4\sqrt{2x} - 3\sqrt{2x}^{-1}$.
Consider the statements

(I) The curve $y = f(x)$ intersects the x-axis exactly at one point

(II) The curve $y = f(x)$ intersects the x-axis at $x = \cos \pi/12$

Then

[29-Jan-2024 Shift 1]

Options:

A.

Only (II) is correct

B.

Both (I) and (II) are incorrect

C.

Only (I) is correct

D.

Both (I) and (II) are correct

Answer: D

Solution:

$$f(x) = 12\sqrt{2}x^2 - 3\sqrt{2} \geq 0 \text{ for } \left[\frac{1}{2}, 1 \right]$$

$$f\left(\frac{1}{2}\right) < 0$$

$f(1) > 0 \Rightarrow$ (A) is correct.

$$f(x) = \sqrt{2}(4x^3 - 3x) - 1 = 0$$

Let $\cos \alpha = x$,

$$\cos 3\alpha = \cos \frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{12}$$

$$x = \cos \frac{\pi}{12}$$

(4) is correct.

Question 37

$$\text{If } f(x) = \begin{cases} 2+2x & -1 \leq x < 0 \\ 1 - \frac{x}{3} & 0 \leq x \leq 3 \end{cases}$$

$$g(x) = \begin{cases} -x & -3 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases} \dots$$

then range of $(f \circ g(x))$ is

[29-Jan-2024 Shift 1]

Options:

A.

$(0, 1]$

B.

$[0, 3)$

C.

$[0, 1]$

D.

$[0, 1)$

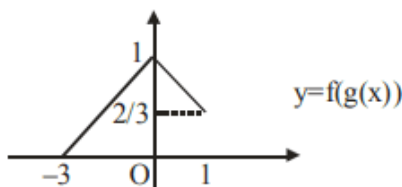
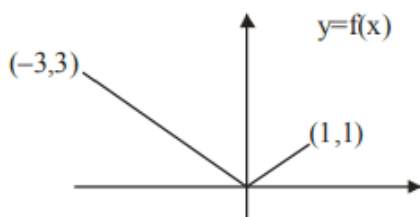
Answer: C

Solution:

$$f(g(x)) = \begin{cases} 2 + 2g(x), & -1 \leq g(x) < 0 \dots\dots (1) \\ 1 - \frac{g(x)}{3}, & 0 \leq g(x) \leq 3 \dots\dots (2) \end{cases}$$

By (1) $x \in \phi$

And by (2) $x \in [-3, 0]$ and $x \in [0, 1]$



Range of $f(g(x))$ is $[0, 1]$

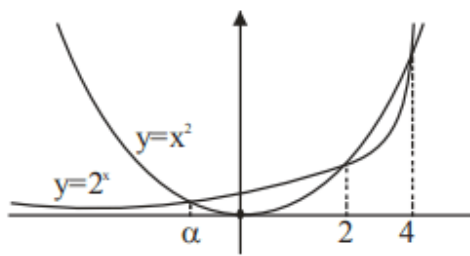
Question38

Let $f(x) = 2^x - x^2$, $x \in \mathbb{R}$. If m and n are respectively the number of points at which the curves $y = f(x)$ and $y = f'(x)$ intersects the x -axis, then the value of $m + n$ is

[29-Jan-2024 Shift 1]

Answer: 5

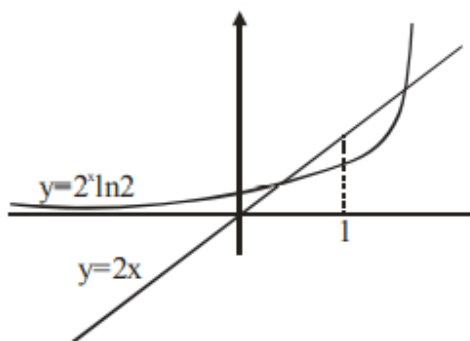
Solution:



$$\therefore m = 3$$

$$f'(x) = 2^x \ln 2 - 2x = 0$$

$$2^x \ln 2 = 2x$$



$$\therefore n = 2$$

$$\Rightarrow m + n = 5$$

Question39

If the domain of the function $f(x) = \cos^{-1}(2 - |x|/4) + (\log_e(3 - x))^{-1}$ is $[-\alpha, \beta] - \{y\}$, then $\alpha + \beta + \gamma$ is equal to :

[30-Jan-2024 Shift 1]

Options:

A.

12

B.

9

C.

11

D.

8

Answer: C**Solution:**

$$-1 \leq \left| \frac{2-x}{4} \right| \leq 1$$

$$\Rightarrow \left| \frac{2-x}{4} \right| \leq 1$$

$$-4 \leq 2-x \leq 4$$

$$-6 \leq -x \leq 2$$

$$-2 \leq x \leq 6$$

$$|x| \leq 6$$

$$\Rightarrow x \in [-6, 6] \dots\dots(1)$$

$$\text{Now, } 3-x \neq 1$$

$$\text{And } x \neq 2 \dots\dots(2)$$

$$\text{and } 3-x > 0$$

$$x < 3 \dots\dots(3)$$

$$\text{From (1), (2) and (3)}$$

$$\Rightarrow x \in [-6, 3) - \{2\}$$

$$\alpha = 6$$

$$\beta = 3$$

$$\gamma = 2$$

$$\alpha + \beta + \gamma = 11$$

Question40

Let $A = \{1, 2, 3, \dots, 7\}$ and let $P(A)$ denote the power set of A . If the number of functions $f: A \rightarrow P(A)$ such that $a \in f(a), \forall a \in A$ is m , m and $n \in \mathbb{N}$ and m is least, then $m + n$ is equal to _____

[30-Jan-2024 Shift 1]

Answer: 44

Solution:

$$f: A \rightarrow P(A)$$

$$a \in f(a)$$

That means 'a' will connect with subset which contain element 'a'.

Total options for 1 will be 2^6 . (Because 2^6 subsets contains 1)

Similarly, for every other element

$$\text{Hence, total is } 2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 = 2^{42}$$

$$\text{Ans. } 2 + 42 = 44$$

Question 41

If the domain of the function $f(x) = \log_e \left(\frac{2x+3}{4x^2+x-3} \right) + \cos^{-1} \left(\frac{2x-1}{x+2} \right)$ is $(\alpha, \beta]$, then the value of $5\beta - 4\alpha$ is equal to

[30-Jan-2024 Shift 2]

Options:

A.

10

B.

12

C.

11

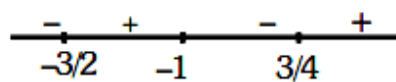
D.

9

Answer: B**Solution:**

$$\frac{2x+3}{4x^2+x-3} > 0 \text{ and } -1 \leq \frac{2x-1}{x+2} \leq 1$$

$$\frac{2x+3}{(4x-3)(x+1)} > 0 \quad \frac{3x+1}{x+2} \geq 0 \text{ \& } \frac{x-3}{x+2} \leq 0$$



$$(-\infty, -2) \cup \left[\frac{-1}{3}, \infty \right) \dots\dots (1)$$

$$(-2, 3] \dots\dots (2)$$

$$\left[\frac{-1}{3}, 3 \right] \dots\dots (3) \quad (1) \cap (2) \cap (3)$$

$$\left(\frac{3}{4}, 3 \right]$$

$$\alpha = \frac{3}{4}\beta = 3$$

$$5\beta - 4\alpha = 15 - 3 = 12$$

Question42

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined $f(x) = \frac{x}{(1+x^4)^{1/4}}$ and $g(x) =$

$f(f(f(x)))$ then $18 \int_0^{\sqrt{2\sqrt{5}}} x^2 g(x) dx$

[30-Jan-2024 Shift 2]

Options:

A.

33

B.

36

C.

42

D.

39

Answer: D

Solution:

$$f(x) = \frac{x}{(1+x^4)^{1/4}}$$

$$f \circ f(x) = \frac{f(x)}{(1+f(x)^4)^{1/4}} = \frac{\frac{x}{(1+x^4)^{1/4}}}{\left(1 + \frac{x^4}{1+x^4}\right)^{1/4}} = \frac{x}{(1+2x^4)^{1/4}}$$

$$f(f(f(f(x)))) = \frac{x}{(1+4x^4)^{1/4}}$$

$$18 \int_0^{\sqrt[4]{2\sqrt{5}}} \frac{x^3}{(1+4x^4)^{1/4}} dx$$

$$\text{Let } 1+4x^4 = t^4$$

$$16x^3 dx = 4t^3 dt$$

$$\frac{18}{4} \int_1^3 \frac{t^3 dt}{t}$$

$$= \frac{9}{2} \left(\frac{t^3}{3} \right)_1^3$$

$$= \frac{3}{2} [26] = 39$$

Question43

If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$ and $(f \circ f)(x) = g(x)$, where $g: \mathbb{R} - \left\{\frac{2}{3}\right\} \rightarrow \mathbb{R} - \left\{\frac{2}{3}\right\}$, then $(g \circ g)(4)$ is equal to

[31-Jan-2024 Shift 1]

Options:

A.

$$-\frac{19}{20}$$

B.

$$19/20$$

C.

$$-4$$

D.

$$4$$

Answer: D

Solution:

$$f(x) = \frac{4x+3}{6x-4}$$

$$g(x) = \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} = \frac{34x}{34} = x$$

$$g(x) = x \therefore g(g(4)) = 4$$

Question44

If the function $f : (-\infty, -1] \rightarrow (a, b]$ defined by $f(x) = e^{x^3 - 3x + 1}$ is one-one and onto, then the distance of the point

$P(2b + 4, a + 2)$ from the line $x + e^{-3}y = 4$ is :

[31-Jan-2024 Shift 2]

Options:

A.

$$2\sqrt{1+e^6}$$

B.

$$4\sqrt{1+e^6}$$

C.

$$3\sqrt{1+e^6}$$

D.

$$\sqrt{1+e^6}$$

Answer: A

Solution:

$$f(x) = e^{x^3 - 3x + 1}$$

$$f'(x) = e^{x^3 - 3x + 1} \cdot (3x^2 - 3)$$

$$= e^{x^3 - 3x + 1} \cdot 3(x - 1)(x + 1)$$

For $f'(x) \geq 0$

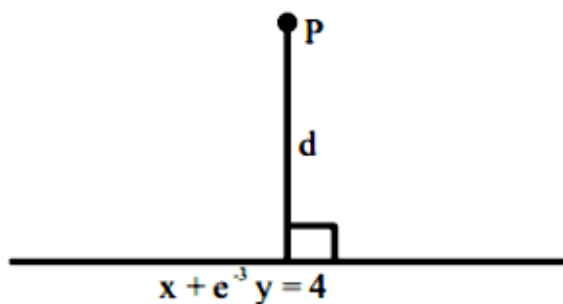
$\therefore f(x)$ is increasing function

$$\therefore a = e^{-\infty} = 0 = f(-\infty)$$

$$b = e^{-1 + 3 + 1} = e^3 = f(-1)$$

$$P(2b + 4, a + 2)$$

$$\therefore P(2e^3 + 4, 2)$$



$$d = \frac{(2e^3 + 4) + 2e^{-3} - 4}{\sqrt{1 + e^{-6}}} = 2\sqrt{1 + e^6}$$

Question 45

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \log_e x & , \quad x > 0 \\ e^{-x} & , \quad x \leq 0 \end{cases}$$

and

$$g(x) = \begin{cases} x & , \quad x \geq 0 \\ e^x & , \quad x < 0 \end{cases}$$

Then, $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ is :

[1-Feb-2024 Shift 1]

Options:

A.

one-one but not onto

B.

neither one-one nor onto

C.

onto but not one-one

D.

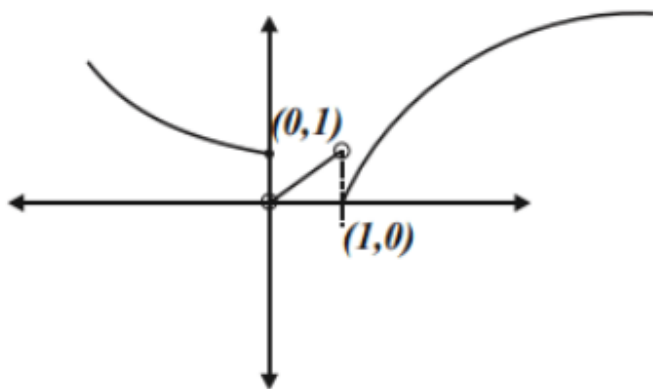
both one-one and onto

Answer: B

Solution:

$$g(f(x)) = \begin{cases} f(x) & f(x) \geq 0 \\ e^{f(x)} & f(x) < 0 \end{cases}$$

$$g(f(x)) = \begin{cases} e^{-x} & (-\infty, 0] \\ e^{\ln x} & (0, 1) \\ \ln x & [1, \infty) \end{cases}$$



Graph of $g(f(x))$

$g(f(x)) \Rightarrow$ Many one into

Question 46

If the domain of the function $f(x) = \frac{\sqrt{x^2 - 25}}{(4 - x^2)} + \log_{10}(x^2 + 2x - 15)$ is $(-\infty, \alpha) \cup [\beta, \infty)$, then $\alpha^2 + \beta^3$ is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

140

B.

175

C.

150

D.

125

Answer: C

Solution:

$$f(x) = \frac{\sqrt{x^2 - 25}}{4 - x^2} + \log_{10}(x^2 + 2x - 15)$$

$$\text{Domain : } x^2 - 25 \geq 0 \Rightarrow x \in (-\infty, -5] \cup [5, \infty)$$

$$4 - x^2 \neq 0 \Rightarrow x \neq \{-2, 2\}$$

$$x^2 + 2x - 15 > 0 \Rightarrow (x + 5)(x - 3) > 0$$

$$\Rightarrow x \in (-\infty, -5) \cup (3, \infty)$$

$$\therefore x \in (-\infty, -5) \cup [5, \infty)$$

$$\alpha = -5; \beta = 5$$

$$\therefore \alpha^2 + \beta^3 = 150$$

Question47

Let $A = \{1, 2, 3, 4, \dots, 10\}$ and $B = \{0, 1, 2, 3, 4\}$. The number of elements in the relation $R = \{(a, b) \in A \times A : 2(a-b)^2 + 3(a-b) \in B\}$ is _____.

[6-Apr-2023 shift 1]

Answer: 18

Solution:

$$A = \{1, 2, 3, \dots, 10\}$$

$$B = \{0, 1, 2, 3, 4\}$$

$$R = \{(a, b) \in A \times A : 2(a-b)^2 + 3(a-b) \in B\}$$

$$\text{Now } 2(a-b)^2 + 3(a-b) = (a-b)(2(a-b) + 3)$$

$$\Rightarrow a=b \text{ or } a-b=-2$$

$$\text{When } a=b \Rightarrow 10 \text{ order pairs}$$

$$\text{When } a-b=-2 \Rightarrow 8 \text{ order pairs}$$

$$\text{Total} = 18$$

Question48

Let $A = \{0, 3, 4, 6, 7, 8, 9, 10\}$ and R be the relation defined on A such that $R = \{(x, y) \in A \times A : x-y \text{ is odd positive integer or } x-y=2\}$. The minimum number of elements that must be added to the relation R , so that it is a symmetric relation, is equal to _____.

[8-Apr-2023 shift 1]

Answer: 19

Solution:

$$A = \{0, 3, 4, 6, 7, 8, 9, 10\} \quad 3, 7, 9 \rightarrow \text{odd}$$

$$R = \{x-y = \text{odd} + \text{ve or } x-y=2\} \quad 0, 4, 6, 8, 10 \rightarrow \text{even}$$

$${}^3C_1 \cdot {}^5C_1 = 15 + (6, 4), (8, 6), (10, 8), (9, 7)$$

Min^m ordered pairs to be added must be

$$:15 + 4 = 19$$

Question49

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then the relation

$R = \{(x, y) \in A \times A : x + y = 7\}$ is

[8-Apr-2023 shift 2]

Options:

- A. Symmetric but neither reflexive nor transitive
- B. Transitive but neither symmetric nor reflexive
- C. An equivalence relation
- D. Reflexive but neither symmetric nor transitive

Answer: A

Solution:

$$R = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

Question50

Let $A = \{2, 3, 4\}$ and $B = \{8, 9, 12\}$. Then the number of elements in the relation $R = \{((a_1, b_1), (a_2, b_2)) \in (A \times B, A \times B) : a_1 \text{ divides } b_2 \text{ and } a_2 \text{ divides } b_1\}$ is

[10-Apr-2023 shift 2]

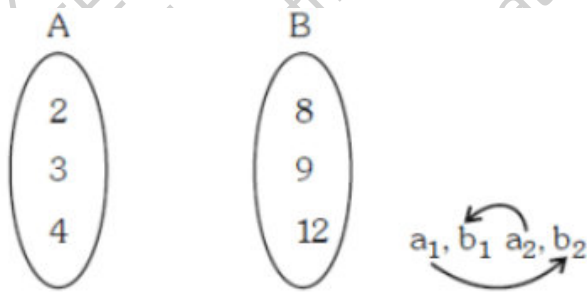
Options:

- A. 18
- B. 24
- C. 12

D. 36

Answer: D

Solution:



a_1 divides b_2

Each element has 2 choices

$$\Rightarrow 3 \times 2 = 6$$

a_2 divides b_1

Each element has 2 choices

$$\Rightarrow 3 \times 2 = 6$$

$$\text{Total} = 6 \times 6 = 36$$

Question 51

Let $A = \{1, 3, 4, 6, 9\}$ and $B = \{2, 4, 5, 8, 10\}$. Let R be a relation defined on $A \times B$ such that $R = \{((a_1, b_1), (a_2, b_2, \dots)) : a_1 \leq b_2 \text{ and } b_1 \leq a_2\}$. Then the number of elements in the set R is [11-Apr-2023 shift 2]

Options:

A. 52

B. 160

C. 26

D. 180

Answer: B

Solution:

Let $a_1 = 1 \Rightarrow 5$ choices of b_2

$a_1 = 3 \Rightarrow 4$ choices of b_2

$a_1 = 4 \Rightarrow 4$ choices of b_2

$a_1 = 6 \Rightarrow 2$ choices of b_2

$a_1 = 9 \Rightarrow 1$ choices of b_2

For (a_1, b_2) 16 ways.

