



Limits and Derivatives

Question1

Let for a differentiable function $f: (0, \infty) \rightarrow \mathbb{R}$, $f(x) - f(y) \geq \log_e \left(\frac{x}{y} \right) + x - y$, $\forall x, y \in (0, \infty)$. Then $\sum_{n=1}^{20} f\left(\frac{1}{n^2}\right)$ is equal to _____

[27-Jan-2024 Shift 1]

Answer: 2890

Solution:

$$f(x) - f(y) \geq \ln x - \ln y + x - y$$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{\ln x - \ln y}{x - y} + 1$$

Let $x > y$

$$\lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y} \geq \frac{1}{x} + 1 \quad \dots (1)$$

Let $x < y$

$$\lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y} \leq \frac{1}{x} + 1 \quad \dots (2)$$

$$f'(x^-) = f'(x^+)$$

$$f'(x) = \frac{1}{x} + 1$$

$$f\left(\frac{1}{x^2}\right) = x^2 + 1$$

$$\sum_{x=1}^{20} (x^2 + 1) = \sum_{x=1}^{20} x^2 + 20$$

$$= \frac{20 \times 21 \times 41}{6} + 20$$

$$= 2890$$

Question2

Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, $x \in \mathbb{R}$. Then $f(10)$ is equal to _____

[27-Jan-2024 Shift 1]

Answer: 202

Solution:

$$f(x) = x^3 + x^2 \cdot f'(1) + x \cdot f''(2) + f'''(3)$$

$$f'(x) = 3x^2 + 2xf'(1) + f''(2)$$

$$f''(x) = 6x + 2f'(1)$$

$$f'''(x) = 6$$

$$f'(1) = -5, f''(2) = 2, f'''(3) = 6$$

$$f(x) = x^3 + x^2 \cdot (-5) + x \cdot (2) + 6$$

$$f'(x) = 3x^2 - 10x + 2$$

$$f'(10) = 300 - 100 + 2 = 202$$

Question3

Suppose

$$f(x) = \frac{(2^x + 2^{-x}) \tan x \sqrt{\tan^{-1}(x^2 - x + 1)}}{(7x^2 + 3x + 1)^3}$$

Then the value of $f'(0)$ is equal to

[29-Jan-2024 Shift 1]

Options:

A.

π

B.

0

C.

$\sqrt{\pi}$

D.

$\pi/2$

Answer: C

Solution:

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2^h + 2^{-h}) \tan h \sqrt{\tan^{-1}(h^2 - h + 1)} - 0}{(7h^2 + 3h + 1)^3 h} \\ &= \sqrt{\pi} \end{aligned}$$

Question4

Let $y = \log_e \left(\frac{1-x^2}{1+x^2} \right)$, $-1 < x < 1$. Then at $x = \frac{1}{2}$, the value of $225(y' - y'')$ is equal to

[29-Jan-2024 Shift 2]

Options:

A.

732

B.

746

C.

742

D.

736

Answer: D

Solution:

$$y = \log_e \left(\frac{1-x^2}{1+x^2} \right)$$

$$\frac{dy}{dx} = y' = \frac{-4x}{1-x^4}$$

Again,

$$\frac{d^2y}{dx^2} = y'' = \frac{-4(1+3x^4)}{(1-x^4)^2}$$

Again

$$y' - y'' = \frac{-4x}{1-x^4} + \frac{4(1+3x^4)}{(1-x^4)^2}$$

$$\text{at } x = \frac{1}{2},$$

$$y' - y'' = \frac{736}{225}$$

$$\text{Thus } 225(y' - y'') = 225 \times \frac{736}{225} = 736$$

Question5

Let $g(x)$ be a linear function and $f(x) = \begin{cases} g(x), & x \leq 0 \\ \left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}, & x > 0 \end{cases}$ is continuous at $x = 0$. If $f'(1) = f'(-1)$, then the value of $g(3)$ is

[31-Jan-2024 Shift 1]

Options:

A.

$$\frac{1}{3} \log_e \left(\frac{4}{9e^{1/3}} \right)$$

B.

$$\frac{1}{3} \log_e \left(\frac{4}{9} \right) + 1$$

C.

$$\log_e \left(\frac{4}{9} \right) - 1$$

D.

$$\log_e \left(\frac{4}{9e^{1/3}} \right)$$

Answer: D

Solution:

Let $g(x) = ax + b$

Now function $f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}} = b$$

$$\Rightarrow 0 = b$$

$$\therefore g(x) = ax$$

Now, for $x > 0$

$$f'(x) = \frac{1}{x} \cdot \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}-1} \cdot \frac{1}{(2+x)^2} + \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}} \cdot \ln \left(\frac{1+x}{2+x} \right) \cdot \left(-\frac{1}{x^2} \right)$$

$$\therefore f(1) = \frac{1}{9} - \frac{2}{3} \cdot \ln \left(\frac{2}{3} \right)$$

$$\text{And } f(-1) = g(-1) = -a$$

$$\therefore a = \frac{2}{3} \ln \left(\frac{2}{3} \right) - \frac{1}{9}$$

$$\therefore g(3) = 2 \ln \left(\frac{2}{3} \right) - \frac{1}{3}$$

$$= \ln \left(\frac{4}{9 \cdot e^{1/3}} \right)$$

Question6

If $a = \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sqrt{1 + x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$, then the value of ab^3 is :

[27-Jan-2024 Shift 1]

Options:

A.

36

B.

32

C.

25

D.