Similarly, $b_1 = 2 \Rightarrow 4$ choices of a_2

$b_1 = 4 \Rightarrow 3$ choices of a_2

$b_1 = 5 \Rightarrow 2$ choices of a_2

$b_1 = 8 \Rightarrow 1$ choices of a_2

Required elements in $R = 160$

Question 52

The number of the relations, on the set $\{1, 2, 3\}$ containing $(1, 2)$ and $(2, 3)$, which are reflexive and transitive but not symmetric, is

 .
[12-Apr-2023 shift 1]

Answer: 3

Solution:

$A = \{1, 2, 3\}$

For Reflexive $(1, 1), (2, 2), (3, 3) \in R$

For transitive : $(1, 2)$ and $(2, 3) \in R \Rightarrow (1, 3) \in R$

Not symmetric : $(2, 1)$ and $(3, 2) \notin R$

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$

$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$

Question 53

Let $A = \{-4, -3, -2, 0, 1, 3, 4\}$ and $R = \{(a, b) \in A \times A : b = |a| \text{ or } b^2 = a + 1\}$ be a relation on A . Then the minimum number of elements, that must be added to the relation R so that it becomes reflexive and

symmetric, is _____
[13-Apr-2023 shift 2]

Answer: 7

Solution:

$$R = \{(-4, 4), (-3, 3), (3, -2), (0, 1), (0, 0), (1, 1), (4, 4), (3, 3)\}$$

For reflexive, add $\Rightarrow (-2, -2), (-4, -4), (-3, -3)$

For symmetric, add $\Rightarrow (4, -4), (3, -3), (-2, 3), (1, 0)$

Question 54

Let $A = \{1, 2, 3, 4\}$ and R be a relation on the set $A \times A$ defined by $R = \{(a, b, (c, d)) : 2a + 3b = 4c + 5d\}$. Then the number of elements in R is _____
[15-Apr-2023 shift 1]

Answer: 6

Solution:

$$A = \{1, 2, 3, 4\}$$

$$R = \{(a, b), (c, d)\}$$

$$2a + 3b = 4c + 5d = \alpha \text{ let}$$

$$2a = \{2, 4, 6, 8\} \quad 4c = \{4, 8, 12, 16\}$$

$$3b = \{3, 6, 9, 12\} \quad 5d = \{5, 10, 15, 20\}$$

$$2a + 3b = \left\{ \begin{array}{cccc} 5 & 8 & 11 & 14 \\ 7 & 10 & 13 & 16 \\ 9 & 12 & 15 & 18 \\ 11 & 14 & 17 & 20 \end{array} \right\} \quad 4c + 5d = \left\{ \begin{array}{cccc} 9 & 14 & 19 & 24 \\ 13 & 18 & & \\ 17 & 22 & & \\ 21 & 26 & & \end{array} \right\}$$

Possible value of $\alpha = 9, 13, 14, 14, 17, 18$

Pairs of $\{(a, b), (c, d)\} = 6$

Question55

Let $5f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} + 3, x > 0$. Then $18 \int_1^2 f(x) dx$ is equal to :

[6-Apr-2023 shift 1]

Options:

A. $10\log_e 2 - 6$

B. $10\log_e 2 + 6$

C. $5\log_e 2 - 3$

D. $5\log_e 2 + 3$

Answer: A

Solution:

$$5f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} + 3 \dots (1)$$

$$x \rightarrow \frac{1}{x}$$

$$5f\left(\frac{1}{x}\right) + 4f(x) = x + 3 \dots (2)$$

$$(1) \times 5 - (2) \times 4$$

$$\Rightarrow f(x) = \frac{5}{9x} - \frac{4}{9}x + \frac{1}{3}$$

$$\Rightarrow 18 \int_1^2 f(x) dx = 18 \left(\frac{5}{9} \ln 2 - \frac{4}{9} \times \frac{3}{2} + \frac{1}{3} \right)$$

$$= 10 \ln 2 - 6$$

Question56

Let $A = \{x \in \mathbb{R} : [x+3] + [x+4] \leq 3\}$,

$B = \left\{ x \in \mathbb{R} : 3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^x} \right)^{x-3} < 3^{-3x} \right\}$, where $[t]$ denotes greatest

integer function. Then,

[6-Apr-2023 shift 1]

Options:

A. $A \subset B, A \neq B$

B. $A \cap B = \emptyset$

C. $A = B$

D. $B \subset C, A \neq B$

Answer: C

Solution:

$$A = \{x \in \mathbb{R} : [x+3] + [x+4] \leq 3\}$$

$$2[x] + 7 \leq 3$$

$$2[x] \leq -4$$

$$[x] \leq -2 \Rightarrow x < -1 \dots (A)$$

$$B = \left\{ x \in \mathbb{R} : 3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^x} \right)^{x-3} < 3^{-3x} \right\}$$

$$3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^x} \right)^{x-3} < 3^{-3x}$$

$$3^{2x-3} \left(\frac{10}{10} \right)^{x-3}$$

$$\Rightarrow \left(\frac{1}{9} \right)^{x-3} < 3^{-5x}$$

$$\Rightarrow 3^{6-2x} < 3^{3-5x}$$

$$\Rightarrow 6-2x < 3-5x$$

$$\Rightarrow 3 < -3x$$

$$\Rightarrow \left(\frac{1}{10} \right)^x < -1 \dots (B)$$

$$A = B$$

Question 57

Let the sets A and B denote the domain and range respectively of the function $f(x) = \frac{1}{\sqrt{[x]-x}}$, where $[x]$ denotes the smallest integer greater than or equal to x . Then among the statements :

(S1) : $A \cap B = (1, \infty) - \mathbb{N}$ and

(S2) : $A \cup B = (1, \infty)$

[6-Apr-2023 shift 2]

Options:

- A. only (S1) is true
- B. neither (S1) nor (S2) is true
- C. only (S2) is true
- D. both (S1) and (S2) are true

Answer: A

Solution:

$$f(x) = \frac{1}{\sqrt{[x] - x}}$$

If $x \in I$ $[x] = [x]$ (greatest integer function)

If $x \notin I$ $[x] = [x] + 1$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{\sqrt{[x] - x}} & x \in I \\ \frac{1}{\sqrt{[x] + 1 - x}} & x \notin I \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{\sqrt{-\{x\}}} & x \in I \text{ (does not exist)} \\ \frac{1}{\sqrt{1 - \{x\}}} & x \notin I \end{cases}$$

$\Rightarrow$ domain of $f(x) = \mathbb{R} - I$

Now, $f(x) = \frac{1}{\sqrt{1 - \{x\}}}$, $x \notin I$

$\Rightarrow x < \{x\} < 1$

$\Rightarrow 0 < 1 - \{x\} < 1$

$\Rightarrow \frac{1}{\sqrt{1 - \{x\}}} > 1$

$\Rightarrow \text{Range}(1, \infty)$

$\Rightarrow A = \mathbb{R} - I$

$B = (1, \infty)$

So, $A \cap B = (1, \infty) - \mathbb{N}$

$A \cup B \neq (1, \infty)$

$\Rightarrow S1$ is only correct.

Question 58

Let $f, g : \mathbb{N} - \{1\} \rightarrow \mathbb{N}$ be functions defined by $f(a) = \alpha$, where α is the maximum of the powers of those primes p such that p^α divides a , and

$g(a) = a + 1$, for all $a \in \mathbb{N} - \{1\}$. Then, the function $f + g$ is
[27-Jul-2022-Shift-1]

Options:

- A. one-one but not onto
- B. onto but not one-one
- C. both one-one and onto
- D. neither one-one nor onto

Answer: D

Solution:

$f, g: \mathbb{N} - \{1\} \rightarrow \mathbb{N}$ defined as

$f(a) = \alpha$, where α is the maximum power of those primes p such that p^α divides a .

$g(a) = a + 1$

Now,

$$f(2) = 1, g(2) = 3 \Rightarrow (f + g)(2) = 4$$

$$f(3) = 1, g(3) = 4 \Rightarrow (f + g)(3) = 5$$

$$f(4) = 2, g(4) = 5 \Rightarrow (f + g)(4) = 7$$

$$f(5) = 1, g(5) = 6 \Rightarrow (f + g)(5) = 7$$

$$\therefore (f + g)(5) = (f + g)(4)$$

$\therefore f + g$ is not one-one

$$\text{Now, } \therefore f_{\min} = 1, g_{\min} = 3$$

So, there does not exist any $x \in \mathbb{N} - \{1\}$ such that $(f + g)(x) = 1, 2, 3$

$\therefore f + g$ is not onto

Question 59

If domain of the function $\log_e \left(\frac{6x^2 + 5x + 1}{2x - 1} \right) + \cos^{-1} \left(\frac{2x^2 - 3x + 4}{3x - 5} \right)$ is

$(\alpha, \beta) \cup (\gamma, \delta]$, then, $18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ is equal to
[8-Apr-2023 shift 2]

Answer: 20

Solution:

$$\frac{6x^2 + 5x + 1}{2x - 1} > 0$$

$$\frac{(3x + 1)(2x + 1)}{2x - 1} > 0$$

$$x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \cup \left(\frac{5}{3}, \infty \right) \dots (B)$$

$$x < \frac{5}{3} \dots (C)$$

$$A \cap B \cap C \equiv \left(\frac{-1}{2}, \frac{-1}{3} \right) \cup \left(\frac{1}{2}, \frac{1}{\sqrt{2}} \right]$$

$$\text{So } 18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 18 \left(\frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{2} \right)$$

$$= 18 + 2 = 20$$

Question60

Let $R = \{a, b, c, d, e\}$ and $S = \{1, 2, 3, 4\}$. Total number of onto functions $f: R \rightarrow S$ such that $f(a) \neq 1$, is equal to _____.

[8-Apr-2023 shift 2]

Answer: 180

Solution:

Total onto function

$$\frac{4!}{1!} \times 4 = 240$$

Now when $f(a) = 1$

$$\left[4 + \frac{4!}{2!} \right] \times 3 = 24 + 36 = 60.$$

$$\text{so required } f^n = 240 - 60 = 180$$

Question61

If the domain of the function $f(x) = \sec^{-1}\left(\frac{2x}{5x+3}\right)$ is $[\alpha, \beta) \cup (\gamma, \delta]$, then

$|3\alpha + 10(\beta + \gamma) + 21\delta|$ is equal to _____.

[10-Apr-2023 shift 2]

Answer: 24

Solution:

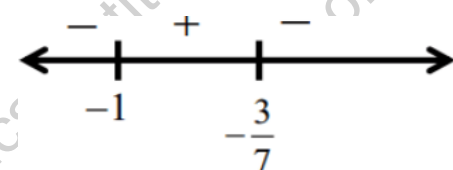
$$f(x) = \sec^{-1} \frac{2x}{5x+3}$$

$$\left| \frac{2x}{5x+3} \right| \leq 1$$

$$\left| \frac{2x}{5x+3} \right| \geq 1 \Rightarrow |2x| \geq |5x+3|$$

$$(2x)^2 - (5x+3)^2 \geq 0$$

$$(7x+3)(-3x-3) \geq 0$$



$$\therefore \text{domain} \left[-1, \frac{-3}{5} \right) \cup \left(\frac{-3}{5}, \frac{-3}{7} \right]$$

$$\alpha = -1, \beta = \frac{-3}{5}, \gamma = \frac{-3}{5}, \delta = \frac{-3}{7}$$

$$3\alpha + 10(\beta + \gamma) + 21\delta = -3$$

$$-3 + 10\left(\frac{-6}{5}\right) + \left(\frac{-3}{7}\right)21 = -24$$

Question 62

The domain of the function $f(x) = \frac{1}{\sqrt{[x]^2 - 3[x] - 10}}$ is (where $[x]$ denotes the greatest integer less than or equal to x)

[11-Apr-2023 shift 2]

Options:

A. $(-\infty, -3] \cup [6, \infty)$

B. $(-\infty, -2) \cup (5, \infty)$

C. $(-\infty, -3] \cup (5, \infty)$

D. $(-\infty, -2) \cup [6, \infty)$

Answer: D

Solution:

$$F(x) = \frac{1}{\sqrt{[x]^2 - 3[x] - 10}}$$

$$[x]^2 - 3[x] - 10 > 0$$

$$([x] + 2)([x] - 5) > 0$$



$$[x] < -2 \text{ or } [x] > 5$$

$$[x] \leq -3 \text{ or } [x] \geq 6$$

$$x < -2 \text{ or } x \geq 6$$

$$x \in (-\infty, -2) \cup [6, \infty)$$

Question63

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then the number of functions $f : A \rightarrow B$ satisfying $f(1) + f(2) = f(4) - 1$ is equal to

 .
[11-Apr-2023 shift 2]

Answer: 360

Solution:

$$f(1) + f(2) + 1 = f(4) \leq 6$$

$$f(1) + f(2) \leq 5$$

$$\text{Case (i) } f(1) = 1 \Rightarrow f(2) = 1, 2, 3, 4 \Rightarrow 4 \text{ mappings}$$

$$\text{Case (ii) } f(1) = 2 \Rightarrow f(2) = 1, 2, 3 \Rightarrow 3 \text{ mappings}$$

$$\text{Case (iii) } f(1) = 3 \Rightarrow f(2) = 1, 2 \Rightarrow 2 \text{ mappings}$$

$$\text{Case (iv) } f(1) = 4 \Rightarrow f(2) = 1 \Rightarrow 1 \text{ mapping}$$

$f(5)$ & $f(6)$ both have 6 mappings each
 Number of functions $= (4 + 3 + 2 + 1) \times 6 \times 6 = 360$

Question 64

Let D be the domain of the function $f(x) = \sin^{-1} \left(\log_{3x} \left(\frac{6 + 2\log_3 x}{-5x} \right) \right)$. If the range of the function $g : D \rightarrow \mathbb{R}$ defined by $g(x) = x - [x]$, ($[x]$ is the greatest integer function), is (α, β) , then $\alpha^2 + \frac{5}{\beta}$ is equal to
 [12-Apr-2023 shift 1]

Options:

- A. 46
- B. 135
- C. 136
- D. 45

Answer: B

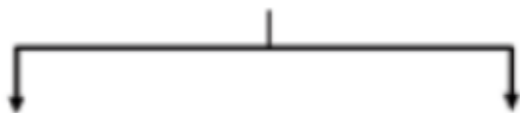
Solution:

$$\frac{6 + 2\log_3 x}{-5x} > 0 \text{ \& } x > 0 \text{ \& } x \neq \frac{1}{3}$$

$$\text{this gives } x \in \left(0, \frac{1}{27} \right) \dots (1)$$

$$-1 \leq \log_{3x} \left(\frac{6 + 2\log_3 x}{-5x} \right) \leq 1$$

$$3x \leq \frac{6 + 2\log_3 x}{-5x} \leq \frac{1}{3x}$$



$$15x^2 + 6 + 2\log_3 x \geq 0 \quad 6 + 2\log_3 x + \frac{5}{3} \geq 0$$

$$x \in \left(0, \frac{1}{27} \right) \dots (2) \quad x \geq 3 - \frac{23}{6} \dots (3)$$

from (1), (2) & (3)

$$x \in \left[3 - \frac{23}{6}, \frac{1}{27} \right)$$

$\therefore \alpha$ is small positive quantity

$$\&\beta = \frac{1}{27}$$

$\therefore \alpha^2 + \frac{5}{\beta}$ is just greater than 135

Question65

For $x \in \mathbf{R}$, two real valued functions $f(x)$ and $g(x)$ are such that, $g(x) = \sqrt{x} + 1$ and $f \circ g(x) = x + 3 - \sqrt{x}$. Then $f(0)$ is equal to
[13-Apr-2023 shift 1]

Options:

A. 5

B. 0

C. -3

D. 1

Answer: A

Solution:

$$\begin{aligned} g(x) &= \sqrt{x} + 1 \\ f \circ g(x) &= x + 3 - \sqrt{x} \\ &= (\sqrt{x} + 1)^2 - 3(\sqrt{x} + 1) + 5 \\ &= g^2(x) - 3g(x) + 5 \\ \Rightarrow f(x) &= x^2 - 3x + 5 \\ \therefore f(0) &= 5 \end{aligned}$$

But, if we consider the domain of the composite function $f \circ g(x)$ then in that case $f(0)$ will be not defined as $g(x)$ cannot be equal to zero.

Question66

For the differentiable function $f : \mathbf{R} - \{0\} \rightarrow \mathbf{R}$, let

$$3f(x) + 2f\left(\frac{1}{x}\right) = \frac{1}{x} - 10, \text{ then } \left| f(3) + f'\left(\frac{1}{4}\right) \right| \text{ is equal to}$$

[13-Apr-2023 shift 1]

Options:

A. 13

B. $\frac{29}{5}$

C. $\frac{33}{5}$

D. 7

Answer: A

Solution:

$$\left[3f(x) + 2f\left(\frac{1}{x}\right) = \frac{1}{x} - 10 \right] \times 3$$

$$\left[2f(x) + 3f\left(\frac{1}{x}\right) = x - 10 \right] \times 2$$

$$5f(x) = \frac{3}{x} - 2x - 10$$

$$f(x) = \frac{1}{5} \left(\frac{3}{x} - 2x - 10 \right)$$

$$f'(x) = \frac{1}{5} \left(-\frac{3}{x^2} - 2 \right)$$

$$\left| f(3) + f'\left(\frac{1}{4}\right) \right| = \left| \frac{1}{5}(1 - 6 - 10) + \frac{1}{5}(-48 - 2) \right|$$
$$= |-3 - 10| = 13$$

Question 67

The range of $f(x) = 4\sin^{-1}\left(\frac{x^2}{x^2+1}\right)$ is

[13-Apr-2023 shift 2]

Options:

A. $[0, \pi)$

B. $[0, \pi]$

C. $[0, 2\pi)$

D. $[0, 2\pi]$

Answer: C

Solution:

$$f(x) = 4\sin^{-1}\left(\frac{x^2}{1+x^2}\right)$$

$$0 \leq \frac{x^2}{1+x^2} < 1$$

$$\Rightarrow 0 \leq \sin^{-1}\left(\frac{x^2}{1+x^2}\right) < \frac{\pi}{2}$$

$$\Rightarrow 0 \leq 4\sin^{-1}\left(\frac{x^2}{1+x^2}\right) < 2\pi$$

Range : $[0, 2\pi)$

Question 68

If the domain of the function

$f(x) = \log_e(4x^2 + 11x + 6) + \sin^{-1}(4x + 3) + \cos^{-1}\left(\frac{10x+6}{3}\right)$ is $(\alpha, \beta]$, then

$36 \mid \alpha + \beta$ is equal to

[15-Apr-2023 shift 1]

Options:

A. 72

B. 63

C. 45

D. 54

Answer: C

Solution:

$$f(x) = \ln(4x^2 + 11x + 6) + \sin^{-1}(4x + 3) + \cos^{-1}\left(\frac{10x+6}{3}\right)$$

$$(i) \quad 4x^2 + 11x + 6 > 0$$

$$4x^2 + 8x + 3x + 6 > 0$$

$$(4x+3)(x+2) > 0$$

$$x \in (-\infty, -2) \cup \left(-\frac{3}{4}, \infty\right)$$

$$(ii) 4x + 3 \in [-1, 1]$$

$$x \in [-1, -1/2]$$

$$(iii) \frac{10x+6}{3} \in [-1, 1]$$

$$x \in \left[-\frac{9}{10}, -\frac{3}{10}\right]$$

$$x \in \left(-\frac{3}{4}, -\frac{1}{2}\right] \quad \alpha = -\frac{3}{4}, \beta = -\frac{1}{2}$$

$$\alpha + \beta = -\frac{5}{4}$$

$$36|\alpha + \beta| = 45$$

Question69

The relation $R = \{(a, b) : \gcd(a, b) = 1, 2a \neq b, a, b \in \mathbb{Z}\}$ is:
[24-Jan-2023 Shift 1]

Options:

- A. transitive but not reflexive
- B. symmetric but not transitive
- C. reflexive but not symmetric
- D. neither symmetric nor transitive

Answer: D

Solution:

Reflexive : $(a, a) \quad \gcd(a, a) = a$

Which is not true for every $a \in \mathbb{Z}$.

Symmetric:

Take $a = 2, b = 1 \quad \gcd(2, 1) = 1$

Also $2a = 4 \neq b$

Now when $a = 1, b = 2 \quad \gcd(1, 2) = 1$

Also now $2a = 2 = b$

Hence $a = 2b$

$\Rightarrow R$ is not Symmetric

Transitive:

Let $a = 14, b = 19, c = 21$

$\gcd(a, b) = 1$

$\gcd(b, c) = 1$

$\gcd(a, c) = 7$

Hence not transitive

R is neither symmetric nor transitive.

Question 70

Let R be a relation defined on $\mathbb{N}$ as $a R b$ is $2a + 3b$ is a multiple of 5, $a, b \in \mathbb{N}$. Then R is
[29-Jan-2023 Shift 2]

Options:

- A. not reflexive
- B. transitive but not symmetric
- C. symmetric but not transitive
- D. an equivalence relation

Answer: D

Solution:

$a R a \Rightarrow 5a$ is multiple of 5

So reflexive

$a R b \Rightarrow 2a + 3b = 5\alpha$,

Now $b R a$

$$2b + 3a = 2b + \left(\frac{5\alpha - 3b}{2} \right) \cdot 3$$

$$= \frac{15}{2}\alpha - \frac{5}{2}b = \frac{5}{2}(3\alpha - b)$$

$$= \frac{5}{2}(2a + 2b - 2a)$$

$$= 5(a + b - a)$$

Hence symmetric

$$a R b \Rightarrow 2a + 3b = 5\alpha$$

$$b R c \Rightarrow 2b + 3c = 5\beta$$

$$\text{Now } 2a + 5b + 3c = 5(\alpha + \beta)$$

$$\Rightarrow 2a + 5b + 3c = 5(\alpha + \beta)$$

$$\Rightarrow 2a + 3c = 5(\alpha + \beta - b)$$

$$\Rightarrow a R c$$

Hence relation is equivalence relation.

Question 71

The minimum number of elements that must be added to the relation $R = \{(a, b), (b, c)\}$ on the set $\{a, b, c\}$ so that it becomes symmetric and transitive is:

[30-Jan-2023 Shift 1]

Options:

- A. 4
- B. 7
- C. 5
- D. 3

Answer: B

Solution:

For Symmetric $(a, b), (b, c) \in R$

$\Rightarrow (b, a), (c, b) \in R$

For Transitive $(a, b), (b, c) \in R$

$\Rightarrow (a, c) \in R$

Now

1. Symmetric

$\therefore (a, c) \in R \Rightarrow (c, a) \in R$

2. Transitive

$\therefore (a, b), (b, a) \in R$

$\Rightarrow (a, a) \in R \text{ \& } (b, c), (c, b) \in R$

$\Rightarrow (b, b) \text{ \& } (c, c) \in R$

$\therefore$ Elements to be added

$\left\{ \begin{array}{l} (b, a) \quad (c, b) \quad (a, c) \quad (c, a) \\ (a, a) \quad (b, b) \quad (c, c) \end{array} \right\}$

Number of elements to be added = 7

Question 72

Let R be a relation on $N \times N$ defined by $(a, b)R (c, d)$ if and only if $ad(b - c) = bc(a - d)$. Then R is

[31-Jan-2023 Shift 1]

Options:

- A. symmetric but neither reflexive nor transitive

- B. transitive but neither reflexive nor symmetric
- C. reflexive and symmetric but not transitive
- D. symmetric and transitive but not reflexive

Answer: A

Solution:

$$(a, b)R(c, d) \Rightarrow ad(b-c) = bc(a-d)$$

Symmetric:

$$(c, d)R(a, b) \Rightarrow cb(d-a) = da(c-b) \Rightarrow$$

Symmetric

Reflexive:

$$(a, b)R(a, b) \Rightarrow ab(b-a) \neq ba(a-b) \Rightarrow$$

Not reflexive

Transitive: $(2, 3)R(3, 2)$ and $(3, 2)R(5, 30)$ but

$((2, 3), (5, 30)) \notin R \Rightarrow$ Not transitive

Question 73

Among the relations

$$S = \left\{ (a, b) : a, b \in \mathbb{R} - \{0\}, 2 + \frac{a}{b} > 0 \right\}$$

$$\text{And } T = \{(a, b) : a, b \in \mathbb{R}, a^2 - b^2 \in \mathbb{Z}\},$$

[31-Jan-2023 Shift 2]

Options:

- A. S is transitive but T is not
- B. T is symmetric but S is not
- C. Neither S nor T is transitive
- D. Both S and T are symmetric

Answer: B

Solution:

For relation $T = a^2 - b^2 = -1$

Then, (b, a) on relation R

$$\Rightarrow b^2 - a^2 = -1$$

$\therefore T$ is symmetric

$$S = \left\{ (a, b) : a, b \in \mathbb{R} - \{0\}, 2 + \frac{a}{b} > 0 \right\}$$

$$2 + \frac{a}{b} > 0 \Rightarrow \frac{a}{b} > -2, \Rightarrow \frac{b}{a} < \frac{-1}{2}$$

If $(b, a) \in S$ then

$$2 + \frac{b}{a} \text{ not necessarily positive}$$

$\therefore S$ is not symmetric

Question 74

Let R be a relation on $\mathbb{R}$, given by $R = \{(a, b) : 3a - 3b + \sqrt{7} \text{ is an irrational number}\}$. Then R is
[1-Feb-2023 Shift 1]

Options:

- A. Reflexive but neither symmetric nor transitive
- B. Reflexive and transitive but not symmetric
- C. Reflexive and symmetric but not transitive
- D. An equivalence relation

Answer: A

Solution:

Check for reflexivity:

As $3(a-a) + \sqrt{7} = \sqrt{7}$ which belongs to relation so relation is reflexive

Check for symmetric:

$$\text{Take } a = \frac{\sqrt{7}}{3}, b = 0$$

Now $(a, b) \in R$ but $(b, a) \notin R$

As $3(b-a) + \sqrt{7} = 0$ which is rational so relation is not symmetric.

Check for Transitivity:

$$\text{Take } (a, b) \text{ as } \left(\frac{\sqrt{7}}{3}, 1 \right)$$

$$\&(b, c) \text{ as } \left(1, \frac{2\sqrt{7}}{3} \right)$$

So now $(a, b) \in R$ & $(b, c) \in R$ but $(a, c) \notin R$ which means relation is not transitive

Question 75

Let $P(S)$ denote the power set of $S = \{1, 2, 3, \dots, 10\}$. Define the relations R_1 and R_2 on $P(S)$ as $A R_1 B$ if $(A \cap B^c) \cup (B \cap A^c) = \emptyset$ and $A R_2 B$ if $A \cup B^c = B \cup A^c, \forall A, B \in P(S)$. Then :

[1-Feb-2023 Shift 2]

Options:

- A. both R_1 and R_2 are equivalence relations
- B. only R_1 is an equivalence relation
- C. only R_2 is an equivalence relation
- D. both R_1 and R_2 are not equivalence relations

Answer: A

Solution:

$$S = \{1, 2, 3, \dots, 10\}$$

$P(S)$ = power set of S

$$A R_1 B \Rightarrow (A \cap \overrightarrow{B}) \cup (\overrightarrow{A} \cap B) = \emptyset$$

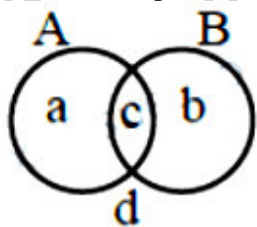
R_1 is reflexive, symmetric

For transitive

$$(A \cap \overrightarrow{B}) \cup (\overrightarrow{A} \cap B) = \emptyset; \{a\} = \emptyset = \{b\} \Rightarrow A = B$$

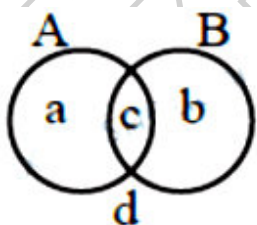
$$(B \cap \overrightarrow{C}) \cup (\overrightarrow{B} \cap C) = \emptyset \therefore B = C$$

$\therefore A = C$ equivalence.



$$R_2 \equiv A \cup \overrightarrow{B} = \overrightarrow{A} \cup B$$

$R_2 \rightarrow$ Reflexive, symmetric
for transitive



$$\begin{aligned}
 A \cup B &= \overrightarrow{A} \cup \overrightarrow{B} \Rightarrow \{a, c, d\} = \{b, c, d\} \\
 \{a\} &= \{b\} \therefore A = B \\
 B \cup C &= \overrightarrow{B} \cup \overrightarrow{C} \Rightarrow B = C \\
 \therefore A &= C \therefore A \cup C = \overrightarrow{A} \cup \overrightarrow{C} \therefore \text{Equivalence}
 \end{aligned}$$

Question 76

The equation $x^2 - 4x + [x] + 3 = x[x]$, where $[x]$ denotes the greatest integer function, has:
[24-Jan-2023 Shift 1]

Options:

- A. exactly two solutions in $(-\infty, \infty)$
- B. no solution
- C. a unique solution in $(-\infty, 1)$
- D. a unique solution in $(-\infty, \infty)$

Answer: D

Solution:

$$\begin{aligned}
 x^2 - 4x + [x] + 3 &= x[x] \\
 \Rightarrow x^2 - 4x + 3 &= x[x] - [x] \\
 (x-1)(x-3) &= [x] \cdot (x-1) \\
 \Rightarrow x=1 \text{ or } x-3 &= [x] \\
 \Rightarrow x - [x] &= 3 \\
 \{x\} &= 3 \text{ (Not Possible)} \\
 \text{Only one solution } x &= 1 \text{ in } (-\infty, \infty)
 \end{aligned}$$

Question 77

Let $f(x)$ be a function such that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{N}$. If $f(1) = 3$ and $\sum_{k=1}^n f(k) = 3279$, then the value of n is
[24-Jan-2023 Shift 2]

Options:

A. 6

B. 8

C. 7

D. 9

Answer: C

Solution:

$$f(x+y) = f(x) \cdot f(y) \quad \forall x, y \in \mathbb{N}, f(1) = 3$$

$$f(2) = f^2(1) = 3^2$$

$$f(3) = f(1)f(2) = 3^3$$

$$f(4) = 3^4$$

$$f(k) = 3^k$$

$$\sum_{k=1}^n f(k) = 3279$$

$$f(1) + f(2) + f(3) + \dots + f(k) = 3279$$

$$3 + 3^2 + 3^3 + \dots + 3^k = 3279$$

$$\frac{3(3^k - 1)}{3 - 1} = 3279$$

$$\frac{3^k - 1}{2} = 1093$$

$$3^k - 1 = 2186$$

$$3^k = 2187$$

$$k = 7$$

Question 78

If $f(x) = \frac{2^{2x}}{2^{2x} + 2}$, $x \in \mathbb{R}$ then $f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \dots + f\left(\frac{2022}{2023}\right)$ equal to

[24-Jan-2023 Shift 2]

Options:

A. 2011

B. 1010

C. 2010

D. 1011

Answer: D

Solution:

$$f(x) = \frac{4^x}{4^x + 2}$$

$$f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2}$$

$$= \frac{4^x}{4^x + 2} + \frac{4}{4 + 2(4^x)}$$

$$= \frac{4^x}{4^x + 2} + \frac{2}{2 + 4^x}$$

$$= 1$$

$$\Rightarrow f(x) + f(1-x) = 1$$

$$\text{Now } f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + f\left(\frac{3}{2023}\right) + \dots +$$

$$\dots + f\left(1 - \frac{3}{2023}\right) + f\left(1 - \frac{2}{2023}\right) + f\left(1 - \frac{1}{2023}\right)$$

Now sum of terms equidistant from beginning and end is 1

$$\text{Sum} = 1 + 1 + 1 + \dots + 1 \text{ (1011 times)}$$

$$= 1011$$

Question 79

For some $a, b, c \in \mathbb{N}$, let $f(x) = ax - 3$ and $g(x) = x^b + c$, $x \in \mathbb{R}$. If

$(f \circ g)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$ then $(f \circ g)(ac) + (g \circ f)(b)$ is equal to

[25-Jan-2023 Shift 1]

Answer: 2039

Solution:

$$\text{Let } f \circ g(x) = h(x)$$

$$\Rightarrow h^{-1}(x) = \left(\frac{x-7}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow h(x) = f \circ g(x) = 2x^3 + 7$$

$$f \circ g(x) = a(x^b + c) - 3$$

$$\begin{aligned}\Rightarrow a &= 2, b = 3, c = 5 \\ \Rightarrow \text{fog}(ac) &= \text{fog}(10) = 2007 \\ g(f(x)) &= (2x - 3)^3 + 5. \\ \Rightarrow \text{gof}(b) &= \text{gof}(3) = 32 \\ \Rightarrow \text{sum} &= 2039\end{aligned}$$

Question80

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \log_{\sqrt{m}}\{\sqrt{2}(\sin x - \cos x) + m - 2\}$, for some m , such that the range of f is $[0, 2]$. Then the value of m is _____
[25-Jan-2023 Shift 2]

Options:

- A. 5
- B. 3
- C. 2
- D. 4

Answer: A

Solution:

Since,

$$-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2}$$

$$\therefore -2 \leq \sqrt{2}(\sin x - \cos x) \leq 2$$

(Assume $\sqrt{2}(\sin x - \cos x) = k$)

$$-2 \leq k \leq 2 \quad \dots (i)$$

$$f(x) = \log_{\sqrt{m}}(k + m - 2)$$

Given,

$$0 \leq f(x) \leq 2$$

$$0 \leq \log_{\sqrt{m}}(k + m - 2) \leq 2$$

$$1 \leq k + m - 2 \leq m$$

$$-m + 3 \leq k \leq 2 \dots \dots (ii)$$

From eq. (i) & (ii), we get $-m + 3 = -2$

$$\Rightarrow m = 5$$

Question81

The number of functions $f : \{1, 2, 3, 4\} \rightarrow \{a \in \mathbb{Z} : |a| \leq 8\}$ satisfying $f(n) + \frac{1}{n}f(n+1) = 1, \forall n \in \{1, 2, 3\}$ is
[25-Jan-2023 Shift 2]

Options:

- A. 3
- B. 4
- C. 1
- D. 2

Answer: D

Solution:

$$f : \{1, 2, 3, 4\} \rightarrow \{a \in \mathbb{Z} : |a| \leq 8\}$$

$$f(n) + \frac{1}{n}f(n+1) = 1, \forall n \in \{1, 2, 3\}$$

$f(n+1)$ must be divisible by n

$$f(4) \Rightarrow -6, -3, 0, 3, 6$$

$$f(3) \Rightarrow -8, -6, -4, -2, 0, 2, 4, 6, 8$$

$$f(2) \Rightarrow -8, \dots, 8$$

$$f(1) \Rightarrow -8, \dots, 8$$

$\frac{f(4)}{3}$ must be odd since $f(3)$ should be even therefore 2 solution possible.

Question 82

Let $f(x) = 2x^n + \lambda, \lambda \in \mathbb{R}, n \in \mathbb{N}$, and $f(4) = 133, f(5) = 255$. Then the sum of all the positive integer divisors of $(f(3) - f(2))$ is
[25-Jan-2023 Shift 2]

Options:

- A. 61
- B. 60
- C. 58
- D. 59

Answer: B

Solution:

$$f(x) = 2x^n + \lambda$$

$$f(4) = 133$$

$$f(5) = 255$$

$$133 = 2 \times 4^n + \lambda \dots (1)$$

$$255 = 2 \times 5^n + \lambda \dots (2)$$

$$(2) - (1)$$

$$122 = 2(5^n - 4^n)$$

$$\Rightarrow 5^n - 4^n = 61$$

$$\therefore n = 3 \text{ and } \lambda = 5$$

$$\text{Now, } f(3) - f(2) = 2(3^3 - 2^3) = 38$$

Number of Divisors is 1, 2, 19, 38; & their sum is 60

Question 83

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}$. Then

[29-Jan-2023 Shift 1]

Options:

A. $f(x)$ is many-one in $(-\infty, -1)$

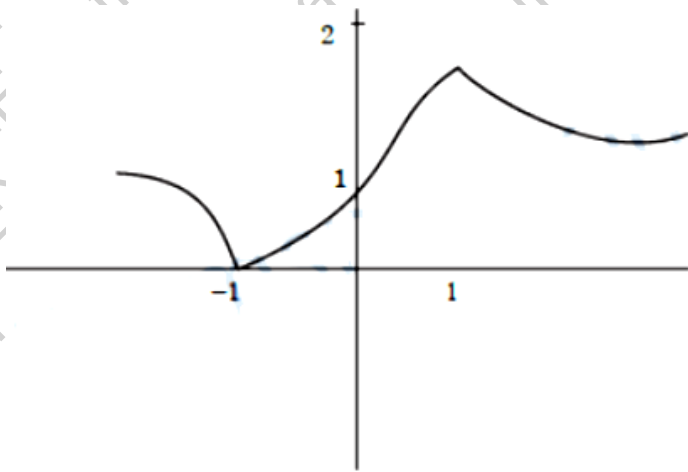
B. $f(x)$ is many-one in $(1, \infty)$

C. $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

D. $f(x)$ is one-one in $(-\infty, \infty)$

Answer: C

Solution:



$$f(x) = \frac{(x+1)^2}{x^2+1} = 1 + \frac{2x}{x^2+1}$$

$$f(x) = 1 + \frac{2}{x + \frac{1}{x}}$$

Question 84

The domain of $f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}$, $x \in \mathbb{R}$ is

[29-Jan-2023 Shift 1]

Options:

- A. $\mathbb{R} - \{1-3\}$
- B. $(2, \infty) - \{3\}$
- C. $(-1, \infty) - \{3\}$
- D. $\mathbb{R} - \{3\}$

Answer: B

Solution:

$$x-2 > 0 \Rightarrow x > 2$$

$$x+1 > 0 \Rightarrow x > -1$$

$$x+1 \neq 1 \Rightarrow x \neq 0 \text{ and } x > 0$$

Denominator

$$x^2 - 2x - 3 \neq 0$$

$$(x-3)(x+1) \neq 0$$

$$x \neq -1, 3$$

So Ans $(2, \infty) - \{3\}$

Question85

Consider a function $f : \mathbb{N} \rightarrow \mathbb{R}$, satisfying

$f(1) + 2f(2) + 3f(3) + \dots + xf(x) = x(x+1)f(x)$; $x \geq 2$ with $f(1) = 1$. Then

$\frac{1}{f(2022)} + \frac{1}{f(2028)}$ is equal to

[29-Jan-2023 Shift 2]

Options:

A. 8200

B. 8000

C. 8400

D. 8100

Answer: D

Solution:

Given for $x \geq 2$

$$f(1) + 2f(2) + \dots + xf(x) = x(x+1)f(x)$$

replace x by $x+1$

$$\Rightarrow x(x+1)f(x) + (x+1)f(x+1)$$

$$= (x+1)(x+2)f(x+1)$$

$$\Rightarrow \frac{x}{f(x+1)} + \frac{1}{f(x)} = \frac{(x+2)}{f(x)}$$

$$\Rightarrow xf(x) = (x+1)f(x+1) = \frac{1}{2}, x \geq 2$$

$$f(2) = \frac{1}{4}, f(3) = \frac{1}{6}$$

$$\text{Now } f(2022) = \frac{1}{4044}$$

$$f(2028) = \frac{1}{4056}$$

$$\text{So, } \frac{1}{f(2022)} + \frac{1}{f(2028)} = 4044 + 4056 = 8100$$

Question86

Suppose $f : \mathbb{R} \rightarrow (0, \infty)$ be a differentiable function such that $5f(x+y) = f(x) \cdot f(y)$, $\forall x, y \in \mathbb{R}$. If $f(3) = 320$, then $\sum_{n=0}^5 f(n)$ is equal to:
[30-Jan-2023 Shift 1]

Options:

A. 6875

B. 6575

C. 6825

D. 6528

Answer: C

Solution:

Option (3)

$$5f(x+y) = f(x) \cdot f(y)$$

$$5f(0) = f(0)^2 \Rightarrow f(0) = 5$$

$$5f(x+1) = f(x) \cdot f(1)$$

$$\Rightarrow \frac{f(x+1)}{f(x)} = \frac{f(1)}{5}$$

$$\Rightarrow \frac{f(1)}{f(0)} \cdot \frac{f(2)}{f(1)} \cdot \frac{f(3)}{f(2)} = \left(\frac{f(1)}{5}\right)^3$$

$$\Rightarrow \frac{320}{5} = \frac{(f(1))^3}{5^3} \Rightarrow f(1) = 20$$

$$\therefore 5f(x+1) = 20 \cdot f(x) \Rightarrow f(x+1) = 4f(x)$$

$$\sum_{n=0}^5 f(n) = 5 + 5.4 + 5.4^2 + 5.4^3 + 5.4^4 + 5.4^5$$

$$= \frac{5[4^6 - 1]}{3} = 6825$$

Question 87

Let $S = \{1, 2, 3, 4, 5, 6\}$. Then the number of one functions $f : S \rightarrow P(S)$, where $P(S)$ denote the power set of S , such that $f(n) \subset f(m)$ where $n < m$ is _____.