30

Answer: B

Solution:

$$\begin{aligned} a &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sqrt{1 + x^4}} - \sqrt{2}}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + x^4} - 1}{x^4 (\sqrt{1 + \sqrt{1 + x^4}} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{1}{x^4 (\sqrt{1 + \sqrt{1 + x^4}} + \sqrt{2}) (\sqrt{1 + x^4} + 1)} \end{aligned}$$

$$\text{Applying limit } a = \frac{1}{4\sqrt{2}}$$

$$\begin{aligned} b &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{2 - (1 + \cos x)} \\ b &= \lim_{x \rightarrow 0} (1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x}) \end{aligned}$$

$$\text{Applying limits } b = 2(\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$$

$$\text{Now, } ab^3 = \frac{1}{4\sqrt{2}} \times (4\sqrt{2})^3 = 32$$

Question7

If $\lim_{x \rightarrow 0} \frac{3 + a \sin x + \beta \cos x + \log_e(1 - x)}{3 \tan^2 x} = \frac{1}{3}$, then $2\alpha - \beta$ is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

2

B.

7

C.

5

D.

1

Answer: C

Solution:

$$\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{3 + \alpha \left[x - \frac{x^3}{3!} + \dots \right] + \beta \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right)}{3 \tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(3 + \beta) + (\alpha - 1)x + \left(-\frac{1}{2} - \frac{\beta}{2} \right) x^2 + \dots}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0, \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3, \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

Question8

$$\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\left(x - \frac{\pi}{2}\right)^2} \int_{\frac{\pi}{3}}^{\left(\frac{\pi}{2}\right)^3} \cos\left(\frac{1}{t^3}\right) dt \right) \text{ is equal to}$$

[29-Jan-2024 Shift 1]

Options:

A.

$3\pi/8$

B.

$3\pi^2/4$

C.

$$3\pi^2/8$$

D.

$$3\pi/4$$

Answer: C

Solution:

Using L'hôpital rule

$$\begin{aligned} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{0 - \cos x \times 3x^2}{2\left(x - \frac{\pi}{2}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin\left(x - \frac{\pi}{2}\right)}{2\left(x - \frac{\pi}{2}\right)} \times \frac{3\pi^2}{4} \\ &= \frac{3\pi^2}{8} \end{aligned}$$

Question9

Let the slope of the line $45x + 5y + 3 = 0$ be $27r_1 + \frac{9r_2}{2}$ for some $r_1, r_2 \in \mathbb{R}$.

Then $\lim_{x \rightarrow 3} \left(\int_3^x \frac{8t^2}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x} dt \right)$ is equal

[29-Jan-2024 Shift 2]

Answer: 12

Solution:

According to the question,

$$27r_1 + \frac{9r_2}{2} = -9$$

$$\begin{aligned} &\lim_{x \rightarrow 3} \frac{\int_3^x 8t^2 dt}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x} \\ &= \lim_{x \rightarrow 3} \frac{8x^2}{\frac{3r_2}{2} - 2r_2x - 3r_1x^2 - 3} \quad (\text{using LH' Rule}) \\ &= \frac{72}{\frac{3r_2}{2} - 6r_2 - 27r_1 - 3} \\ &= \frac{72}{-\frac{9r_2}{2} - 27r_1 - 3} \\ &= \frac{72}{9 - 3} = 12 \end{aligned}$$

Question10

Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$ be a differentiable function such that $f(0) = 1/2$, If the

$\lim_{x \rightarrow 0} \frac{\frac{x \int_0^x f(t) dt}{e^{x^2} - 1}}{x^2} = \alpha$, then $8\alpha^2$ is equal to :

[30-Jan-2024 Shift 1]

Options:

A.

16

B.

2

C.

1

D.

4

Answer: B

Solution:

$$\lim_{x \rightarrow 0} \frac{\frac{x \int_0^x f(t) dt}{e^{x^2} - 1}}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{\frac{\int_0^x f(t) dt}{x}}{\left(\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} = 1 \right)}$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{1} \text{ (using L Hospital)}$$

$$f(0) = \frac{1}{2}$$

$$\alpha = \frac{1}{2}$$

$$8\alpha^2 = 2$$

Question 11

Let a be the sum of all coefficients in the expansion of

$$(1 - 2x + 2x^2)^{2023} (3 - 4x^2 + 2x^3)^{2024} \text{ and } b = \lim_{x \rightarrow 0} \left(\frac{\int_0^x \frac{\log(1+t)}{1+t^{2024}} dt}{x^2} \right).$$

If the equations $cx^2 + dx + e = 0$ and $2bx^2 + ax + 4 = 0$ have a common root, where $c, d, e \in \mathbb{R}$, then $d:c : e$ equals

[31-Jan-2024 Shift 1]

Options:

A.

$$2:1:4$$

B.

$$4:1:4$$

C.

$$1:2:4$$

D.

$$1:1:4$$

Answer: D

Solution:

$$\text{Put } x = 1$$

$$\therefore a = 1$$

$$b = \lim_{x \rightarrow 0} \frac{\int_0^x \frac{\ln(1+t)}{1+t^{2024}} dt}{x^2}$$

Using L'HOPITAL Rule

$$b = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{(1+x^{2024})} \times \frac{1}{2x} = \frac{1}{2}$$

$$\text{Now, } cx^2 + dx + e = 0, x^2 + x + 4 = 0$$

(D < 0)

$$\therefore \frac{c}{1} = \frac{d}{1} = \frac{e}{4}$$

Question 12

$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$

[31-Jan-2024 Shift 1]

Options:

- A.
is equal to -1
- B.
does not exist
- C.
is equal to 1
- D.
is equal to 2

Answer: D

Solution:

$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$
$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{|\sin x|^2} \times \frac{\sin^2 x}{x^2}$$

Let $|\sin x| = t$

$$\lim_{t \rightarrow 0} \frac{e^{2t} - 2t - 1}{t^2} \times \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$$
$$= \lim_{t \rightarrow 0} \frac{2e^{2t} - 2}{2t} \times 1 = 2 \times 1 = 2$$

.....