[30-Jan-2023 Shift 1]

Answer: 3240

Solution:

Let $S = \{1, 2, 3, 4, 5, 6\}$, then the number of one-one functions, $f: S \rightarrow P(S)$, where $P(S)$ denotes the power set of S , such that $f(n) < f(m)$ where $n < m$ is

$$n(S) = 6$$

$$P(S) = \left\{ \begin{array}{ccccccc} \phi & \{1\} & \dots & \{6\} & \{1, 2\} & \dots & \\ \{5, 6\} & \dots & \{1, 2, 3, 4, 5, 6\} & & & & \end{array} \right\}$$

- 64 elements

case - 1

$f(6) = S$ i.e. 1 option,

$f(5) =$ any 5 element subset A of S i.e. 6 options,

$f(4) =$ any 4 element subset B of A i.e. 5 options,

$f(3) =$ any 3 element subset C of B i.e. 4 options,

$f(2) =$ any 2 element subset D of C i.e. 3 options,

$f(1) =$ any 1 element subset E of D or empty subset i.e. 3 options,

Total functions = 1080

Case - 2

$f(6) =$ any 5 element subset A of S i.e. 6 options,

$f(5) =$ any 4 element subset B of A i.e. 5 options,

$f(4) =$ any 3 element subset C of B i.e. 4 options,

$f(3) =$ any 2 element subset D of C i.e. 3 options,

$f(2) =$ any 1 element subset E of D i.e. 2 options,

$f(1) =$ empty subset i.e. 1 option

Total functions = 720

Case - 3

$f(6) = S$

$f(5) =$ any 4 element subset A of S i.e. 15 options,

$f(4) =$ any 3 element subset B of A i.e. 4 options,

$f(3) =$ any 2 element subset C of B i.e. 3 options,

$f(2) =$ any 1 element subset D of C i.e. 2 options,

$f(1) =$ empty subset i.e. 1 option

Total functions = 360

Case - 4

$f(6) = S$

$f(5) =$ any 5 element subset A of S i.e. 6 options,

$f(4) =$ any 3 element subset B of A i.e. 10 options,

$f(3) =$ any 2 element subset C of B i.e. 3 options,

$f(2) =$ any 1 element subset D of C i.e. 2 options,

$f(1) =$ empty subset i.e. 1 option

Total functions = 360

Case - 5

$f(6) = S$

$f(5) =$ any 5 element subset A of S i.e. 6 options,

$f(4) =$ any 4 element subset B of A i.e. 5 options,

$f(3) =$ any 2 element subset C of B i.e. 6 options,

$f(2) =$ any 1 element subset D of C i.e. 2 options,

$f(1) =$ empty subset i.e. 1 option

Total functions = 360

Case - 6

$f(6) = S$
 $f(5) =$ any 5 element subset A of S i.e. 6 options,
 $f(4) =$ any 4 element subset B of A i.e. 5 options,
 $f(3) =$ any 3 element subset C of B i.e. 4 options,
 $f(2) =$ any 1 element subset D of C i.e. 3 options,
 $f(1) =$ empty subset i.e. 1 option
 Total functions = 360
 $\therefore$ Number of such functions = 3240

Question88

Let $f^1(x) = \frac{3x+2}{2x+3}$, $x \in \mathbb{R} - \left\{ -\frac{3}{2} \right\}$

For $n \geq 2$, define $f^n(x) = f^1 \circ f^{n-1}(x)$.

If $f^5(x) = \frac{ax+b}{bx+a}$, $\gcd(a, b) = 1$, then $a + b$ is equal to _____.

[30-Jan-2023 Shift 1]

Answer: 3125

Solution:

$$\begin{aligned}
 f^1(x) &= \frac{3x+2}{2x+3} \\
 \Rightarrow f^2(x) &= \frac{13x+12}{12x+13} \\
 \Rightarrow f^3(x) &= \frac{63x+62}{62x+63} \\
 \therefore f^5(x) &= \frac{1563x+1562}{1562x+1563} \\
 a+b &= 3125
 \end{aligned}$$

Question89

The range of the function $f(x) = \sqrt{3-x} + \sqrt{2+x}$ is

[30-Jan-2023 Shift 2]

Options:

A. $[\sqrt{5}, \sqrt{10}]$

B. $[2\sqrt{2}, \sqrt{11}]$

C. $[\sqrt{5}, \sqrt{13}]$

D. $[\sqrt{2}, \sqrt{7}]$

Answer: A

Solution:

$$y^2 = 3 - x + 2 + x + 2\sqrt{(3-x)(2+x)}$$

$$= 5 + 2\sqrt{6 + x - x^2}$$

$$y^2 = 5 + 2\sqrt{\frac{25}{4} - \left(x - \frac{1}{2}\right)^2}$$

$$y_{\max} = \sqrt{5+5} = \sqrt{10}$$

$$y_{\min} = \sqrt{5}$$

Question90

Let $A = \{1, 2, 3, 5, 8, 9\}$. Then the number of possible functions $f: A \rightarrow A$ such that $f(m \cdot n) = f(m) \cdot f(n)$ for every $m, n \in A$ with $m \cdot n \in A$ is equal to _____.

[30-Jan-2023 Shift 2]

Answer: 432

Solution:

$$f(1) = 1; f(9) = f(3) \times f(3)$$

$$\text{i.e., } f(3) = 1 \text{ or } 3$$

$$\text{Total function} = 1 \times 6 \times 2 \times 6 \times 6 \times 1 = 432$$

Question91

If the domain of the function $f(x) = \frac{[x]}{1+x^2}$, where $[x]$ is greatest integer $\leq x$, is $[2, 6)$, then its range is
[31-Jan-2023 Shift 1]

Options:

A. $\left(\frac{5}{26}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$

B. $\left(\frac{5}{26}, \frac{2}{5}\right]$

C. $\left(\frac{5}{37}, \frac{2}{5}\right] - \left\{\frac{9}{29}, \frac{27}{109}, \frac{18}{89}, \frac{9}{53}\right\}$

D. $\left(\frac{5}{37}, \frac{2}{5}\right]$

Answer: D

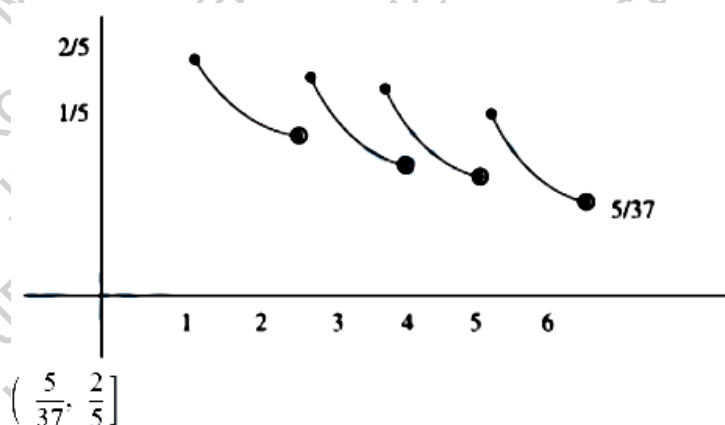
Solution:

$$f(x) = \frac{2}{1+x^2} \quad x \in [2, 3)$$

$$f(x) = \frac{3}{1+x^2} \quad x \in [3, 4)$$

$$f(x) = \frac{4}{1+x^2} \quad x \in [4, 5)$$

$$f(x) = \frac{5}{1+x^2} \quad x \in [5, 6)$$



Question92

The absolute minimum value, of the function

$f(x) = |x^2 - x + 1| + [x^2 - x + 1]$, where $[t]$ denotes the greatest integer function, in the interval $[-1, 2]$, is :

[31-Jan-2023 Shift 2]

Options:

A. $\frac{3}{4}$

B. $\frac{3}{2}$

C. $\frac{1}{4}$

D. $\frac{5}{4}$

Answer: A

Solution:

$$f(x) = |x^2 - x + 1| + [x^2 - x + 1]; x \in [-1, 2]$$

$$\text{Let } g(x) = x^2 - x + 1$$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\because |x^2 - x + 1| \text{ and } [x^2 - x + 1]$$

Both have minimum value at $x = 1/2$

$$\Rightarrow \text{Minimum } f(x) = \frac{3}{4} + 0$$

$$= \frac{3}{4}$$

Question93

Let $f : \mathbb{R} - \{2, 6\} \rightarrow \mathbb{R}$ be real valued function defined as

$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$. Then range of f is

[31-Jan-2023 Shift 2]

Options:

A. $\left(-\infty, -\frac{21}{4}\right] \cup [0, \infty)$

B. $\left(-\infty, -\frac{21}{4}\right) \cup (0, \infty)$

C. $\left(-\infty, -\frac{21}{4}\right] \cup \left[\frac{21}{4}, \infty\right)$

D. $\left(-\infty, -\frac{21}{4}\right] \cup [1, \infty)$

Answer: A

Solution:

Let $y = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$

By cross multiplying

$$yx^2 - 8xy + 12y - x^2 - 2x - 1 = 0$$

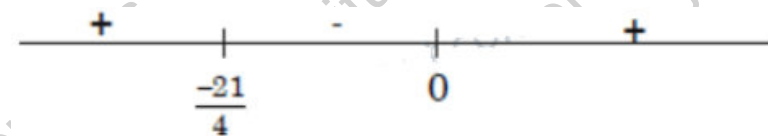
$$x^2(y - 1) - x(8y + 2) + (12y - 1) = 0$$

Case 1, $y \neq 1$

$$D \geq 0$$

$$\Rightarrow (8y + 2)^2 - 4(y - 1)(12y - 1) \geq 0$$

$$\Rightarrow y(4y + 21) \geq 0$$



$$y \in \left(-\infty, -\frac{21}{4}\right] \cup [0, \infty) - \{1\}$$

Case 2, $y = 1$

$$x^2 + 2x + 1 = x^2 - 8x + 12$$

$$10x = 11$$

$$x = \frac{11}{10} \quad \text{So, } y \text{ can be } 1$$

$$\text{Hence } y \in \left(-\infty, -\frac{21}{4}\right] \cup [0, \infty)$$

Question 94

Let $f: \mathbb{R} - \{0, 1\} \rightarrow \mathbb{R}$ be a function such that $f(x) + f\left(\frac{1}{1-x}\right) = 1 + x$.

Then f (2) is equal to :
[1-Feb-2023 Shift 2]

Options:

A. $\frac{9}{2}$

B. $\frac{9}{4}$

C. $\frac{7}{4}$

D. $\frac{7}{3}$

Answer: B

Solution:

$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + x$$

$$x = 2 \Rightarrow f(2) + f(-1) = 3$$

$$x = -1 \Rightarrow f(-1) + f\left(\frac{1}{2}\right) = 0 \dots (2)$$

$$x = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) + f(2) = \frac{3}{2} \dots (3)$$

$$(1) + (3) - (2) \Rightarrow 2f(2) = \frac{9}{2}$$

$$\therefore f(2) = \frac{9}{4}$$

Question95

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x - 1$ and $g : \mathbb{R} - \{1, -1\} \rightarrow \mathbb{R}$ be defined as $g(x) = \frac{x^2}{x^2 - 1}$. Then the function fog is :

[26-Jun-2022-Shift-2]

Options:

- A. one-one but not onto
- B. onto but not one-one
- C. both one-one and onto
- D. neither one-one nor onto

Answer: D

Solution:

$f : R \rightarrow R$ defined as

$$f(x) = x - 1 \text{ and } g : R \rightarrow \{1, -1\} \rightarrow R, g(x) = \frac{x^2}{x^2 - 1}$$

$$\text{Now } f \circ g(x) = \frac{x^2}{x^2 - 1} - 1 = \frac{1}{x^2 - 1}$$

$$\therefore \text{Domain of } f \circ g(x) = R - \{-1, 1\}$$

$$\text{And range of } f \circ g(x) = (-\infty, -1] \cup (0, \infty)$$

$$\text{Now, } \frac{d}{dx}(f \circ g(x)) = \frac{-1}{x^2 - 1} \cdot 2x = \frac{2x}{1 - x^2}$$

$$\therefore \frac{d}{dx}(f \circ g(x)) > 0 \text{ for } \frac{2x}{(1-x)(1+x)} > 0$$

$$\Rightarrow \frac{x}{(x-1)(x+1)} < 0$$

$$\therefore x \in (-\infty, -1) \cup (0, 1)$$

$$\text{and } \frac{d}{dx}(f \circ g(x)) < 0 \text{ for } x \in (-1, 0) \cup (1, \infty)$$

$\therefore f \circ g(x)$ is neither one-one nor onto.

Question 96

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{2e^{2x}}{e^{2x} + e}$.

Then

$$f\left(\frac{1}{100}\right) + f\left(\frac{2}{100}\right) + f\left(\frac{3}{100}\right) + \dots + f\left(\frac{99}{100}\right)$$

is equal to _____

[27-Jun-2022-Shift-1]

Answer: 99

Solution:

Given,

$$f(x) = \frac{2e^{2x}}{e^{2x} + e}$$

$$\therefore f(1-x) = \frac{2e^{2(1-x)}}{e^{2(1-x)} + e}$$

$$= \frac{2 \cdot \frac{e^2}{e^{2x}}}{\frac{e^2}{e^{2x}} + e}$$

$$= \frac{2e^2}{e^2 + e^{2x} \cdot e}$$

$$= \frac{2e^2}{e^2 + e^{2x+1}}$$

$$= \frac{2e^2}{e(e + e^{2x})}$$

$$= \frac{2e}{e + e^{2x}}$$

$$\therefore f(x) + f(1-x) = \frac{2e^{2x}}{e^{2x} + e} + \frac{2e}{e^{2x} + e}$$

$$= \frac{2(e^{2x} + e)}{e^{2x} + e}$$

$$= 2 \dots (1)$$

Now,

$$f\left(\frac{1}{100}\right) + f\left(\frac{99}{100}\right)$$

$$= f\left(\frac{1}{100}\right) + f\left(1 - \frac{1}{100}\right)$$

$$= 2 \text{ [as } f(x) + f(1-x) = 2 \text{]}$$

$$f\left(\frac{2}{100}\right) + f\left(1 - \frac{2}{100}\right) = 2$$

⋮

$$f\left(\frac{49}{100}\right) + f\left(1 - \frac{49}{100}\right) = 2$$

$$\therefore \text{Total sum} = 49 \times 2$$

$$\text{Remaining term} = f\left(\frac{50}{100}\right) = f\left(\frac{1}{2}\right)$$

Put $x = \frac{1}{2}$ in equation (1), we get

$$f\left(\frac{1}{2}\right) + f\left(1 - \frac{1}{2}\right) = 2$$

$$\Rightarrow 2f\left(\frac{1}{2}\right) = 2$$

$$\Rightarrow f\left(\frac{1}{2}\right) = 1$$

$$\therefore \text{Sum} = 49 \times 2 + 1 = 99$$

Question 97

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define $f : S \rightarrow S$ as

$$f(n) = \begin{cases} 2n & , \text{ if } n = 1, 2, 3, 4, 5 \\ 2n - 11 & , \text{ if } n = 6, 7, 8, 9, 10, \end{cases}$$

Let $g : S \rightarrow S$ be a function such that $fg(n) = \begin{cases} n+1 & , \text{ if } n \text{ is odd} \\ n-1 & , \text{ if } n \text{ is even} \end{cases}$

Then $g(10)g(1) + g(2) + g(3) + g(4) + g(5)$ is equal to
[27-Jun-2022-Shift-2]

Answer: 190

Solution:

$$\therefore f(n) = \begin{cases} 2n & n = 1, 2, 3, 4, 5 \\ 2n - 11 & n = 6, 7, 8, 9, 10 \end{cases}$$

$$\therefore f(1) = 2, f(2) = 4, \dots, f(5) = 10$$

$$\text{and } f(6) = 1, f(7) = 3, f(8) = 5, \dots, f(10) = 9$$

$$\text{Now, } f(g(n)) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$$

$$\begin{aligned}
 f(g(10)) &= 9 \Rightarrow g(10) = 10 \\
 f(g(1)) &= 2 \Rightarrow g(1) = 1 \\
 \therefore f(g(2)) &= 1 \Rightarrow g(2) = 6 \\
 \therefore f(g(3)) &= 4 \Rightarrow g(3) = 2 \\
 f(g(4)) &= 3 \Rightarrow g(4) = 7 \\
 f(g(5)) &= 6 \Rightarrow g(5) = 3 \\
 \therefore g(10)g(1) + g(2) + g(3) + g(4) + g(5) &= 190
 \end{aligned}$$

Question 98

Let a function $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f(n) = \begin{cases} 2n & n = 2, 4, 6, 8, \dots \\ n-1 & n = 3, 7, 11, 15, \dots \\ \frac{n+1}{2} & n = 1, 5, 9, 13, \dots \end{cases}$$

then, f is

[28-Jun-2022-Shift-1]

Options:

- A. one-one but not onto
- B. onto but not one-one
- C. neither one-one nor onto
- D. one-one and onto

Answer: D

Solution:

When $n = 1, 5, 9, 13$ then $\frac{n+1}{2}$ will give all odd numbers.

When $n = 3, 7, 11, 15, \dots$

$n-1$ will be even but not divisible by 4

When $n = 2, 4, 6, 8, \dots$

Then $2n$ will give all multiples of 4

So range will be $\mathbb{N}$.

And no two values of n give same y , so function is one-one and onto.

Question99

The probability that a randomly chosen one-one function from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3, 4, 5\}$ satisfies $f(a) + 2f(b) - f(c) = f(d)$ is :
[28-Jun-2022-Shift-2]

Options:

A. $\frac{1}{24}$

B. $\frac{1}{40}$

C. $\frac{1}{30}$

D. $\frac{1}{20}$

Answer: D

Solution:

Number of one-one function from $\{a, b, c, d\}$ to set $\{1, 2, 3, 4, 5\}$ is ${}^5P_4 = 120$ (s).

The required possible set of value $(f(a), f(b), f(c), f(d))$ such that $f(a) + 2f(b) - f(c) = f(d)$ are $(5, 3, 2, 1), (5, 1, 2, 3), (4, 1, 3, 5), (3, 1, 4, 5), (5, 4, 3, 2)$ and $(3, 4, 5, 2)$

$$\therefore n(E) = 6$$

$$\therefore \text{Required probability} = \frac{n(E)}{n(S)} = \frac{6}{120} = \frac{1}{20}$$

Question100

Let $S = \{1, 2, 3, 4\}$. Then the number of elements in the set $\{f : S \times S \rightarrow S : f \text{ is onto and } f(a, b) = f(b, a) \geq a \forall (a, b) \in S \times S\}$ is _____
[28-Jun-2022-Shift-2]

Answer: 37

Solution:

There are 16 ordered pairs in $S \times S$. We write all these ordered pairs in 4 sets as follows.

$$A = \{(1, 1)\}$$

$$B = \{(1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1)\}$$

$$C = \{(1, 3), (2, 3), (3, 3), (3, 2), (3, 1)\}$$

$$D = \{(1, 2), (2, 2), (2, 1)\}$$

All elements of set B have image 4 and only element of A has image 1.

All elements of set C have image 3 or 4 and all elements of set D have image 2 or 3 or 4.

We will solve this question in two cases.

Case I: When no element of set C has image 3.