Question 13

Let $f : \mathbb{R} \rightarrow (0, \infty)$ be strictly increasing function such that

$\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$. Then, the value of $\lim_{x \rightarrow \infty} \left[\frac{f(5x)}{f(x)} - 1 \right]$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

4

B.

0

C.

7/5

D.

1

Answer: B

Solution:

$$f : \mathbb{R} \rightarrow (0, \infty)$$

$$\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$$

$\because f$ is increasing

$$\therefore f(x) < f(5x) < f(7x)$$

$$\therefore \frac{f(x)}{f(x)} < \frac{f(5x)}{f(x)} < \frac{f(7x)}{f(x)}$$

$$1 < \lim_{x \rightarrow \infty} \frac{f(5x)}{f(x)} < 1$$

$$\therefore \left[\frac{f(5x)}{f(x)} - 1 \right]$$

$$\Rightarrow 1 - 1 = 0$$

Question 14

If $\lim_{x \rightarrow 0} \frac{ax^2 e^x - b \log_e(1+x) + cxe^{-x}}{x^2 \sin x} = 1$, then $16(a^2 + b^2 + c^2)$ is equal to _____

[31-Jan-2024 Shift 2]

Answer: 81

Solution:

$$\lim_{x \rightarrow 0} \frac{ax^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - b \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) + cx \left(1 - x + \frac{x^2}{x!} - \frac{x^3}{3!} + \dots \right)}{x^3 \cdot \frac{\sin x}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{(c-b)x + \left(\frac{b}{2} - c + a \right)x^2 + \left(a - \frac{b}{3} + \frac{c}{2} \right)x^3 + \dots}{x^3} = 1$$

$$c-b=0, \quad \frac{b}{2} - c + a = 0$$

$$a - \frac{b}{3} + \frac{c}{2} = 1 \quad a = \frac{3}{4} \quad b = c = \frac{3}{2}$$

$$a^2 + b^2 + c^2 = \frac{9}{16} + \frac{9}{4} + \frac{9}{4}$$

$$16(a^2 + b^2 + c^2) = 81$$

Question15

Let $\{x\}$ denote the fractional part of x and $f(x) = \frac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\} - \{x\}^3}$, $x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of $f(x)$ at $x = 0$, then $\frac{32}{\pi^2}(L^2 + R^2)$ is equal to ____

[1-Feb-2024 Shift 1]

Answer: 18

Solution:

Finding right hand limit

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h)$$

$$= \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2) \sin^{-1}(1-h)}{h(1-h^2)}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2)}{h} \left(\frac{\sin^{-1} 1}{1} \right)$$

$$\text{Let } \cos^{-1}(1-h^2) = \theta \Rightarrow \cos \theta = 1-h^2$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sqrt{1-\cos \theta}}$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{\frac{1-\cos \theta}{\theta^2}}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$L = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-\{-h\}^2) \sin^{-1}(1-\{-h\})}{\{-h\} - \{-h\}^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-(-h+1)^2) \sin^{-1}(1-(-h+1))}{(-h+1) - (-h+1)^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(-h^2+2h) \sin^{-1}h}{(1-h)(1-(1-h)^2)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\pi}{2} \right) \frac{\sin^{-1}h}{(1-(1-h)^2)}$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1}h}{-h^2+2h} \right)$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1}h}{h} \right) \left(\frac{1}{-h+2} \right)$$

$$L = \frac{\pi}{4}$$

$$\frac{32}{\pi^2}(L^2+R^2) = \frac{32}{\pi^2} \left(\frac{\pi^2}{2} + \frac{\pi^2}{16} \right)$$

$$= 16+2$$

$$= 18$$

Question 16

Let $f(x) = \begin{cases} x-1 & x \text{ is even,} \\ 2x & x \text{ is odd,} \end{cases}$. $a \in N$, $f(f(f(a))) = 21$, then $\lim_{x \rightarrow a^-} \left\{ \frac{|x|^3}{a} - \left[\frac{x}{a} \right] \right\}$, where $[t]$ denotes the greatest integer less than or equal to t , is equal to :

[1-Feb-2024 Shift 2]

Options:

- A. 121
- B. 144
- C. 169
- D. 225

Answer: B

Solution:

$$f(x) = \begin{cases} x-1; & x = \text{even} \\ 2x; & x = \text{odd} \end{cases}$$

$$f(f(f(a))) = 21$$

C - 1 : If $a = \text{even}$

$$f(a) = a - 1 = \text{odd}$$

$$f(f(a)) = 2(a - 1) = \text{even}$$

$$f(f(f(a))) = 2a - 3 = 21 \Rightarrow a = 12$$

C - 2 : If $a = \text{odd}$

$$f(a) = 2a = \text{even}$$

$$f(f(a)) = 2a - 1 = \text{odd}$$

$$f(f(f(a))) = 4a - 2 = 21 \text{ (Not possible)}$$

Hence $a = 12$.