Number of onto functions = 2 (when elements of set D have images 2 or 3)

Case II: When atleast one element of set C has image 3.

$$\text{Number of onto functions} = (2^3 - 1)(1 + 2 + 2) = 35$$

$$\text{Total number of functions} = 37$$

Question 101

The domain of the function $\cos^{-1} \left(\frac{2\sin^{-1} \left(\frac{1}{4x^2 - 1} \right)}{\pi} \right)$ is :

[29-Jun-2022-Shift-1]

Options:

A. $\mathbb{R} - \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$

B. $(-\infty, -1] \cup [1, \infty) \cup \{0\}$

C. $\left(-\infty, -\frac{1}{2} \right) \cup \left(\frac{1}{2}, \infty \right) \cup \{0\}$

D. $\left(-\infty, -\frac{1}{\sqrt{2}} \right] \cup \left[\frac{1}{\sqrt{2}}, \infty \right) \cup \{0\}$

Answer: D

Solution:

$$-1 \leq \frac{2\sin^{-1} \left(\frac{1}{4x^2 - 1} \right)}{\pi} \leq 1$$

$$\Rightarrow -\frac{\pi}{2} \leq \sin^{-1} \left(\frac{1}{4x^2 - 1} \right) \leq \frac{\pi}{2}$$

$$\Rightarrow -1 \leq \frac{1}{4x^2 - 1} \leq 1$$

$$\therefore \frac{1}{4x^2 - 1} + 1 \geq 0$$

$$\Rightarrow \frac{1 + 4x^2 - 1}{4x^2 - 1} \geq 0$$

$$\Rightarrow \frac{4x^2}{4x^2 - 1} \geq 0$$

$$\Rightarrow \frac{4x^2}{(2x+1)(2x-1)} \geq 0 \dots (1)$$

$$\therefore x \in \left(-\alpha, -\frac{1}{2}\right) \cup \{0\} \cup \left(\frac{1}{2}, \alpha\right)$$

$$\text{And } \frac{1}{4x^2 - 1} - 1 \leq 0$$

$$\Rightarrow \frac{1 - 4x^2 + 1}{4x^2 - 1} \leq 0$$

$$\Rightarrow \frac{2 - 4x^2}{4x^2 - 1} \leq 0$$

$$\Rightarrow \frac{2x^2 - 1}{4x^2 - 1} \geq 0$$

$$\Rightarrow \frac{(\sqrt{2}x+1)(\sqrt{2}x-1)}{(2x+1)(2x-1)} \geq 0$$

$$x \in \left(-\alpha, -\frac{1}{\sqrt{2}}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{\sqrt{2}}, \alpha\right)$$

From (3) and (4), we get

$$\therefore x \in \left[-\alpha, -\frac{1}{\sqrt{2}}\right) \cup \left[\frac{1}{\sqrt{2}}, \alpha\right) \cup \{0\}$$

Question 102

Let $c, k \in \mathbb{R}$. If $f(x) = (c+1)x^2 + (1-c^2)x + 2k$ and $f(x+y) = f(x) + f(y) - xy$, for all $x, y \in \mathbb{R}$, then the value of $|2(f(1) + f(2) + f(3) + \dots + f(20))|$ is equal to _____
[29-Jun-2022-Shift-1]

Answer: 3395

Solution:

$f(x)$ is polynomial

Put $y = 1/x$ in given functional equation we get

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) - 1$$

$$\Rightarrow (c+1)\left(x + \frac{1}{x}\right)^2 + (1-c^2)\left(x + \frac{1}{x}\right) + 2K$$

$$= (c+1)x^2 + (1-c^2)x + 2K + (c+1)\frac{1}{x^2} + (1-c^2)\frac{1}{x} + 2K - 1$$

$$\Rightarrow 2(c+1) = 2K - 1 \dots (1)$$

and put $x = y = 0$ we get

$$f(0) = 2 + f(0) - 0 \Rightarrow f(0) = 0 \Rightarrow k = 0$$

$$\therefore k = 0 \text{ and } 2c = -3 \Rightarrow c = -3/2$$

$$f(x) = -\frac{x^2}{2} - \frac{5x}{4} = \frac{1}{4}(5x + 2x^2)$$

$$\left| 2 \sum_{i=1}^{20} f(i) \right| = \left| \frac{-2}{4} \left(\frac{5 \cdot 20 \cdot 21}{2} + \frac{2 \cdot 20 \cdot 21 \cdot 41}{6} \right) \right|$$

$$= \left| \frac{-1}{2} (2730 + 5740) \right|$$

$$= \left| -\frac{6790}{2} \right| = 3395.$$

Question103

Let $f(x)$ and $g(x)$ be two real polynomials of degree 2 and 1 respectively. If $f(g(x)) = 8x^2 - 2x$ and $g(f(x)) = 4x^2 + 6x + 1$, then the value of $f(2) + g(2)$ is _____
[29-Jun-2022-Shift-2]

Answer: 18

Solution:

$$f(g(x)) = 8x^2 - 2x.$$

$$g(f(x)) = 4x^2 + 6x + 1.$$

$$\text{So, } g(x) = 2x - 1$$

$$\&f(x) = 2x^2 + 3x + 1$$

$$f(2) = 8 + 6 + 1 = 15$$

$$\text{Ans. 18}$$

Question104

The domain of the function

$$f(x) = \frac{\cos^{-1}\left(\frac{x^2-5x+6}{x^2-9}\right)}{\log_e(x^2-3x+2)} \text{ is :}$$

[24-Jun-2022-Shift-1]

Options:

A. $(-\infty, 1) \cup (2, \infty)$

B. $(2, \infty)$

C. $\left[-\frac{1}{2}, 1\right) \cup (2, \infty)$

D. $\left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{3, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$

Answer: D

Solution:

$$-1 \leq \frac{x^2-5x+6}{x^2-9} \leq 1 \text{ and } x^2-3x+2 > 0, \neq 1$$

$$\frac{(x-3)(2x+1)}{x^2-9} \geq 0 \mid \frac{5(x-3)}{x^2-9} \geq 0$$

The solution to this inequality is

$$x \in \left[-\frac{1}{2}, \infty\right) - \{3\}$$

$$\text{for } x^2-3x+2 > 0 \text{ and } \neq 1$$

$$x \in (-\infty, 1) \cup (2, \infty) - \left\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right\}$$

Combining the two solution sets (taking intersection)

$$x \in \left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right\}$$

Question 105

The number of one-one functions $f : \{a, b, c, d\} \rightarrow \{0, 1, 2, \dots, 10\}$ such that $2f(a) - f(b) + 3f(c) + f(d) = 0$ is
[24-Jun-2022-Shift-1]

Answer: 31

Solution:

$$-1 \leq \frac{x^2 - 5x + 6}{x^2 - 9} \leq 1 \text{ and } x^2 - 3x + 2 > 0, \neq 1$$

$$\frac{(x-3)(2x+1)}{x^2-9} \geq 0 \mid \frac{5(x-3)}{x^2-9} \geq 0$$

The solution to this inequality is

$$x \in \left[\frac{-1}{2}, \infty \right) - \{3\}$$

$$\text{for } x^2 - 3x + 2 > 0 \text{ and } \neq 1$$

$$x \in (-\infty, 1) \cup (2, \infty) - \left\{ \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2} \right\}$$

Combining the two solution sets (taking intersection)

$$x \in \left[-\frac{1}{2}, 1 \right) \cup (2, \infty) - \left\{ \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2} \right\}$$

$f(d)$ can't be 9 and 10 as if $f(d) = 9$ or 10 then $f(b) = 2 + 9 = 11$ or $f(b) = 2 + 10 = 12$, which is not possible as here any function's maximum value can be 10.

∴ Total possible functions when $f(c) = 0$ and $f(a) = 1$ are = 7

(2) When $f(c) = 0$ and $f(a) = 2$ then

$$3 \times 0 + 2 \times 2 + f(d) = f(b)$$

$$\Rightarrow 4 + f(d) = f(b)$$

∴ possible value of $f(d) = 1, 3, 4, 5, 6$

∴ Total possible functions in this case = 5

(3) When $f(c) = 0$ and $f(a) = 3$ then

$$3 \times 0 + 2 \times 3 + f(d) = f(b)$$

$$\Rightarrow 6 + f(d) = f(b)$$

∴ Possible value of $f(d) = 1, 2, 4$

∴ Total possible functions in this case = 3

(4) When $f(c) = 0$ and $f(a) = 4$ then

$$3 \times 0 + 2 \times 4 + f(d) = f(b)$$

$$\Rightarrow 8 + f(d) = f(b)$$

∴ Possible value of $f(d) = 1, 2$

∴ Total possible functions in this case = 2

(5) When $f(c) = 0$ and $f(a) = 5$ then

$$3 \times 0 + 2 \times 5 + f(d) = f(b)$$

$$\Rightarrow 10 + f(d) = f(b)$$

Possible value of $f(d)$ can be 0 but $f(c)$ is already zero. So, no value to $f(d)$ can satisfy.

∴ No function is possible in this case.

∴ Total possible functions when $f(c)=0$ and $f(a)=1,2,3$ and 4 are $=7+5+3+2=17$

Case II:

(1) When $f(c)=1$ and $f(a)=0$ then

$$3 \times 1 + 2 \times 0 + f(d) = f(b)$$

$$\Rightarrow 3 + f(d) = f(b)$$

∴ Possible value of $f(d)=2,3,4,5,6,7$

∴ Total possible functions in this case = 6

(2) When $f(c)=1$ and $f(a)=2$ then

$$3 \times 1 + 2 \times 2 + f(d) = f(b)$$

$$\Rightarrow 7 + f(d) = f(b)$$

$\therefore$ Possible value of $f(d)=0,3$

$\therefore$ Total possible functions in this case = 2

(3) When $f(c)=1$ and $f(a)=3$ then

$$3 \times 1 + 2 \times 3 + f(d) = f(b)$$

$$\Rightarrow 9 + f(d) = f(b)$$

$\therefore$ Possible value of $f(d)=0$

$\therefore$ Total possible functions in this case = 1

$\therefore$ Total possible functions when $f(c)=1$ and $f(a)=0,2$ and 3 are $=6+2+1=9$

Case III:

(1) When $f(c)=2$ and $f(a)=0$ then

$$3 \times 2 + 2 \times 0 + f(d) = f(b)$$

$$\Rightarrow 6 + f(d) = f(b)$$

$\therefore$ Possible values of $f(d)=1,3,4$

$\therefore$ Total possible functions in this case = 3

(2) When $f(c)=2$ and $f(a)=1$ then,

$$3 \times 2 + 2 \times 1 + f(d) = f(b)$$

$$\Rightarrow 8 + f(d) = f(b)$$

$\therefore$ Possible values of $f(d)=0$

$\therefore$ Total possible function in this case = 1

$\therefore$ Total possible functions when $f(c)=2$ and $f(a)=0,1$ are $=3+1=4$

Case IV:

(1) When $f(c)=3$ and $f(a)=0$ then

$$3 \times 3 + 2 \times 0 + f(d) = f(b)$$

$$\Rightarrow 9 + f(d) = f(b)$$

$\therefore$ Possible values of $f(d)=1$

$\therefore$ Total one-one functions from four cases

$$=17+9+4+1=31$$

Question106

Let R_1 and R_2 be relations on the set $\{1, 2, \dots, 50\}$ such that

$R_1 = \{(p, p^n) : p \text{ is a prime and } n \geq 0 \text{ is an integer}\}$ and

$R_2 = \{(p, p^n) : p \text{ is a prime and } n = 0 \text{ or } 1\}$

Then, the number of elements in $R_1 - R_2$ is ____

[28-Jun-2022-Shift-1]

Answer: 8

Solution:

$$R_1 - R_2 = \{(2, 2^2), (2, 2^3), (2, 2^4), (2, 2^5), (3, 3^2), (3, 3^3), (5, 5^2), (7, 7^2)\}$$

So number of elements = 8

Question107

Let $R_1 = \{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \leq 13\}$ and

$R_2 = \{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \neq 13\}$. Then on $\mathbb{N}$:

[28-Jun-2022-Shift-2]

Options:

- A. Both R_1 and R_2 are equivalence relations
- B. Neither R_1 nor R_2 is an equivalence relation
- C. R_1 is an equivalence relation but R_2 is not
- D. R_2 is an equivalence relation but R_1 is not

Answer: B

Solution:

$R_1 = \{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \leq 13\}$ and

$R_2 = \{(a, b) \in \mathbb{N} \times \mathbb{N} : |a - b| \neq 13\}$

In R_1 : $|2 - 11| = 9 \leq 13$

$\therefore (2, 11) \in R_1$ and $(11, 19) \in R_1$ but $(2, 19) \notin R_1$

$\therefore R_1$ is not transitive

Hence R_1 is not equivalence

In R_2 : $(13, 3) \in R_2$ and $(3, 26) \in R_2$ but $(13, 26) \notin R_2$ ($\because |13 - 26| = 13$)

$\therefore R_2$ is not transitive

Hence R_2 is not equivalence.

Question 108

The probability that a relation R from $\{x, y\}$ to $\{x, y\}$ is both symmetric and transitive, is equal to
[29-Jun-2022-Shift-2]

Options:

A. $\frac{5}{16}$

B. $\frac{9}{16}$

C. $\frac{11}{16}$

D. $\frac{13}{16}$

Answer: A

Solution:

Total no. of relations = $2^{2 \times 2} = 16$

Fav. relation = $\emptyset, \{(x, x)\}, \{(y, y)\}, \{(x, x)(y, y)\}$

$\{(x, x), (y, y), (x, y)(y, x)\}$

Prob. = $\frac{5}{16}$

Question 109

The number of bijective functions

$f : \{1, 3, 5, 7, \dots, 99\} \rightarrow \{2, 4, 6, 8, \dots, 100\}$, such that $f(3) \geq f(9) \geq f(15) \geq f(21) \geq \dots \geq f(99)$, is
[25-Jul-2022-Shift-2]

Options:

A. ${}^{50}P_{17}$

B. ${}^{50}P_{33}$

C. $33! \times 17!$

D. $\frac{50!}{2}$

Answer: B

Solution:

As function is one-one and onto, out of 50 elements of domain set 17 elements are following restriction $f(3) > f(9) > f(15) > \dots > f(99)$.

So number of ways $= {}^{50}C_{17} \cdot 1.33!$
 $= {}^{50}P_{33}$

Question110

Let $f(x)$ be a quadratic polynomial with leading coefficient 1 such that $f(0) = p$, $p \neq 0$, and $f(1) = \frac{1}{3}$. If the equations $f(x) = 0$ and $f \circ f \circ f \circ f(x) = 0$ have a common real root, then $f(-3)$ is equal to
[25-Jul-2022-Shift-2]

Answer: 25

Solution:

$$\text{Let } f(x) = (x - \alpha)(x - \beta)$$

$$\text{It is given that } f(0) = p \Rightarrow \alpha\beta = p$$

$$\text{and } f(1) = \frac{1}{3} \Rightarrow (1 - \alpha)(1 - \beta) = \frac{1}{3}$$

Now, let us assume that, α is the common root of $f(x) = 0$ and $f(f(x)) = 0$

$$f(f(x)) = 0$$

$$\Rightarrow f(f(0)) = 0$$

$$\Rightarrow f(p) = 0$$

So, $f(p)$ is either α or β .

$$(p - \alpha)(p - \beta) = \alpha$$

$$(\alpha\beta - \alpha)(\alpha\beta - \beta) = \alpha \Rightarrow (\beta - 1)(\alpha - 1)\beta = 1 (\because \alpha \neq 0)$$

$$\text{So, } \beta = 3$$

$$(1 - \alpha)(1 - 3) = \frac{1}{3}$$

$$\alpha = \frac{7}{6}$$

$$f(x) = \left(x - \frac{7}{6}\right)(x - 3)$$

$$f(-3) = \left(-3 - \frac{7}{6}\right)(-3 - 3) = 25$$

Question 11

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(3x) - f(x) = x$. If $f(8) = 7$, then $f(14)$ is equal to:
[26-Jul-2022-Shift-1]

Options:

A. 4

B. 10

C. 11

D. 16

Answer: B

Solution:

$$f(3x) - f(x) = x \dots (1)$$

$$x \rightarrow \frac{x}{3}$$

$$f(x) - f\left(\frac{x}{3}\right) = \frac{x}{3} \dots (2)$$

$$\text{Again } x \rightarrow \frac{x}{3}$$

$$f\left(\frac{x}{3}\right) - f\left(\frac{x}{9}\right) = \frac{x}{3^2} \dots$$

Similarly

$$f\left(\frac{x}{3^{n-2}}\right) - f\left(\frac{x}{3^{n-1}}\right) = \frac{x}{3^{n-1}} \dots (n)$$

Adding all these and applying $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \left(f(3x) - f\left(\frac{x}{3^{n-1}}\right) \right) = x \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots \right)$$

$$f(3x) - f(0) = \frac{3x}{2}$$

$$\text{Putting } x = \frac{8}{3}$$

$$f(8) - f(0) = 4$$

$$\Rightarrow f(0) = 3$$

$$\text{Putting } x = \frac{14}{3}$$

$$f(14) - 3 = 7 \Rightarrow f(14) = 10$$

Question 112

The domain of the function

$$f(x) = \sin^{-1}[2x^2 - 3] + \log_2 \left(\log_{\frac{1}{2}}(x^2 - 5x + 5) \right), \text{ where } [t] \text{ is the greatest}$$

integer function, is :

[27-Jul-2022-Shift-2]

Options:

A. $\left(-\sqrt{\frac{5}{2}}, \frac{5-\sqrt{5}}{2} \right)$

B. $\left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2} \right)$

C. $\left(1, \frac{5-\sqrt{5}}{2} \right)$

D. $\left[1, \frac{5+\sqrt{5}}{2} \right)$

Answer: C

Solution:

$$-1 \leq 2x^2 - 3 < 2$$

$$\text{or } 2 \leq 2x^2 < 5$$

$$\text{or } 1 \leq x^2 < \frac{5}{2}$$

$$x \in \left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$$

$$\log_{\frac{1}{2}}(x^2 - 5x + 5) > 0$$

$$0 < x^2 - 5x + 5 < 1$$

$$x^2 - 5x + 5 > 0 \text{ \& } x^2 - 5x + 4 < 0$$

$$x \in \left(-\infty, \frac{5 - \sqrt{5}}{2}\right) \cup \left(\frac{5 + \sqrt{5}}{2}, \infty\right)$$

$$\text{\& } x \in (-\infty, 1) \cup (4, \infty)$$

Taking intersection

$$x \in \left(1, \frac{5 - \sqrt{5}}{2}\right)$$

Question 113

The number of functions f , from the set $A = \{x \in \mathbb{N} : x^2 - 10x + 9 \leq 0\}$ to the set $B = \{n^2 : n \in \mathbb{N}\}$ such that $f(x) \leq (x - 3)^2 + 1$, for every $x \in A$, is _____.

[27-Jul-2022-Shift-2]

Answer: 1440

Solution:

$$A = \{x \in \mathbb{N}, x^2 - 10x + 9 \leq 0\}$$

$$= \{1, 2, 3, \dots, 9\}$$

$$B = \{1, 4, 9, 16, \dots\}$$

$$f(x) \leq (x - 3)^2 + 1$$

$$f(1) \leq 5, f(2) \leq 2, \dots, f(9) \leq 37$$

$x = 1$ has 2 choices

$x = 2$ has 1 choice

$x = 3$ has 1 choice

$x = 4$ has 1 choice

$x = 5$ has 2 choices

$x = 6$ has 3 choices

$x = 7$ has 4 choices

$x = 8$ has 5 choices

$x = 9$ has 6 choices

$\therefore$ Total functions $= 2 \times 1 \times 1 \times 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 1440$

Question114

Considering only the principal values of the inverse trigonometric functions, the domain of the function $f(x) = \cos^{-1}\left(\frac{x^2 - 4x + 2}{x^2 + 3}\right)$ is :

[28-Jul-2022-Shift-1]

Options:

A. $\left(-\infty, \frac{1}{4}\right]$

B. $\left[-\frac{1}{4}, \infty\right)$

C. $(-1/3, \infty)$

D. $\left(-\infty, \frac{1}{3}\right]$

Answer: B

Solution:

$$-1 \leq \frac{x^2 - 4x + 2}{x^2 + 3} \leq 1$$

$$\Rightarrow -x^2 - 3 \leq x^2 - 4x + 2 \leq x^2 + 3$$

$$\Rightarrow 2x^2 - 4x + 5 \geq 0 \quad -4x \leq 1$$

$$x \in \mathbb{R} \text{ and } x \geq -\frac{1}{4}$$

$$\text{So domain is } \left[-\frac{1}{4}, \infty\right)$$

Question115

Let α , β and γ be three positive real numbers. Let

$f(x) = \alpha x^5 + \beta x^3 + \gamma x$, $x \in \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g(f(x)) = x$ for

all $x \in \mathbb{R}$. If $a_1, a_2, a_3, \dots, a_n$ be in arithmetic progression with mean zero, then the value of $f\left(g\left(\frac{1}{n} \sum_{i=1}^n f(a_i)\right)\right)$ is equal to:

[28-Jul-2022-Shift-1]

Options:

- A. 0
- B. 3
- C. 9
- D. 27

Answer: A

Solution:

$$f\left(g\left(\frac{1}{n} \sum_{i=1}^n f(a_i)\right)\right)$$

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = 0$$

$\therefore$ First and last term, second and second last and so on are equal in magnitude but opposite in sign.

$$f(x) = \alpha x^5 + \beta x^3 + \gamma x$$

$$\sum_{i=1}^n f(a_i) = \alpha(a_1^5 + a_2^5 + a_3^5 + \dots + a_n^5) + \beta(a_1^3 + a_2^3 + \dots + a_n^3) + \gamma(a_1 + a_2 + \dots + a_n)$$

$$= 0\alpha + 0\beta + 0\gamma$$

$$= 0$$

$$\therefore f\left(g\left(\frac{1}{n} \sum_{i=1}^n f(a_i)\right)\right) = \frac{1}{n} \sum_{i=1}^n f(a_i) = 0$$

Question 116

The number of elements in the set

$$S = \left\{ x \in \mathbb{R} : 2 \cos\left(\frac{x^2 + x}{6}\right) = 4^x + 4^{-x} \right\} \text{ is:}$$

[29-Jul-2022-Shift-2]

Options:

- A. 1

- B. 3
C. 0
D. infinite

Answer: A

Solution:

$$2 \cos \left(\frac{x^2 + x}{6} \right) = 4^x + 4^{-x}$$

$$\text{L.H.S} \leq 2 \text{ \& R.H.S.} \geq 2$$

$$\text{Hence L.H.S} = 2 \text{ \& R.H.S} = 2$$

$$2 \cos \left(\frac{x^2 + x}{6} \right) = 2 \quad 4^x + 4^{-x} = 2$$

Check $x = 0$ Possible hence only one solution.