Now

$$\begin{aligned} & \lim_{x \rightarrow 12^-} \left(\frac{|x|^3}{2} - \left[\frac{x}{12} \right] \right) \\ &= \lim_{x \rightarrow 12^-} \frac{|x|^3}{12} - \lim_{x \rightarrow 12^-} \left[\frac{x}{12} \right] \\ &= 144 - 0 = 144. \end{aligned}$$

Question 17

If $y = \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \frac{1}{15}(3\cos^2 x - 5)\cos^3 x$ then $96y' \left(\frac{\pi}{6} \right)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 105

Solution:

$$y = \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \frac{1}{15}(3\cos^2 x - 5)\cos^3 x$$

$$y = \frac{(\sqrt{x}+1)(\sqrt{x})((\sqrt{x})^3-1)}{(\sqrt{x})((\sqrt{x})^2+(\sqrt{x})+1)} + \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

$$y = (\sqrt{x}+1)(\sqrt{x}-1) + \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

$$y' = 1 - \cos^4 x \cdot (\sin x) + \cos^2 x (\sin x)$$

$$y' \left(\frac{\pi}{6} \right) = 1 - \frac{9}{16} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}$$

$$= \frac{32-9+12}{32} = \frac{35}{32}$$

$$= 96y' \left(\frac{\pi}{6} \right) = 105$$

Question 18

$\lim_{t \rightarrow 0} \left(1 \frac{1}{\sin^2 t} + 2 \frac{1}{\sin^2 t} + \dots + n \frac{1}{\sin^2 t} \right) \sin^2 t$ is equal to

[24-Jan-2023 Shift 1]

Options:

A. $n^2 + n$

B. n

C. $\frac{n(n+1)}{2}$

D. n^2

Answer: B

Solution:

Solution:

$$\lim_{t \rightarrow 0} \left(1^{\csc^2 t} + 2^{\csc^2 t} + \dots + n^{\csc^2 t} \right) \sin^2 t$$

$$= \lim_{t \rightarrow 0} \left(\left(\frac{1}{n} \right)^{\csc^2 t} + \left(\frac{2}{n} \right)^{\csc^2 t} + \dots + 1 \right)^{\sin^2 t}$$

$$= n$$

Question19

The set of all values of a for which $\lim_{x \rightarrow a} ([x - 5] - [2x + 2]) = 0$, where $[\alpha]$ denotes the greater integer less than or equal to α is equal to [24-Jan-2023 Shift 2]

Options:

- A. $(-7.5, -6.5)$
- B. $(-7.5, -6.5]$
- C. $[-7.5, -6.5]$
- D. $[-7.5, -6.5)$

Answer: A

Solution:

Solution:

$$\lim_{x \rightarrow a} ([x - 5] - [2x + 2]) = 0$$

$$\lim_{x \rightarrow a} ([x] - 5 - [2x] - 2) = 0$$

$$\lim_{x \rightarrow a} ([x] - [2x]) = 7$$

$$[a] - [2a] = 7$$

$$a \in I, a = -7$$

$$a \notin I, a = I + f$$

$$\text{Now, } [a] - [2a] = 7$$

$$-I - [2f] = 7$$

$$\text{Case-I: } f \in \left(0, \frac{1}{2}\right)$$

$$2f \in (0, 1)$$

$$-I = 7$$

$$I = -7 \Rightarrow a \in (-7, -6.5)$$

$$\text{Case-II: } f \in \left(\frac{1}{2}, 1\right)$$

$$2f \in (1, 2)$$

$$-I - 1 = 7$$

$$I = -8 \Rightarrow a \in (-7.5, -7)$$

$$\text{Hence, } a \in (-7.5, -6.5)$$

Question20

Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$$f(x) = \begin{cases} \frac{1 - \cos(x^2 - 4p + q^2 + 8q + 16)}{(x - 2p)^4} & x \neq 2p \\ 0 & x = 2p \end{cases}$$

Then $\lim_{x \rightarrow 22^+} [f(x)]$ where $[.]$ denotes greatest integer function, is

[29-Jan-2023 Shift 1]

Options:

- A. 2
- B. 1
- C. 0
- D. -1

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 2p^+} \left(\frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x^2 - 4px + q^2 + 8q + 16)^2} \right) \left(\frac{(x^2 - 4px + q^2 + 8q + 16)^2}{(x - 2p)^2} \right)$$

$$\lim_{h \rightarrow 0} \frac{1}{2} \left(\frac{(2p + h)^2 - 4p(2p + h) + q^2 + 8q + 16}{h^2} \right)^2 = \frac{1}{2}$$

Using L'Hospital's

$$\lim_{x \rightarrow 2p^+} [f(x)] = 0$$

Question 21

Let f , g and h be the real valued functions defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}, \quad g(x) = \begin{cases} \frac{\sin(x+1)}{(x+1)} & x \neq -1 \\ 1 & x = -1 \end{cases}, \quad \text{and } h(x) = 2[x] - f(x),$$

where $[x]$ is the greatest integer $\leq x$. Then the value of $\lim_{x \rightarrow 1} g(h(x-1))$ is

[30-Jan-2023 Shift 2]

Options:

- A. 1
- B. $\sin(1)$
- C. -1
- D. 0

Answer: A

Solution:

Solution:

$$\text{LHL} = \lim_{k \rightarrow 0} g(h(-k)), \quad k > 0$$

$$= \lim_{k \rightarrow 0} g(-2+1) \because f(x) = -1 \quad \forall x < 0$$

$$= g(-1) = 1$$

$$\text{RHL} = \lim_{k \rightarrow 0} g(h(k)), \quad k > 0$$

$$= \lim_{k \rightarrow 0} g(-1), \because f(x) = 1, \forall x > 0$$

$$= 1$$

Question22

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} x^3$$

[31-Jan-2023 Shift 2]

Options:

- A. is equal to 9
- B. is equal to 27
- C. does not exist
- D. is equal to $\frac{27}{2}$

Answer: B

Solution:

Solution:

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} x^3$$

$$\lim_{x \rightarrow \infty} x^3 \times \left\{ \frac{x^3 \left\{ \left(\sqrt{3 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}} \right)^6 + \left(\sqrt{3 + \frac{1}{x}} - \sqrt{3 - \frac{1}{x}} \right)^6 \right\}}{x^6 \left\{ \left(1 + \sqrt{1 - \frac{1}{x^2}} \right)^6 + \left(1 - \sqrt{1 - \frac{1}{x^2}} \right)^6 \right\}} \right\}$$

$$= \frac{(2\sqrt{3})^6 + 0}{2^6 + 0} = 3^3 = (27)$$

Question23

If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, $x \in \mathbb{R}$, then
[24-Jan-2023 Shift 2]

Options:

- A. $3f(1) + f(2) = f(3)$
- B. $f(3) - f(2) = f(1)$
- C. $2f(0) - f(1) + f(3) = f(2)$
- D. $f(1) + f(2) + f(3) = f(0)$

Answer: C

Solution:

Solution:

$$f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3), x \in \mathbb{R}$$

$$\text{Let } f(1) = a, f(2) = b, f'''(3) = c$$

$$f(x) = x^3 - x^2 + bx - c$$

$$f(x) = 3x^2 - 2ax + b$$

$$f'(x) = 6x - 2a$$

$$f''(x) = 6$$

$$c = 6, a = 3, b = 6$$

$$f(x) = x^3 - 3x^2 + 6x - 6$$

$$f(1) = -2, f(2) = 2, f(3) = 12, f(0) = -6$$

$$2f(0) - f(1) + f(3) = 2 = f(2)$$

Question24

Let

$$y(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}).$$

Then $y - y''$ at $x = -1$ is equal to

[25-Jan-2023 Shift 1]

Options:

A. 976

B. 464

C. 496

D. 944

Answer: C

Solution:

Solution:

$$y = \frac{1-x^{32}}{1-x} \Rightarrow y - xy = 1 - x^{32}$$

$$y' - xy' - y = -32x^{31}$$

$$y'' - xy'' - y' = -(32)(31)x^{30}$$

$$\text{at } x = -1 \Rightarrow y - y'' = 496$$

Question25

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function that satisfies the relation

$f(x+y) = f(x) + f(y) - 1, \forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to

[29-Jan-2023 Shift 1]

Answer: 3

Solution:

Solution:

$$f(x+y) = f(x) + f(y) - 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) = 2$$

$$f'(x) = 2 \Rightarrow dy = 2 dx$$

$$y = 2x + C$$

$$x = 0, y = 1, C = 1$$

$$y = 2x + 1$$

$$|f(-2)| = |-4 + 1| = |-3| = 3$$

Question 26

Let f and g be twice differentiable functions on $\mathbb{R}$ such that

$$f''(x) = g''(x) + 6x$$

$$f'(1) = 4g'(1) - 3 = 9$$

$$f(2) = 3g(2) = 12$$

Then which of the following is NOT true?

[29-Jan-2023 Shift 2]

Options:

A. $g(-2) - f(-2) = 20$

B. If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

C. $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$

D. There exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Answer: B

Solution:

Solution:

$$f''(x) = g''(x) + 6x \dots (1)$$

$$f'(1) = 4g'(1) - 3 = 9 \dots (2)$$

$$f(2) = 3g(2) = 12 \dots (3)$$

By integrating (1)

$$f'(x) = g'(x) + 6 \frac{x^2}{2} + C$$

$$\text{At } x = 1$$

$$f'(1) = g'(1) + 3 + C$$

$$\Rightarrow 9 = 4 + 3 + C \Rightarrow C = 3$$

$$\therefore f'(x) = g'(x) + 3x^2 + 3$$

Again by integrating,

$$f(x) = g(x) + \frac{3x^3}{3} + 3x + D$$

$$\text{At } x = 2$$

$$f(2) = g(2) + 8 + 3(2) + D$$

$$\Rightarrow 12 = 4 + 8 + 6 + D \Rightarrow D = -6$$

$$\text{So, } f(x) = g(x) + x^3 + 3x - 6$$

$$\Rightarrow f(x) - g(x) = x^3 + 3x - 6$$

$$\text{At } x = -2$$

$$\Rightarrow g(-2) - f(-2) = 20 \quad (\text{Option (1) is true})$$

$$\text{Now, for } -1 < x < 2$$

$$h(x) = f(x) - g(x) = x^3 + 3x - 6$$

$\Rightarrow h(x) = 3x^2 + 3$
 $\Rightarrow h(x) \uparrow$
 So, $h(-1) < h(x) < h(2)$
 $\Rightarrow -10 < h(x) < 8$
 $\Rightarrow |h(x)| < 10$ (option (2) is NOT true)
 Now, $h(x) = f(x) - g(x) = 3x^2 + 3$
 If $|h(x)| < 6 \Rightarrow |3x^2 + 3| < 6$
 $\Rightarrow 3x^2 + 3 < 6$
 $\Rightarrow x^2 < 1$
 $\Rightarrow -1 < x < 1$ (option (3) is True)
 If $x \in (-1, 1) \mid f(x) - g(x) < 6$
 option (3) is true and now to solve
 $f(x) - g(x) = 0$
 $\Rightarrow x^3 + 3x - 6 = 0$
 $h(x) = x^3 + 3x - 6$
 here, $h(1) = -ve$ and $h\left(\frac{3}{2}\right) = +ve$
 So there exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$
 (option (4) is true)