Question 117

The domain of the function $f(x) = \sin^{-1} \left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \right)$ is:

[29-Jul-2022-Shift-2]

Options:

- A. $[1, \infty)$
B. $[-1, 2]$
C. $[-1, \infty)$
D. $(-\infty, 2]$

Answer: C

Solution:

$$f(x) = \sin^{-1} \left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \right)$$

$$-1 \leq \frac{x^2 - 3x + 2}{x^2 + 2x + 7} \leq 1$$

$$\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \leq 1$$

$$x^2 - 3x + 2 \leq x^2 + 2x + 7$$

$$5x \geq -5$$

$$x \geq -1$$

$$\text{And } \frac{x^2 - 3x + 2}{x^2 + 2x + 7} \geq -1$$

$$x^2 - 3x + 2 \geq -x^2 - 2x - 7$$

$$2x^2 - x + 9 \geq 0$$

$$x \in \mathbb{R}$$

$$(i) \cap (ii)$$

$$\text{Domain} \in [-1, \infty)$$

Question 118

The total number of functions,

$f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ such that $f(1) + f(2) = f(3)$, is equal to

[25-Jul-2022-Shift-1]

Options:

A. 60

B. 90

C. 108

D. 126

Answer: B

Solution:

Given, $f(1) + f(2) = f(3)$

It means $f(1)$, $f(2)$ and $f(3)$ are dependent on each other. But there is no condition on $f(4)$, so $f(4)$ can be $f(4) = 1, 2, 3, 4, 5, 6$.

For $f(1)$, $f(2)$ and we have to find how many functions possible which will satisfy the condition

$$f(1) + f(2) = f(3)$$

Case 1:

When $f(3) = 2$ then possible values of $f(1)$ and $f(2)$ which satisfy $f(1) + f(2) = f(3)$ is $f(1) = 1$ and $f(2) = 1$.

And $f(4)$ can be $= 1, 2, 3, 4, 5, 6$

$$\therefore \text{Total possible functions} = 1 \times 6 = 6$$

Case 2 :

When $f(3) = 3$ then possible values

(1) $f(1) = 1$ and $f(2) = 2$

(2) $f(1) = 2$ and $f(2) = 1$

And $f(4)$ can be $= 1, 2, 3, 4, 5, 6$.

$\therefore$ Total functions $= 2 \times 6 = 12$

Case 3 :

When $f(3) = 4$ then

(1) $f(1) = 1$ and $f(2) = 3$

(2) $f(1) = 2$ and $f(2) = 2$

(3) $f(1) = 3$ and $f(2) = 1$

And $f(4)$ can be $= 1, 2, 3, 4, 5, 6$

$\therefore$ Total functions $= 3 \times 6 = 18$

Case 4 :

When $f(3) = 5$ then

(1) $f(1) = 1$ and $f(4) = 4$

(2) $f(1) = 2$ and $f(4) = 3$

(3) $f(1) = 3$ and $f(4) = 2$

(4) $f(1) = 4$ and $f(4) = 1$

And $f(4)$ can be $= 1, 2, 3, 4, 5$ and 6

$\therefore$ Total functions $= 4 \times 6 = 24$

Case 5 :

When $f(3) = 6$ then

(1) $f(1) = 1$ and $f(2) = 5$

(2) $f(1) = 2$ and $f(2) = 4$

(3) $f(1) = 3$ and $f(2) = 3$

(4) $f(1) = 4$ and $f(2) = 2$

(5) $f(1) = 5$ and $f(2) = 1$

And $f(4)$ can be $= 1, 2, 3, 4, 5$ and 6

$\therefore$ Total possible functions $= 5 \times 6 = 30$

$\therefore$ Total functions from those 5 cases we get $= 6 + 12 + 18 + 24 + 30 = 90$

Question 119

Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function such that $f(x+y) = 2f(x)f(y)$ for natural numbers x and y . If $f(1) = 2$, then the value of α for which

$$\sum_{k=1}^{10} f(\alpha + k) = \frac{512}{3}(2^{20} - 1)$$

holds, is :

[25-Jun-2022-Shift-1]

Options:

A. 2

B. 3

C. 4

D. 6

Answer: C

Solution:

Given,

$$f(x+y) = 2f(x)f(y)$$

$$\text{and } f(1) = 2$$

For $x = 1$ and $y = 1$,

$$f(1+1) = 2f(1)f(1)$$

$$\Rightarrow f(2) = 2(f(1))^2 = 2(2)^2 = 2^3$$

For $x = 1, y = 2$

$$f(1+2) = 2f(1)f(2)$$

$$\Rightarrow f(3) = 2 \cdot 2 \cdot 2^3 = 2^5$$

For $x = 1, y = 3$

$$f(1+3) = 2f(1)f(3)$$

$$\Rightarrow f(4) = 2 \cdot 2 \cdot 2^5 = 2^7$$

For $x = 1, y = 4$

$$f(1+4) = 2f(1)f(4)$$

$$\Rightarrow f(5) = 2 \cdot 2 \cdot 2^7 = 2^9 \dots$$

Also given

$$\sum_{k=1}^{10} f(a+k) = \frac{512}{3}(2^{20} - 1)$$

$$\Rightarrow f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \dots + f(\alpha+10) = \frac{512}{3}(2^{20} - 1)$$

$$\Rightarrow f(\alpha+1) + f(\alpha+2) + f(\alpha+3) + \dots + f(\alpha+10) = \frac{2^9((2^2)^{10} - 1)}{2^2 - 1}$$

This represent a G.P with first term $= 2^9$ and common ratio $= 2^2$

$$\therefore \text{First term} = f(\alpha+1) = 2^9 \dots (2)$$

$$\text{From equation (1), } f(5) = 2^9$$

$\therefore$ From (1) and (2), we get

$$f(\alpha+1) = 2^9 = f(5)$$

$$\Rightarrow f(\alpha+1) = f(5)$$

$$\Rightarrow f(\alpha+1) = f(4+1)$$

Comparing both sides we get, $\alpha = 4$

Question 120

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \left(2 \left(1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right)^{\frac{1}{50}}. \text{ If the function}$$

$g(x) = f(f(f(x)) + f(f(x)))$, then the greatest integer less than or equal to $g(1)$ is ____

[25-Jun-2022-Shift-1]

Answer: 2

Solution:

Given,

$$f(x) = \left(2 \left(1 - \frac{x^{25}}{2} \right) (2 + x^{25}) \right)^{\frac{1}{50}}$$

$$\text{and } g(x) = f(f(f(x))) + f(f(x))$$

$$\therefore g(1) = f(f(f(1))) + f(f(1))$$

$$\text{Now, } f(1) = \left(2 \left(1 - \frac{1^{25}}{2} \right) (2 + 1^{25}) \right)^{\frac{1}{50}}$$

$$= \left(2 \left(1 - \frac{1}{2} \right) (2 + 1) \right)^{\frac{1}{50}}$$

$$= (3)^{\frac{1}{50}}$$

$$\therefore f(f(1)) = f\left(3^{\frac{1}{50}}\right)$$

$$= \left(2 \left(1 - \frac{\left(3^{\frac{1}{50}}\right)^{25}}{2} \right) \left(2 + \left(3^{\frac{1}{50}}\right)^{25} \right) \right)^{\frac{1}{50}}$$

$$= \left(2 \left(1 - \frac{3^{\frac{1}{2}}}{2} \right) \left(2 + 3^{\frac{1}{2}} \right) \right)^{\frac{1}{50}}$$

$$= \left(2 \times \left(\frac{2 - \sqrt{3}}{2} \right) (2 + \sqrt{3}) \right)^{\frac{1}{50}}$$

$$= [(2 - \sqrt{3})(2 + \sqrt{3})]^{\frac{1}{50}}$$

$$= (4 - 3)^{\frac{1}{50}}$$

$$= 1^{\frac{1}{50}} = 1$$

$$\text{Now, } f(f(f(1))) = f(1) = 3^{\frac{1}{50}}$$

$$\therefore g(1) = f(f(f(1))) + f(f(1))$$

$$= 3^{\frac{1}{50}} + 1$$

Now, greatest integer less than or equal to $g(1)$

$$= [g(1)]$$

$$= \left[3^{\frac{1}{50}} + 1 \right]$$

$$= \left[3^{\frac{1}{50}} \right] + [1]$$

$$= [1.02] + 1$$

$$= 1 + 1 = 2$$

Question 121

Let $f(x) = \frac{x-1}{x+1}$, $x \in \mathbb{R} - \{0, -1, 1\}$. If $f^{n+1}(x) = f(f^n(x))$ for all $n \in \mathbb{N}$,

then $f^6(6) + f^7(7)$ is equal to :

[26-Jun-2022-Shift-1]

Options:

A. $\frac{7}{6}$

B. $-\frac{3}{2}$

C. $\frac{7}{12}$

D. $-\frac{11}{12}$

Answer: B

Solution:

Given,

$$f(x) = \frac{x-1}{x+1}$$

Also given,

$$f^{n+1}(x) = f(f^n(x)) \dots (1)$$

$\therefore$ For $n = 1$

$$f^{1+1}(x) = f(f^1(x))$$

$$\Rightarrow f^2(x) = f(f(x))$$

$$= f\left(\frac{x-1}{x+1}\right)$$

$$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1}$$

$$= \frac{\frac{x-1-x-1}{x+1}}{\frac{x-1+x+1}{x+1}}$$

$$= \frac{-2}{2x} = -\frac{1}{x}$$

From equation (1), when $n = 2$ $f^{2+1}(x) = f(f^2(x))$

$$\Rightarrow f^3(x) = f(f^2(x))$$

$$= f\left(-\frac{1}{x}\right)$$

$$= \frac{-\frac{1}{x} - 1}{-\frac{1}{x} + 1}$$

$$= \frac{\frac{-1-x}{x}}{\frac{-1+x}{x}}$$

$$= \frac{-1-x}{-1+x} = \frac{-(x+1)}{x-1}$$

Similarly,

$$f^4(x) = f(f^3(x))$$

$$= f\left(\frac{-(x+1)}{x-1}\right)$$

$$= \frac{\frac{-(x+1)}{x-1} - 1}{\frac{-(x+1)}{x-1} + 1}$$

$$= \frac{\frac{x-1-x+1}{x-1}}{\frac{-x-1+x-1}{x-1}}$$

$$= \frac{-2x}{-2} = x$$

$$\therefore f^5(x) = f(f^4(x))$$

$$= f(x)$$

$$= \frac{x-1}{x+1}$$

$$f^6(x) = f(f^5(x))$$

$$= f\left(\frac{x-1}{x+1}\right)$$

$$= -\frac{1}{x} \text{ (Already calculated earlier)}$$

$$f^7(x) = f(f^6(x))$$

$$= f\left(-\frac{1}{x}\right)$$

$$= \frac{-\frac{1}{x} - 1}{-\frac{1}{x} + 1}$$

$$= \frac{-(x+1)}{x-1}$$

$$\therefore f^6(6) = -\frac{1}{6}$$

$$\text{and } f^7(7) = \frac{-(7+1)}{7-1} = -\frac{8}{6}$$

$$\text{So, } f^6(6) + f^7(7)$$

$$= -\frac{1}{6} - \frac{8}{6}$$

$$= -\frac{3}{2}$$

Question 122

The range of the function,

$$f(x) = \log_{\sqrt{5}} \left(3 + \cos \left(\frac{3\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} - x \right) - \cos \left(\frac{3\pi}{4} - x \right) \right) \text{ is}$$

[2021, 01 Sep. Shift-II]

Options:

A. $(0, \sqrt{5})$

B. $[-2, 2]$

C. $\left[\frac{1}{\sqrt{5}}, \sqrt{5} \right]$

D. $[0, 2]$

Answer: D

Solution:

$$\begin{aligned} f(x) &= \log_{\sqrt{5}} \left(3 + \cos \left(\frac{3\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} + x \right) \right. \\ &\quad \left. + \cos \left(\frac{\pi}{4} - x \right) - \cos \left(\frac{3\pi}{4} - x \right) \right) \\ &= \log_{\sqrt{5}} (3 - \sqrt{2} \sin x + \sqrt{2} \cos x) \\ \therefore -2 &\leq -\sqrt{2} \sin x + \sqrt{2} \cos x \leq 2 \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 &\leq 3 - \sqrt{2} \sin x + \sqrt{2} \cos x \leq 5 \\ \Rightarrow \log_{\sqrt{5}} 1 &\leq \log_{\sqrt{5}} (3 - \sqrt{2} \sin x + \sqrt{2} \cos x) \\ \Rightarrow 0 &\leq f(x) \leq 2 \\ \Rightarrow f(x) &\in [0, 2] \end{aligned}$$

Question 123

The domain of the function $f(x) = \sin^{-1} \left(\frac{3x^2 + x - 1}{(x-1)^2} \right) + \cos^{-1} \left(\frac{x-1}{x+1} \right)$ is
[2021, 31 Aug. Shift-II]

Options:

- A. $\left[0, \frac{1}{4} \right]$
- B. $[-2, 0] \cup \left[\frac{1}{4}, \frac{1}{2} \right]$
- C. $\left[\frac{1}{4}, \frac{1}{2} \right] \cup \{0\}$
- D. $\left[0, \frac{1}{2} \right]$

Answer: C

Solution:

$$\begin{aligned} f(x) &= \sin^{-1} \left(\frac{3x^2 + x - 1}{(x-1)^2} \right) + \cos^{-1} \left(\frac{x-1}{x+1} \right) \\ -1 &\leq \frac{x-1}{x+1} \leq 1 \Rightarrow -1 - 1 \leq \frac{x-1}{x+1} - 1 \leq 1 - 1 \\ \Rightarrow -2 &\leq \frac{-2}{x+1} \leq 0 \Rightarrow 0 \leq \frac{1}{x+1} \leq 1 \\ \Rightarrow 1 &\leq x+1 < \infty \\ \Rightarrow 0 &\leq x < \infty \end{aligned}$$

$$\Rightarrow x \in [0, \infty)$$

$$\text{and } -1 \leq \frac{3x^2 + x - 1}{(x-1)^2} \leq 1$$

$$\Rightarrow -(x-1)^2 \leq 3x^2 + x - 1 \leq (x-1)^2, x \neq 1$$

$$\Rightarrow -(x^2 - 2x + 1) \leq 3x^2 + x - 1$$

$$\text{and } 3x^2 + x - 1 \leq x^2 - 2x + 1$$

$$\Rightarrow 4x^2 - x \geq 0$$

$$\text{and } 2x^2 + 3x - 2 \leq 0$$

$$\Rightarrow x(4x-1) \geq 0$$

$$\text{and } (x+2)(2x-1) \leq 0$$

$$\Rightarrow x \in (-\infty, 0] \cup \left[\frac{1}{4}, \infty \right)$$

$$\text{and } x \in \left[-2, \frac{1}{2} \right]$$

$$\Rightarrow x \in (-2, 0] \cup \left[\frac{1}{4}, \frac{1}{2} \right]$$

Domain of f in Eq. (i) $\cap$ Eq. (ii)

$$\therefore x \in \{0\} \cup \left[\frac{1}{4}, \frac{1}{2} \right]$$

Question 124

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $f(m+n) = f(m) + f(n)$ for every $m, n \in \mathbb{N}$. If $f(6) = 18$, then $f(2) \cdot f(3)$ is equal to
[2021, 31 Aug. Shift-11]

Options:

A. 6

B. 54

C. 18

D. 36

Answer: B

Solution:

$$f(m+n) = f(m) + f(n), m, n \in \mathbb{N}$$

$$\therefore f(3+3) = f(3) + f(3)$$

$$\Rightarrow f(6) = 2f(3) = 18 \quad [\because f(6) = 18]$$

$$\text{Also } f(3) = f(2+1) = f(2) + f(1)$$

$$= f(1+1) + f(1)$$

$$f(3) = f(1) + f(1) + f(1)$$

$$\Rightarrow 9 = 3f(1) \Rightarrow f(1) = 3$$

$$\therefore f(2) = f(1+1) = f(1) + f(1) = 3 + 3 = 6$$

$$\text{Hence, } f(2) \cdot f(3) = 6 \cdot 9 = 54$$

Question 125

The domain of the function $\operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is

[2021, 26 Aug. Shift-II]

Options:

A. $\left(-1, -\frac{1}{2}\right] \cup (0, \infty)$

B. $\left[-\frac{1}{2}, 0\right) \cup [1, \infty)$

C. $\left(-\frac{1}{2}, \infty\right) - \{0\}$

D. $\left[-\frac{1}{2}, \infty\right) - \{0\}$

Answer: D

Solution:

$$f(x) = \operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right) \quad \left| \frac{1+x}{x} \right| \geq 1$$

Clearly, $x \neq 0$

$$r |1+x|^2 \geq |x|^2$$

$$1+x^2+2x \geq x^2$$

$$2x+1 \geq 0$$

$$x \geq -\frac{1}{2}$$

So,

$$x \in \left[-\frac{1}{2}, \infty\right) - \{0\}$$

Question 126

Which of the following is not correct for relation R on the set of real numbers ?

[2021, 31 Aug. Shift-1]

Options:

A. $(x, y) \in R \Leftrightarrow 0 < |x| - |y| \leq 1$ is neither transitive nor symmetric.

B. $(x, y) \in R \Leftrightarrow 0 < |x - y| \leq 1$ is symmetric and transitive.

C. $(x, y) \in R \Leftrightarrow x| - |y| \leq 1$ is reflexive but not symmetric.

D. $(x, y) \in R \Leftrightarrow x - y| \leq 1$ is reflexive and symmetric.

Answer: B

Solution:

According to the question, let's consider option (b) (2, 3) and (3, 4) satisfy $0 < |x - y| \leq 1$ but (2, 4) does not satisfy it.

Question 127

Let N be the set of natural numbers and a relation R on N be defined by $R = \{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$. Then the relation R is

[2021, 27 July Shift-11]

Options:

A. symmetric but neither reflexive nor transitive.

B. reflexive but neither symmetric nor transitive.

C. reflexive and symmetric, but not transitive.

D. an equivalence relation.

Answer: B

Solution:

Given, relation R on N is defined by

$$R = \{(x, y) \in N \times N : x^3 - 3x^2 - xy^2 + 3y^3 = 0\}$$

$$x^3 - 3x^2y - xy^2 + 3y^3 = 0$$

$$\Rightarrow x^3 - xy^2 - 3x^2y + 3y^3 = 0$$

$$\Rightarrow x(x^2 - y^2) - 3y(x^2 - y^2) = 0$$

$$\Rightarrow (x - 3y)(x^2 - y^2) = 0$$

$$\Rightarrow (x - 3y)(x - y)(x + y) = 0$$

$$\text{Now, } x - x = 0$$

$$\Rightarrow x = x, \forall (x, x) \in N \times N$$

So, R is a reflexive relation.