Question 27

Let

$$y = f(x) = \sin^3 \left(\frac{\pi}{3} \left(\cos \left(\frac{\pi}{3\sqrt{2}} (-4x^3 + 5x^2 + 1) \right)^{\frac{3}{2}} \right) \right)$$

Then, at $x = 1$,
[31-Jan-2023 Shift 1]

Options:

A. $2y' + \sqrt{3}\pi^2 y = 0$

B. $2y' + 3\pi^2 y = 0$

C. $\sqrt{2}y' - 3\pi^2 y = 0$

D. $y' + 3\pi^2 y = 0$

Answer: B

Solution:

$$y = \sin^3 \left(\frac{\pi}{3} \cos g(x) \right)$$

$$g(x) = \frac{\pi}{3\sqrt{2}} (-4x^3 + 5x^2 + 1)^{3/2}$$

$$g(1) = 2\pi/3$$

$$y' = 3\sin^2 \left(\frac{\pi}{3} \cos g(x) \right) \times \cos \left(\frac{\pi}{3} \cos g(x) \right)$$

$$y'(1) = 3\sin^2 \left(-\frac{\pi}{6} \right) \cdot \cos \left(\frac{\pi}{6} \right) \cdot \frac{\pi}{3} \left(-\sin \frac{2\pi}{3} \right) g'(1)$$

$$g'(x) = \frac{\pi}{3\sqrt{2}} (-4x^3 + 5x^2 + 1)^{1/2} (-12x^2 + 10x)$$

$$g'(1) = \frac{\pi}{2\sqrt{2}} (\sqrt{2})(-2) = -\pi$$

$$y'(1) = \frac{\text{not } 6}{4} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\pi}{\text{not}} \left(-\frac{\sqrt{3}}{2} \right) (-\pi) = \frac{3\pi^2}{16}$$

$$y'(1) = \sin^3 (\pi/3 \cos 2\pi/3) = -\frac{1}{8}$$

$$2y'(1) + 3\pi^2 y(1) = 0$$

Question28

If $\int_0^1 (x^{21} + x^{14} + x^7)(2x^{14} + 3x^7 + 6)^{1/7} dx = \frac{1}{1}(11)^{m/n}$

where $1, m, n \in \mathbb{N}$, m and n are coprime then $l + m + n$ is equal to

 .

[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$$\int (x^{20} + x^{13} + x^6)(2x^{21} + 3x^{14} + 6x^7)^{1/7} dx$$

$$2x^{21} + 3x^{14} + 6x^7 = t$$

$$42(x^{20} + x^{13} + x^6) dx = dt$$

$$\frac{1}{42} \int_0^{11} t^{\frac{1}{7}} dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}} \times \frac{1}{42} \right)_0^{11}$$

$$= \frac{1}{48} \left(t^{\frac{8}{7}} \right)_0^{11} = \frac{1}{48} (11)^{8/7}$$

$$l = 48, m = 8, n = 7$$

$$l + m + n = 63$$

Question29

$\lim_{n \rightarrow \infty} \left\{ \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right) \left(2^{\frac{1}{2}} - 2^{\frac{1}{5}} \right) \dots \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right) \right\}$ is equal to

[6-Apr-2023 shift 2]

Options:

A. $\frac{1}{\sqrt{2}}$

B. $\sqrt{2}$

C. 1

D. 0

Answer: D

Solution:

$$P = \lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right) \left(2^{\frac{1}{2}} - 2^{\frac{1}{5}} \right) \dots \dots \dots \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)$$

Let

$$2^{\frac{1}{2}} - 2^{\frac{1}{3}} \rightarrow \text{Smallest}$$

$$2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \rightarrow \text{Largest}$$

Sandwich th.

$$\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right)^n \leq P \leq \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)^n$$

$$\left(\text{lie b / w} \right)^n$$

$$0 \text{ and } 1$$

$$\lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right)^n = 0$$

$$\lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)^n = 0$$

$$\therefore P = 0$$

Question30

$$\lim_{x \rightarrow 0} \left(\left(\frac{1 - \cos^2(3x)}{\cos^3(4x)} \right) \left(\frac{\sin^3(4x)}{(\log_e(2x + 1))^5} \right) \right) \text{ is equal to } \underline{\hspace{2cm}}$$

[8-Apr-2023 shift 1]

Options:

A. 24

B. 9

C. 18

D. 15

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \left[\frac{1 - \cos^2 3x}{9x^2} \right] \frac{9x^2}{\cos^3 4x} \cdot \frac{\left(\frac{\sin 4x}{4x} \right)^3 \times 64x^3}{\left[\frac{\ln(1 + 2x)}{2x} \right]^5 \times 32x^5}$$

$$\lim_{x \rightarrow 0} 2 \left(\frac{1}{2} \times \frac{9}{1} \times \frac{1 \times 64}{1 \times 32} \right) = 18$$