But not symmetric and transitive relation because,

$(3, 1)$ satisfies but $(1, 3)$ does not. Also, $(3, 1)$ and

$(1, -1)$ satisfies but $(3, -1)$ does not.

Hence, relation R is reflexive but neither symmetric nor transitive.

Question 128

Define a relation R over a class of $n \times n$ real matrices A and B as

" ARB , if there exists a non-singular matrix P such that $PAP^{-1} = B$ ".

Then which of the following is true ?

[2021, 18 March Shift-II]

Options:

A. R is symmetric, transitive but not reflexive.

B. R is reflexive, symmetric but not transitive.

C. R is an equivalence relation.

D. R is reflexive, transitive but not symmetric.

Answer: C

Solution:

For reflexive relation, $\forall (A, A) \in R$ for matrix P .

$$\Rightarrow A = PAP^{-1} \text{ is true for } P = I$$

So, R is reflexive relation.

For symmetric relation,

Let $(A, B) \in R$ for matrix P .

$$\Rightarrow A = PAP^{-1} \text{ After pre-multiply by } P^{-1} \text{ and post-multiply by } P_1$$

we get

$$P^{-1}AP = B$$

So, $(B, A) \in R$ for matrix P^{-1} .

So, R is a symmetric relation.

For transitive relation,

Let ARB and BRC

So, $A = PBP^{-1}$ and $B = PCP^{-1}$

Now, $A = P(PCP^{-1})P^{-1}$

$$\Rightarrow A = (P)^2 C (P^{-1})^2 \Rightarrow A = (P)^2 \cdot C \cdot (P^2)^{-1}$$

$\therefore (A, C) \in R$ for matrix P^2 .

$\therefore R$ is transitive relation.

Hence, R is an equivalence relation.

Question 129

Let $A = \{2, 3, 4, 5, \dots, 30\}$ and $\sim$ be an equivalence relation on $A \times A$, defined by $(a, b) \sim (c, d)$, if and only if $ad = bc$. Then, the number of ordered pairs, which satisfy this equivalence relation with ordered pair $(4, 3)$ is equal to
[2021, 16 March Shift-II]

Options:

A. 5

B. 6

C. 8

D. 7

Answer: D

Solution:

$$A = \{2, 3, 4, 5, \dots, 30\}$$

$$a = bc$$

$$\therefore (a, b)R(4, 3)$$

$$\Rightarrow 3a = 4b$$

$$\Rightarrow a = \left(\frac{4}{3}\right)b$$

b must be a multiple of 3, b can be

$$(3, 6, 9, \dots, 30).$$

Also, a must be less than or equal to 30.

$$(a, b) = (4, 3), (8, 6), (12, 9), (16, 12), (20, 15)$$

$$(24, 18), (28, 21)$$

$\Rightarrow 7$ ordered pairs

Question130

Let $R = \{(P, O) | P \text{ and } Q \text{ are at the same distance from the origin}\}$ be a relation, then the equivalence class of $(1, -1)$ is the set
[2021, 26 Feb. Shift-1]

Options:

A. $S = \{(x, y) | x^2 + y^2 = 4\}$

B. $S = \{(x, y) | x^2 + y^2 = 1\}$

C. $S = \{(x, y) | x^2 + y^2 = \sqrt{2}\}$

D. $S = \{(x, y) | x^2 + y^2 = 2\}$

Answer: D

Solution:

Let $P(a, b)$ and $Q(c, d)$ are any two points.

Given, $OP = OQ$

i.e. $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$

Squaring on both sides,

$$R = \{(a, b), (c, d) : a^2 + b^2 = c^2 + d^2\}$$

$R(x, y), S(1, -1), OR = OS$ (equivalence class)

This gives $OR = \sqrt{x^2 + y^2}$ and $OS = \sqrt{2}$

$$1 \Rightarrow \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\Rightarrow x^2 + y^2 = 2 \quad (\text{Squaring on both sides})$$

$$\therefore S = \{(x, y) : x^2 + y^2 = 2\}$$

Question131

Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Then the number of possible functions $f : S \rightarrow S$ such that $f(m \cdot n) = f(m) \cdot f(n)$ for every $m, n \in S$ and $m \cdot n \in S$ is equal to
[2021, 27 July Shift-I]

Answer: 490

Solution:

$$S = \{1, 2, 3, 4, 5, 6, 7\}$$

$$f: S \rightarrow S$$

$$f(m \cdot n) = f(m)f(n)$$

$$m, n \in S \Rightarrow m, n \in S$$

$$\text{If } mn \in S \Rightarrow mn \leq 7$$

$$\text{So, } (1 \cdot 1, 1 \cdot 2, \dots, 1 \cdot 7) \leq 7$$

$$(2 \cdot 2, 2 \cdot 3) \leq 7$$

$$\text{When } m = 1, f(n) = f(1) \cdot f(n) \Rightarrow f(1) = 1$$

$$\text{When } m = n = 2,$$

$$f(4) = f(2)f(2) = \begin{cases} f(2) = 1 \Rightarrow f(4) = 1 & \text{or} \\ f(2) = 2 \Rightarrow f(4) = 4. \end{cases}$$

$$\text{When, } m = 2, n = 3$$

$$f(6) = f(2)f(3) \begin{cases} \text{When, } f(2) = 1 \\ f(3) = 1 \text{ to } 7 \\ \text{When, } f(2) = 2 \\ f(3) = 1 \text{ or } 2 \text{ or } 3. \end{cases}$$

And $f(5), f(7)$ can take any value $(1-7)$ [$\because f(5) = f(1) \cdot f(5) \leq 7$ and $f(7) = f(1) \cdot f(7) \leq 7$] The possible combination is

$$\text{If } f(1) = 1 \quad f(1) = 1$$

$$f(2) = 1 \quad f(2) = 2$$

$$f(3) = (1-7) \quad f(3) = (1-3)$$

$$f(4) = 1 \quad f(4) = 4$$

$$f(5) = (1-7) \quad f(5) = (1-7)$$

$$f(6) = f(3) \quad f(6) = f(3)$$

$$f(7) = (1-7) \quad f(7) = (1-7)$$

$$\text{So, total} = (1 \times 1 \times 7 \times 1 \times 7 \times 1 \times 7)$$

$$+ (1 \times 1 \times 3 \times 1 \times 7 \times 1 \times 7)$$

$$= 490$$

Question 132

If $[x]$ be the greatest integer less than or equal to x , then $\sum_{n=8}^{100} \left[\frac{(-1)^n n}{2} \right]$ is equal to
[25 July 2021, Shift-III]

Options:

A. 0

B. 4

C. -2

D. 2

Answer: D

Solution:

We have,

$$\sum_{n=8}^{100} \left[\frac{(-1)^n n}{2} \right] (\because [x] \text{ is the greatest integer function})$$

Substitute the values of n

$$\begin{aligned} &= [4] + [-4.5] + [5] + [-5.5] \\ &\quad + \dots + [-49.5] + [50] \\ &= 4 - 5 + 5 - 6 + \dots - 50 + 50 \\ &= 4 \end{aligned}$$

Question 133

If the domain of the function $f(x) = \frac{\cos^{-1} \sqrt{x^2 - x + 1}}{\sqrt{\sin^{-1} \left(\frac{2x-1}{2} \right)}}$ is the interval (α, β) ,

then $\alpha + \beta$ is equal to
[2021, 22 July Shift-II]

Options:

A. $\frac{3}{2}$

B. 2

C. $\frac{1}{2}$

D. 1

Answer: A

Solution:

$$f(x) = \frac{\cos^{-1} \sqrt{x^2 - x + 1}}{\sqrt{\sin^{-1} \left(\frac{2x-1}{2} \right)}}$$

$$\Rightarrow x \in \mathbb{R}, \quad x(x-1) \leq 0$$

$$x^2 - x + 1 \geq 0 \text{ and } x^2 - x + 1 \leq 1$$

$$0 \leq x \leq 1 \quad \dots\dots (i) \Rightarrow 0 < \sin^{-1} \left(\frac{2x-1}{2} \right) < \frac{\pi}{2}$$

$$\Rightarrow 0 < \frac{2x-1}{2} < 1$$

$$\Rightarrow \frac{1}{2} < x < \frac{3}{2} \quad \dots\dots (ii)$$

$$(A) \cap (B) = x \in \left(\frac{1}{2}, 1 \right]$$

$$\therefore \alpha + \beta = \frac{3}{2}$$

Question 134

Let $[x]$ denote the greatest integer $\leq x$, where $x \in \mathbb{R}$. If the domain of the real valued function

$$f(x) = \sqrt{\frac{[x]-2}{[x]-3}}$$
 is $(-\infty, a) \cup [b, c)$

$U[u, \infty)$, $a < b < c$, then the value of $a + b + c$ is

[2021, 20 July Shift-I]

Options:

A. 8

B. 1

C. -2

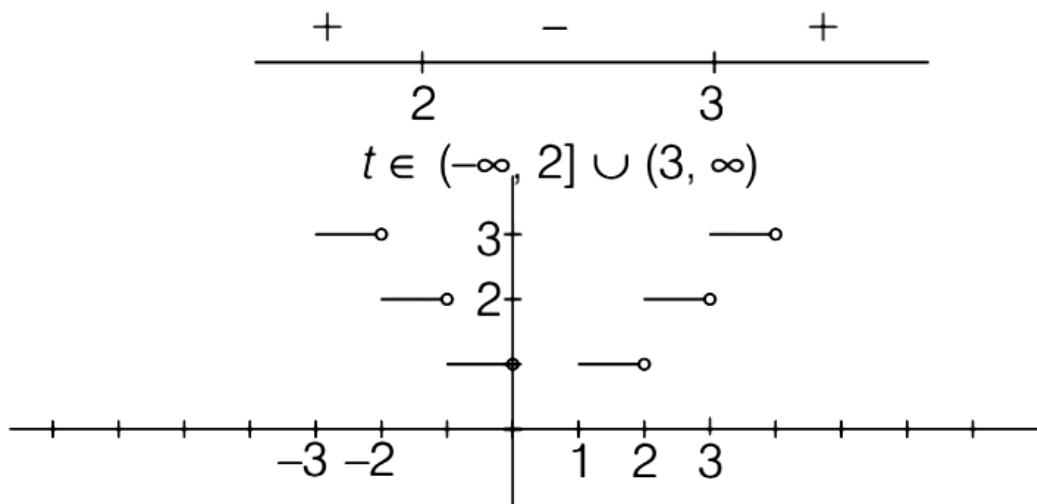
D. -3

Answer: C

Solution:

$$f(x) = \sqrt{\frac{[x]-2}{[x]-3}} \geq 0$$

$$\text{Let } [x] = t$$



$$1 \mid [x] \mid = 3 \Rightarrow x \in [-3, -2) \cup [3, 4)$$

Domain of $x = [-\infty, -3) \cup [-2, 3) \cup [4, \infty)$

$$a = -3$$

$$b = -2$$

$$c = 3$$

$$\therefore a + b + c = -3 + (-2) + 3 = -2$$

Question 135

The real valued function $f(x) = \frac{\operatorname{cosec}^{-1} x}{\sqrt{x - [x]}}$, where $[x]$ denotes the greatest integer less than or equal to x , is defined for all x belonging to [2021, 18 March Shift-I]

Options:

- A. all reals except integers
- B. all non-integers except the interval $[-1, 1]$
- C. all integers except 0, -1, 1
- D. all reals except the interval $[-1, 1]$

Answer: B

Solution:

Given, $f(x) = \frac{\operatorname{cosec}^{-1} x}{\sqrt{x - [x]}}$

$\Rightarrow f(x) = \frac{\operatorname{cosec}^{-1} x}{\sqrt{\{x\}}}$

For $f(x)$ to be defined,

$$\begin{cases} |x| \geq 1 \\ \{x\} > 0. \end{cases} \Rightarrow \begin{cases} x \leq -1 \text{ or } x \geq 1 \\ x \neq 1 \text{ integers} \end{cases}$$

i.e. $x \in (-\infty, -1] \cup [1, \infty) - \{\text{integers}\}$

i.e. all non-integers except the interval $[-1, 1]$

(here, -1 and 1 are included in except case, because of -1 and 1 are integers).

Question 136

If the functions are defined as $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$, then what is the common domain of the following functions?

$f + g, f - g, f / g, g / f, g - f$, where

$(f \pm g)(x) = f(x) \pm g(x), (f / g)(x) = \frac{f(x)}{g(x)}$

[2021, 18 March, Shift-1]

Options:

A. $0 \leq x \leq 1$

B. $0 \leq x < 1$

C. $0 < x < 1$

D. $0 < x \leq 1$

Answer: C

Solution:

Given, $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$

$\therefore$ Domain of $f(x) = D_1$ is $x \geq 0$

i.e. $D_1 : x \in (0, \infty)$

and domain of $g(x) = D_2$ is $1-x \geq 0 \Rightarrow x \leq 1$

i.e. $D_2 : x \in (-\infty, 1]$

As, we know that, the domain of $f + g, f - g, g - f$ will be $D_1 \cap D_2$ as well as the domain for $\frac{f}{g}$ is $D_1 \cap D_2$ except all

those value(s) of x , such that $g(x) = 0$.

Similarly, for $\frac{g}{f}$ is $D_1 \cap D_2$ but $f(x) \neq 0$.

Hence, common domain for $(f + g)$, $(f - g)$, $\left(\frac{f}{g}\right)$, $\left(\frac{g}{f}\right)$ and $(g - f)$ will be $0 < x < 1$.

Question 137

A function $f(x)$ is given by $f(x) = \frac{5^x}{5^x + 5}$, then the sum of the series

$$f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{3}{20}\right) + \dots + f\left(\frac{39}{20}\right)$$

is equal to

[2021, 25 Feb. Shift-II]

Options:

A. $\frac{29}{2}$

B. $\frac{49}{2}$

C. $\frac{39}{2}$

D. $\frac{19}{2}$

Answer: C

Solution:

Given, $f(x) = \frac{5^x}{5^x + 5}$, then,

$$\begin{aligned} f(2-x) &= \frac{5^{2-x}}{5^{2-x} + 5} \\ &= \frac{5}{5^x + 5} \end{aligned}$$

This gives, $f(x) + f(2-x) = \frac{5^x + 5}{5^x + 5} = 1 \Rightarrow f\left(\frac{1}{20}\right) + f\left(2 - \frac{1}{20}\right) = f\left(\frac{1}{20}\right) + f\left(\frac{39}{20}\right) = 1$

Similarly,

$$f\left(\frac{2}{20}\right) + f\left(\frac{38}{20}\right) = 1 \text{ and so on,}$$

$$\begin{aligned}
 &\therefore f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + \dots + f\left(\frac{38}{20}\right) + f\left(\frac{39}{20}\right) \\
 &= 1 + 1 + \dots + 1 + f\left(\frac{20}{20}\right) \\
 &= 19 + f(1) = 19 + \frac{1}{2} = \frac{39}{2}
 \end{aligned}$$

Question 138

If $a + \alpha = 1$, $b + \beta = 2$ and $af(x) + \alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}$, $x \neq 0$, then the value of expression $\frac{f(x) + f\left(\frac{1}{x}\right)}{x + \frac{1}{x}}$ is

[2021, 24 Feb. Shift-II]

Answer: 2

Solution:

$$\begin{aligned}
 \text{Given, } a + \alpha &= 1 \\
 b + \beta &= 2
 \end{aligned}$$

$$\therefore a \cdot f(x) + \alpha \cdot f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x} \quad \dots\dots\dots (i)$$

Replace x by $\frac{1}{x}$,

$$af\left(\frac{1}{x}\right) + af(x) = \frac{b}{x} + \beta x$$

Adding Eqs. (i) and (ii), we get

$$(a + \alpha) \left[f(x) + f\left(\frac{1}{x}\right) \right] = \left(x + \frac{1}{x}\right)(b + \beta) \quad \dots\dots\dots (ii)$$

$$\Rightarrow \frac{f(x) + f\left(\frac{1}{x}\right)}{x + \frac{1}{x}} = \frac{b + \beta}{a + \alpha} = \frac{2}{1} = 2$$

Question 139

Let $f(x) = \sin^{-1} x$ and

$$g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$$

If $g(2) = \lim_{x \rightarrow 2} g(x)$, then the domain of the function $f \circ g$ is

[2021, 26 Feb. Shift-II]

Options:

A. $(-\infty, -2] \cup \left[-\frac{3}{2}, \infty\right)$

B. $(-\infty, -2] \cup [-1, \infty)$

C. $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$

D. $(-\infty, -1] \cup [2, \infty)$

Answer: C

Solution:

Given, $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$, $f(x) = \sin^{-1} x$

$$f(g(x)) = \sin^{-1}(g(x))$$

$$f \circ g(x) = \sin^{-1} \left(\frac{x^2 - x - 2}{2x^2 - x - 6} \right)$$

For the domain of $f \circ g(x)$,

$$|g(x)| \leq 1$$

[$\because$ Domain of $f(x)$ is $[-1, 1]$]

$$\Rightarrow \left| \frac{x^2 - x - 2}{2x^2 - x - 6} \right| \leq 1$$

$$\Rightarrow \left| \frac{(x+1)(x-2)}{(2x+3)(x-2)} \right| \leq 1$$

$$\Rightarrow \left| \frac{x+1}{2x+3} \right| \leq 1$$

$$\Rightarrow -1 \leq \frac{x+1}{2x+3} \leq 1$$

$$\Rightarrow \left(\frac{x+1}{2x+3} \right)^2 \leq 1$$

$$\Rightarrow (x+1)^2 \leq (2x+3)^2$$

$$\Rightarrow 3x^2 + 10x + 8 \geq 0$$

$$\Rightarrow (3x+y)(x+2) \geq 0$$

This implies,

$$x \in (-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$$

Question 140

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as

$$g(3n+1) = 3n+2$$

$$g(3n+2) = 3n+3,$$

$$g(3n+3) = 3n+1, \text{ for all } n \geq 0.$$

Then which of the following statements is true?

[2021, 25 July Shift-1]

Options:

A. There exists an onto function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f \circ g = f$

B. There exists a one-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f \circ g = f$

C. $g \circ g \circ g = g$

D. There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $g \circ f = f$

Answer: A

Solution:

$$g(3n+1) = 3n+2$$

$$g(3n+2) = 3n+3$$

$$g(3n+3) = 3n+1, \text{ for all } n \geq 0$$

$$g : \mathbb{N} \rightarrow \mathbb{N}$$

$$g(1) = 2, g(4) = 5, g(7) = 8$$

$$g(2) = 3, g(5) = 6, g(8) = 9$$

$$g(3) = 1, g(6) = 4, g(9) = 7$$

$$\Rightarrow f[g(1)] = f(1)$$

$$\Rightarrow f(2) = f(1)$$

Clearly, it is not a one - one function.

$$\text{Now, } f[g(2)] = f(2)$$

$$f(3) = f(2)$$

$$\text{And, } f[g(3)] = f(3)$$

$$f(1) = f(3)$$

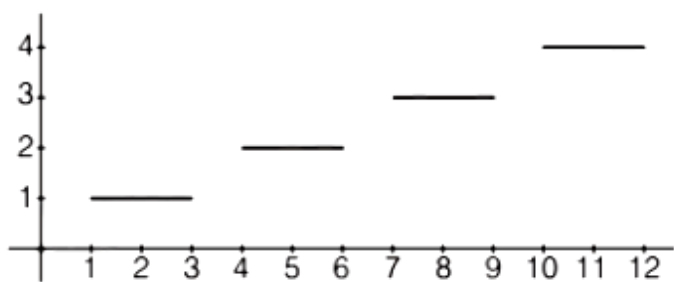
$$\text{Similarly, } f[g(4)] = f(4)$$

$$f(5) = f(4)$$

And, so on

$$f(1) = f(2) = f(3)$$

$$f(4) = f(5) = f(6)$$



Now, there can be a possibility such that

So, $f(x)$ can be onto function.

$$\text{When } f(1) = f(2) = f(3) = 1$$

$$f(4) = f(5) = f(6) = 2$$

and so on.