Question31

If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and

$$\lim_{x \rightarrow \frac{1}{\alpha}} \left(\frac{1 - \cos(x^2 + bx + a)}{2(1 - ax)^2} \right)^{\frac{1}{2}} = \frac{1}{k} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right), \text{ then k is equal to}$$

[8-Apr-2023 shift 2]

Options:

- A. β
- B. 2α
- C. 2β
- D. α

Answer: B

Solution:

Solution:

$$\therefore ax^2 + bx + 1 = a(x - \alpha)(x - \beta) \therefore \alpha\beta = \frac{1}{a}$$

$$\therefore x^2 + bx + a = a(1 - \alpha x)(1 - \beta x)$$

$$\therefore \lim_{x \rightarrow \frac{1}{\alpha}} \left\{ \frac{1 - \cos(x^2 + bx + a)}{2(1 - \alpha x)^2} \right\}^{\frac{1}{2}} = \lim_{x \rightarrow \frac{1}{2}} \left\{ \frac{1 - \cos a(1 - \alpha x)(1 - \beta x)}{2\{a(1 - \alpha x)(1 - \beta x)\}^2} \cdot a^2(1 - \beta x)^2 \right\}^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} \cdot \frac{1}{2} a^2 \left(1 - \frac{\beta}{\alpha} \right)^2 \right]^{\frac{1}{2}}$$

$$= \frac{1}{2} \frac{1}{\alpha\beta} \left(1 - \frac{\beta}{\alpha} \right) = \frac{1}{2} \left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2} \right)$$

$$= \frac{1}{2\alpha} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) = \frac{1}{k} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)$$

$$\therefore k = 2\alpha \text{ Ans.}$$

Question32

Among

(S1): $\lim_{n \rightarrow \infty} \frac{1}{n^2} (2 + 4 + 6 + \dots + 2n) = 1$

(S2): $\lim_{n \rightarrow \infty} \frac{1}{n^{16}} (1^{15} + 2^{15} + 3^{15} + \dots + n^{15}) = \frac{1}{16}$

[13-Apr-2023 shift 1]

Options:

- A. Only (S1) is true
- B. Both (S1) and (S2) are true
- C. Both (S1) and (S2) are false
- D. Only (S2) is true

Answer: B

Solution:

Solution:

$$S_1: \lim_{n \rightarrow \infty} \frac{n(n+1)}{n^2} = 1 \Rightarrow \text{True}$$

$$S_2 : \lim_{n \rightarrow \infty} \frac{1}{n^{16}} (\sum r^{15}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum \left(\frac{r}{n} \right)^{15} \\ = \int_0^1 x^{15} dx = \frac{1}{16} \Rightarrow \text{True}$$

Question33

If $\lim_{x \rightarrow 0} \frac{e^{ax} - \cos(bx) - \frac{cxe^{-cx}}{2}}{1 - \cos(2x)} = 17$, then $5a^2 + b^2$ is equal to

[13-Apr-2023 shift 2]

Options:

- A. 76
- B. 72
- C. 64
- D. 68

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{e^{ax} - \cos bx - \frac{cxe^{-cx}}{2}}{1 - \cos 2x} = 17$$

On expansion

$$\lim_{x \rightarrow 0} \frac{\left(1 + ax + \frac{(ax)^2}{2!} + \dots \right) - \left(1 - \frac{(bx)^2}{2!} + \dots \right) - \frac{cx}{2} \left(1 - cx + \frac{(cx)^2}{2!} \right)}{\left(\frac{1 - \cos 2x}{(2x)^2} \right) \times (2x)^2} = 17$$

$$\lim_{x \rightarrow 0} \frac{x \left(a - \frac{c}{2} \right) + x^2 \left(\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} \right)}{\frac{1}{2}(4x^2)} = 17$$

For limit to be exist

$$a - \frac{c}{2} = 0 \Rightarrow c = 2a$$

$$\Rightarrow \frac{\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}}{2} = 17$$

$$\Rightarrow \frac{a^2}{2} + \frac{b^2}{2} + \frac{4a^2}{2} = 34$$

$$\Rightarrow 5a^2 + b^2 = 68$$

Question34

If $2x^y + 3y^x = 20$, then $\frac{dy}{dx}$ at (2, 2) is equal to:

[6-Apr-2023 shift 1]

Options:

A. $-\left(\frac{3 + \log_e 8}{2 + \log_e 4}\right)$

B. $-\left(\frac{2 + \log_e 8}{3 + \log_e 4}\right)$

C. $-\left(\frac{3 + \log_e 4}{2 + \log_e 8}\right)$

D. $-\left(\frac{3 + \log_e 16}{4 + \log_e 8}\right)$

Answer: B

Solution:

Solution:

$$2x^y + 3y^x = 20$$

$$v_1^{v_2} \left(v_2 \frac{1}{v_1} + \ln v_1 \cdot v_2^{-1} \right)$$

$$2x^y \left(y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} \right) + 3y^x \left(x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \ln y \cdot 1 \right) = 0$$

Put (2, 2)

$$2 \cdot 2 \left(1 + \ln 2 \cdot \frac{dy}{dx} \right) + 3 \cdot 2 \left(1 \cdot \frac{dy}{dx} + \ln 2 \right) = 0$$

$$\frac{dy}{dx} [8 \ln 2 + 12] + 8 + 12 \ln 2 = 0$$

$$\frac{dy}{dx} = - \left[\frac{2 + 3 \ln 2}{3 + 2 \ln 2} \right] = - \left[\frac{2 + \ln 8}{3 + \ln 4} \right]$$