Question 141

Consider function $f : A \rightarrow B$ and $g : B \rightarrow C$ ($A, B, C \subseteq \text{eqR}$) such that $(g \circ f)^{-1}$ exists, then
[2021, 25 July Shift-II]

Options:

- A. f and g both are one-one
- B. f and g both are onto
- C. f is one-one and g is onto
- D. f is onto and g is one-one

Answer: C

Solution:

Given functions, $f : A \rightarrow B$ and $g : B \rightarrow C$ ($A, B, C \subseteq \text{eqR}$)

$\therefore (g \circ f)^{-1}$ exists $\Rightarrow g \circ f$ is a bijective function.

$\Rightarrow 'f'$ must be 'one-one' and ' g ' must be 'onto' function.

Question 142

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then, the number of bijective functions $f : A \rightarrow A$ such that $f(1) + f(2) = 3 - f(3)$ is equal to

[2021, 22 July Shift-III]

Answer: 720

Solution:

$$f(1) + f(2) = 3 - f(3)$$

$$A = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$f : A \rightarrow A$$

$$\text{So, } f(1) + f(2) + f(3) = 3$$

$$0 + 1 + 2 = 3 \text{ is the only possibility.}$$

$$\text{So, } f(0) \text{ can be either } 0 \text{ or } 1 \text{ or } 2.$$

$$\text{Similarly, } f(1) \text{ and } f(2) \text{ can be } 0, 1 \text{ and } 2.$$

$$\text{and } \{3, 4, 5, 6, 7\} \rightarrow \{3, 4, 5, 6, 7\}$$

They have $5!$ choices.

And $\{0, 1, 2\}$

They have $3!$ choices.

Number of bijective functions

$$= 3! \times 5! = 720$$

Question 143

Let $f : \mathbb{R} - \left\{ \frac{\alpha}{6} \right\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{5x + 3}{6x - \alpha}$$

Then, the value of α for which $(f \circ f)(x) = x$, for all $x \in \mathbb{R} - \left\{ \frac{\alpha}{6} \right\}$ is

[2021, 20 July Shift-II]

Options:

A. No such α exists

B. 5

C. 8

D. 6

Answer: B

Solution:

$$f(x) = \frac{5x+3}{6x-\alpha}$$

$$\text{Now, } f \circ f(x) = f\left(\frac{5x+3}{6x-\alpha}\right)$$

$$= \frac{5\left(\frac{5x+3}{6x-\alpha}\right) + 3}{6\left(\frac{5x+3}{6x-\alpha}\right) - \alpha}$$

$$= \frac{5(5x+3) + 3(6x-\alpha)}{6(5x+3) - \alpha(6x-2)}$$

$$= \frac{5(5x+3) + 3(6x-\alpha)}{6(5x+3) - \alpha(6x-2)}$$

$$\text{Given, } f \circ f(x) = x$$

$$\Rightarrow \frac{5(5x+3) + 3(6x-\alpha)}{6(5x+3) - \alpha(6x-2)} = x$$

$$\Rightarrow 25x + 15 + 18x - 3\alpha$$

$$= 30x^2 + 18x - 6\alpha x + \alpha^2 x$$

$$\Rightarrow x^2(30 - 6\alpha) - x(\alpha^2 - 25) + 3\alpha - 15 = 0$$

Comparing coefficients,

$$30 - 6\alpha = 0$$

$$\Rightarrow 6\alpha = 30$$

$$\Rightarrow \alpha = 5$$

Question 144

Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ be defined by $f(x) = \frac{x-2}{x-3}$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be given as $g(x) = 2x - 3$. Then, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$ is equal to
[2021, 18 March Shift-II]

Options:

A. 7

B. 2

C. 5

D. 3

Answer: C

Solution:

$$\text{Given, } f(x) = \frac{x-2}{x-3}$$

$$g(x) = 2x - 3$$

$$\text{Let } y = f(x) = \frac{x-2}{x-3}$$

$$\Rightarrow xy - 3y = x - 2 \Rightarrow xy - x = 3y - 2$$

$$\Rightarrow x(y-1) = 3y-2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

$$\Rightarrow f^{-1}(y) = \frac{3y-2}{y-1}$$

$$\Rightarrow f^{-1}(x) = \frac{3x-2}{x-1}$$

$$\text{Similarly, } g^{-1}(x) = \frac{x+3}{2}$$

$$\text{Given, } f^{-1}(x) + g^{-1}(x) = \frac{13}{2}$$

$$\Rightarrow \frac{3x-2}{x-1} + \frac{x+3}{2} = \frac{13}{2}$$

$$\Rightarrow x^2 + 8x - 7 = 13(x-1)$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x-2)(x-3) = 0$$

$$\Rightarrow x = 2, 3$$

$$\therefore \text{Sum} = 2 + 3 = 5$$

Question 145

**The inverse of $y = 5^{\log x}$ is
[2021, 17 March Shift-I]**

Options:

A. $x = 5^{\log y}$

B. $x = y^{\log 5}$

C. $x = y^{\frac{1}{\log 5}}$

D. $x = 5^{\frac{1}{\log y}}$

Answer: C

Solution:

$$y = 5^{\log x}$$

Taking log on both sides,

$$\Rightarrow \frac{\log y}{\log 5} = \log x \cdot \log 5$$

$$\Rightarrow \frac{1}{\log 5} = \frac{\log x}{\log y}$$

$$\frac{1}{\log 5} = \log_y x$$

$$x = y^{\frac{1}{\log 5}}$$

Question 146

Let $A = \{1, 2, 3, \dots, 10\}$ and $f : A \rightarrow A$ be defined as defined as

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ x & \text{if } x \text{ is even} \end{cases}$$

Then, the number of possible functions $g : A \rightarrow A$, such that $g \circ f = f$ is
[2021, 26 Feb. Shift-II]

Options:

A. 10^5

B. ${}^{10}C_5$

C. 5^5

D. $5!$

Answer: A

Solution:

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ x & \text{if } x \text{ is even} \end{cases}$$

Given, $g : A \rightarrow A$ such that,

$$g(f(x)) = f(x)$$

When x is even, then

$$g(x) = x$$

When x is odd, then

$$g(x+1) = x+1$$

This implies,

$$g(x) = x, x \text{ is even.}$$

$\Rightarrow$ If x is odd, then $g(x)$ can take any value in set A .

So, number of $g(x) = 10^5$

Question 147

Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$, such that $f(n+1) = f(n) + f(1) \forall n \in \mathbb{N}$ and g be any arbitrary function. Which of the following statements is not true?
[2021, 25 Feb. Shift-1]

Options:

A. if $f \circ g$ is one-one, then g is one-one.

B. if f is onto, then $f(n) = n, \forall n \in \mathbb{N}$.

C. f is one-one.

D. if g is onto, then $f \circ g$ is one-one.

Answer: D

Solution:

Given, $f(n+1) = f(n) + f(1), \forall n \in \mathbb{N}$

$$\Rightarrow f(n+1) - f(n) = f(1)$$

It is an AP with common difference $= f(1)$

Also, general term

$$11 = T_n = f(1) + (n-1)f(1) = nf(1)$$

$$\Rightarrow f(n) = nf(1)$$

Clearly, $f(n)$ is one-one.

For $f \circ g$ to be one-one, g must be one-one.

For f to be onto, $f(n)$ should take all the values of natural numbers.

As, $f(x)$ is increasing, $f(1) = 1$

$$\Rightarrow f(n) = n$$

If g is many-one, then $f \circ g$ is many one.

So, if g is onto, then $f \circ g$ is one-one.

Question 148

Let x denote the total number of one-one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one-one functions from the set A to the set $A \times B$. Then,
[2021, 25 Feb. Shift-II]

Options:

A. $2y = 91x$

B. $2y = 273x$

C. $y = 91x$

D. $y = 273x$

Answer: A

Solution:

$$x = \{f : A \rightarrow B, f \text{ is one - one} \}$$

$$y = \{g : A \rightarrow A \times B, g \text{ is one one} \}$$

$$\text{Number of elements in } A = 3 \text{ i.e. } |A| = 3$$

$$\text{Similarly, } |B| = 5$$

$$\text{Then, } |A \times B| = |A| \times |B| = 3 \times 5 = 15$$

Now, number of one-one function from A to B will be

$${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

$$\therefore x = 60$$

Now, number of one-one function from A

$$\text{to } A \times B \text{ will be } = {}^{15}P_3 = \frac{15!}{(15-3)!} = \frac{15!}{12!}$$

$$= 15 \times 14 \times 13 = 2730$$

$$\therefore y = 2730 \text{ i.e. } y = 2730$$

$$\text{Thus, } 2 \times (2730) = 91 \times (60)$$

$$\Rightarrow 2y = 91x$$

Question 149

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x - 1$ and $g : \mathbb{R} - \{1\} \rightarrow \mathbb{R}$ be

defined as $g(x) = \frac{x - \frac{1}{2}}{x - 1}$. Then the composition function $f(g(x))$ is :

24 Feb 2021 Shift 1

Options:

- A. onto but not one-one
- B. both one-one and onto
- C. one-one but not onto
- D. neither one-one nor onto

Answer: C

Solution:

$$f(g(x)) = 2g(x) - 1 = 2\left(\frac{2x-1}{2(x-1)}\right) - 1 = \frac{x}{x-1} = 1 + \frac{1}{x-1}$$

Range of $f(g(x)) = \mathbb{R} - \{1\}$

Range of $f(g(x))$ is not onto

& $f(g(x))$ is one-one

So, $f(g(x))$ is one-one but not onto.

Question 150

Let R_1 and R_2 be two relations defined as follows :

$$R_1 = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \in \mathbb{Q}\} \text{ and } R_2 = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \notin \mathbb{Q}\},$$

where $\mathbb{Q}$ is the set of all rational numbers. Then :

[Sep. 03, 2020 (II)]

Options:

- A. Neither R_1 nor R_2 is transitive.
- B. R_2 is transitive but R_1 is not transitive.
- C. R_1 is transitive but R_2 is not transitive.
- D. R_1 and R_2 are both transitive.

Answer: A

Solution:

(a) For R_1 let $a = 1 + \sqrt{2}$, $b = 1 - \sqrt{2}$, $c = 8^{1/4}$
 $aR_1b \Rightarrow a^2 + b^2 = (1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 = 6 \in \mathbb{Q}$
 $bR_1c \Rightarrow b^2 + c^2 = (1 - \sqrt{2})^2 + (8^{1/4})^2 = 3 \in \mathbb{Q}$
 $aR_1c \Rightarrow a^2 + c^2 = (1 + \sqrt{2})^2 + (8^{1/4})^2 = 3 + 4\sqrt{2} \notin \mathbb{Q}$
 $\therefore R_1$ is not transitive.

For R_2 let $a = 1 + \sqrt{2}$, $b = \sqrt{2}$, $c = 1 - \sqrt{2}$
 $aR_2b \Rightarrow a^2 + b^2 = (1 + \sqrt{2})^2 + (\sqrt{2})^2 = 5 + 2\sqrt{2} \notin \mathbb{Q}$
 $bR_2c \Rightarrow b^2 + c^2 = (\sqrt{2})^2 + (1 - \sqrt{2})^2 = 5 - 2\sqrt{2} \notin \mathbb{Q}$
 $aR_2c \Rightarrow a^2 + c^2 = (1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 = 6 \in \mathbb{Q}$
 $\therefore R_2$ is not transitive.

Question 151

The domain of the function $f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$ is $(-\infty, -a] \cup [a, \infty)$.

Then a is equal to :
[Sep. 02, 2020 (I)]

Options:

- A. $\frac{\sqrt{17}}{2}$
- B. $\frac{\sqrt{17}-1}{2}$
- C. $\frac{1+\sqrt{17}}{2}$
- D. $\frac{\sqrt{17}}{2} + 1$

Answer: C

Solution:

$$\begin{aligned} \because f(x) &= \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right) \\ \therefore -1 &\leq \frac{|x|+5}{x^2+1} \leq 1 \\ \Rightarrow |x|+5 &\leq x^2+1 \\ [\because x^2+1 &\neq 0] \\ \Rightarrow x^2-|x|-4 &\geq 0 \end{aligned}$$

$$\Rightarrow \left(|x| - \frac{1 - \sqrt{17}}{2} \right) \left(|x| - \frac{1 + \sqrt{17}}{2} \right) \geq 0$$

$$\Rightarrow x \in \left(-\infty, -\frac{1 + \sqrt{17}}{2} \right) \cup \left[\frac{1 + \sqrt{17}}{2}, \infty \right)$$

$$\therefore a = \frac{1 + \sqrt{17}}{2}$$

Question 152

If $R = \{(x, y) : x, y \in \mathbb{Z}, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers $\mathbb{Z}$, then the domain of R^{-1} is :
[Sep. 02, 2020 (I)]

Options:

- A. $\{-2, -1, 1, 2\}$
- B. $\{0, 1\}$
- C. $\{-2, -1, 0, 1, 2\}$
- D. $\{-1, 0, 1\}$

Answer: D

Solution:

$$\text{Since, } R = \{(x, y) : x, y \in \mathbb{Z}, x^2 + 3y^2 \leq 8\}$$

$$\therefore R = \{(1, 1), (2, 1), (1, -1), (0, 1), (1, 0)\}$$

$$\Rightarrow D_{R^{-1}} = \{-1, 0, 1\}$$

Question 153

Let $[t]$ denote the greatest integer $\leq t$. Then the equation in x , $[x]^2 + 2[x + 2] - 7 = 0$ has :
[Sep. 04, 2020 (I)]

Options:

- A. exactly two solutions
- B. exactly four integral solutions
- C. no integral solution
- D. infinitely many solutions

Answer: D

Solution:

The given equation $[x]^2 + 2[x] + 4 - 7 = 0$
 $\Rightarrow [x]^2 + 2[x] - 3 = 0$
 $\Rightarrow [x]^2 + 3[x] - [x] - 3 = 0$
 $\Rightarrow ([x] + 3)([x] - 1) = 0 \Rightarrow [x] = 1 \text{ or } -3 \Rightarrow x \in [-3, -2) \cup [1, 2)$
 $\therefore$ The equation has infinitely many solutions.

Question 154

**Let $f(x)$ be a quadratic polynomial such that $f(-1) + f(2) = 0$. If one of the roots of $f(x) = 0$ is 3, then its other root lies in:
[Sep. 02, 2020 (II)]**

Options:

- A. $(-1, 0)$
- B. $(1, 3)$
- C. $(-3, -1)$
- D. $(0, 1)$

Answer: A

Solution:

$$\text{Let } f(x) = ax^2 + bx + c$$

$$\text{Given : } f(-1) + f(2) = 0$$

$$a - b + c + 4a + 2b + c = 0$$

$$\Rightarrow 5a + b + 2c = 0 \dots\dots(i)$$

$$\text{and } f(3) = 0 \Rightarrow 9a + 3b + c = 0 \dots\dots(ii)$$

From equations (i) and (ii),

$$\frac{a}{1-6} = \frac{b}{18-5} = \frac{c}{15-9} \Rightarrow \frac{a}{-5} = \frac{b}{13} = \frac{c}{6}$$

$$\text{Product of roots, } \alpha\beta = \frac{c}{a} = \frac{-6}{5} \text{ and } \alpha = 3$$

$$\Rightarrow \beta = \frac{-2}{5} \in (-1, 0)$$

Question 155

Let $f(1, 3) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$ where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is:
[Jan. 8, 2020 (II)]

Options:

A. $\left(\frac{2}{5}, \frac{3}{5}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right)$

B. $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right)$

C. $\left(\frac{2}{5}, \frac{4}{5}\right)$

D. $\left(\frac{3}{5}, \frac{4}{5}\right)$

Answer: B

Solution:

$$f(x) = \begin{cases} \frac{x}{x^2 + 1} & x \in (1, 2) \\ \frac{2x}{x^2 + 1} & x \in [2, 3) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1 - x^2}{1 + x^2} & x \in (1, 2) \\ \frac{1 - 2x^2}{1 + x^2} & x \in [2, 3) \end{cases}$$

$\therefore f(x)$ is a decreasing function

$$\therefore y \in \left(\frac{2}{5}, \frac{1}{2} \right) \cup \left(\frac{6}{10}, \frac{4}{5} \right]$$

$$\Rightarrow y \in \left(\frac{2}{5}, \frac{1}{2} \right) \cup \left(\frac{3}{5}, \frac{4}{5} \right]$$

Question 156

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then the number of elements in the set $C = \{f : A \rightarrow B \mid 2 \in f(A) \text{ and } f \text{ is not one-one}\}$ is _____.
[NA Sep. 05, 2020 (II)]

Answer: 19

Solution:

The desired functions will contain either one element or two elements in its codomain of which '2' always belongs to $f(A)$.

$\therefore$ The set B can be $\{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}$

Total number of functions $= 1 + (2^3 - 2)3 = 19$

Question 157

The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$, $x \in (-1, 1)$, is _____.

[Jan. 8, 2020 (I)]

Options:

A. $\frac{1}{4} \log_e \left(\frac{1+x}{1-x} \right)$

B. $\frac{1}{4} (\log_8 e) \log_e \left(\frac{1-x}{1+x} \right)$

C. $\frac{1}{4} \log_e \left(\frac{1-x}{1+x} \right)$

D. $\frac{1}{4} (\log_8 e) \log_e \left(\frac{1+x}{1-x} \right)$

Answer: A

Solution:

$$y = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$$

$$\frac{1+y}{1-y} = \frac{8^{2x}}{8^{-2x}} \Rightarrow 8^{4x} = \frac{1+y}{1-y}$$

$$\Rightarrow 4x = \log_8 \left(\frac{1+y}{1-y} \right)$$

$$\Rightarrow x = \frac{1}{4} \log_8 \left(\frac{1+y}{1-y} \right)$$

$$\therefore f^{-1}(x) = \frac{1}{4} \log_8 \left(\frac{1+x}{1-x} \right)$$

Question 158

If $g(x) = x^2 + x - 1$ and $(g \circ f)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to:

[Jan. 7, 2020 (I)]

Options:

A. $\frac{3}{2}$

B. $-\frac{1}{2}$

C. $\frac{1}{2}$

D. $-\frac{3}{2}$

Answer: B

Solution:

$$(g \circ f)(x) = g(f(x)) = f^2(x) + f(x) - 1$$

$$g\left(f\left(\frac{5}{4}\right)\right) = 4\left(\frac{5}{4}\right)^2 - 10 \cdot \frac{5}{4} + 5 = -\frac{5}{4}$$

$$[\because g(f(x)) = 4x^2 - 10x + 5]$$

$$g\left(f\left(\frac{5}{4}\right)\right) = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$$

$$-\frac{5}{4} = f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) - 1$$

$$f^2\left(\frac{5}{4}\right) + f\left(\frac{5}{4}\right) + \frac{1}{4} = 0$$

$$\left(f\left(\frac{5}{4}\right) + \frac{1}{2}\right)^2 = 0$$

$$f\left(\frac{5}{4}\right) = -\frac{1}{2}$$

Question 159

For a suitably chosen real constant a , let a function, $f : \mathbb{R} - \{-a\} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{a-x}{a+x}$. Further suppose that for any real number

$x \neq -a$ and $f(x) \neq -a$, $(f \circ f)(x) = x$. Then $f\left(-\frac{1}{2}\right)$ is equal to:

[Sep. 06, 2020 (II)]

Options:

A. $\frac{1}{3}$

B. $-\frac{1}{3}$

C. -3

D. 3

Answer: D

Solution:

$$f(f(x)) = \frac{a - \left(\frac{a-x}{a+x}\right)}{a + \left(\frac{a-x}{a+x}\right)} = x$$

$$\Rightarrow \frac{a-ax}{1+x} = f(x) \Rightarrow \frac{a(1-x)}{1+x} = \frac{a-x}{a+x} \Rightarrow a = 1$$

$$\therefore f(x) = \frac{1-x}{1+x} \left(\Rightarrow f \right) - \frac{1}{2} = 3$$