Question 35

Let $f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}$, $x \in [0, \pi] - \left\{ \frac{\pi}{4} \right\}$. Then $f\left(\frac{7\pi}{12}\right) f''\left(\frac{7\pi}{12}\right)$ is equal to

[8-Apr-2023 shift 1]

Options:

A. $\frac{-2}{3}$

B. $\frac{2}{9}$

C. $\frac{-1}{3\sqrt{3}}$

D. $\frac{2}{3\sqrt{3}}$

Answer: B

Solution:

Solution:

$$f(x) = -\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f'(x) = -\frac{1}{2} \sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f''(x) = -\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \tan\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \frac{1}{2}$$

$$f\left(\frac{7\pi}{12}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

$$f''\left(\frac{7\pi}{12}\right) = -\frac{1}{2} \sec^2\frac{\pi}{6} \cdot \tan\frac{\pi}{6} = \frac{-1}{2} \cdot \frac{4}{3} \times \frac{1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$$

$$f\left(\frac{7\pi}{12}\right) \cdot f''\left(\frac{7\pi}{12}\right) = \frac{2}{9}$$

Question 36

Let k and m be positive real numbers such that the function

$$f(x) = \begin{cases} 3x^2 + k\sqrt{x+1} & 0 < x < 1 \\ mx^2 + k^2 & x \geq 1 \end{cases} \text{ is differentiable for all } x > 0. \text{ Then } \frac{8f(8)}{f'\left(\frac{1}{8}\right)}$$

is equal to _____.
[8-Apr-2023 shift 2]

Answer: 309

Solution:

Solution:

function is differentiable $\forall x < 0$

so $f(1^-) = f(1) \dots (1)$

$$3 + \sqrt{2}k = m + k^2$$

$$\text{and } f_+^{-1}(1^-) = f_-^{-1}(1^+)$$

$$2m = 6 + \frac{k}{2\sqrt{2}}$$

$$m = 3 + \frac{k}{4\sqrt{2}} \dots (2)$$

$$k^2 + 3 + \frac{k}{4\sqrt{2}} = 3 + \sqrt{2}k$$

$$k = \frac{7}{4\sqrt{2}}, 0$$

$$m = 3 + \frac{7}{32}$$

$$m = \frac{103}{32}$$

$$8 \times 2 \times 8 \times \frac{103}{32}$$

$$= \frac{16}{12}$$

$$= 103 \times 3 = 309$$

Question 37

$\lim_{x \rightarrow \frac{\pi}{2}} \left(\tan^2 x \left((2\sin^2 x + 3\sin x + 4)^{\frac{1}{2}} - (\sin^2 x + 6\sin x + 2)^{\frac{1}{2}} \right) \right)$ is equal to

[25-Jun-2022-Shift-2]

Options:

- A. $\frac{1}{12}$
- B. $-\frac{1}{18}$
- C. $-\frac{1}{12}$
- D. $\frac{1}{6}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \tan^2 x \left\{ \sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2} \right\} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan^2 x (\sin^2 x - 3\sin x + 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}} \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)(2 - \sin x) \cdot \sin^2 x}{\cos^2 x} \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(2 - \sin x)\sin^2 x}{1 + \sin x} \\ &= \frac{1}{12} \end{aligned}$$

Question38

$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1} x) - x}{1 - \tan(\cos^{-1} x)}$ is equal to :

[26-Jun-2022-Shift-1]

Options:

- A. $\sqrt{2}$
- B. $-\sqrt{2}$
- C. $\frac{1}{\sqrt{2}}$
- D. $-\frac{1}{\sqrt{2}}$

Answer: D

Solution:

$$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1}x) - x}{1 - \tan(\cos^{-1}x)}$$

$$\text{Let } \cos^{-1}x = t$$

$$\Rightarrow x = \cos t$$

$$\text{When } x \rightarrow \frac{1}{\sqrt{2}}, \text{ then } t \rightarrow \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \rightarrow \frac{\pi}{4}$$

$$\therefore \lim_{t \rightarrow \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \tan(t)}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \frac{\sin t}{\cos t}}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} \frac{(\sin t - \cos t)(\cos t)}{(\cos t - \sin t)}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} -\cos t$$

$$= -\lim_{t \rightarrow \frac{\pi}{4}} \cos t$$

$$= -\frac{1}{\sqrt{2}}$$

Question39

$\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to :

[26-Jun-2022-Shift-2]

Options:

A. $\frac{1}{3}$

B. $\frac{1}{4}$

C. $\frac{1}{6}$

D. $\frac{1}{12}$

Answer: C

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4} &= \lim_{x \rightarrow 0} \frac{2 \sin(x + \sin x) \cdot \sin\left(\frac{x - \sin x}{2}\right)}{x^4} \\
 &= \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\left(\frac{x + \sin x}{2}\right) \left(\frac{x - \sin x}{2}\right)}{x^4} \right) \\
 &= \lim_{x \rightarrow 0} \frac{1}{2} \cdot \left(\frac{\left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right) \left(x - x + \frac{x^3}{3!} \dots\right)}{x^4} \right) \\
 &= \lim_{x \rightarrow 0} \frac{1}{2} \cdot \left(2 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots \right) \left(\frac{1}{3!} - \frac{x^2}{5!} - 1 \right) \\
 &= \frac{1}{6}
 \end{aligned}$$

Question 40

Let a be an integer such that $\lim_{x \rightarrow 7} \frac{18 - [1 - x]}{[x - 3a]}$ exists, where $[t]$ is greatest integer $\leq t$. Then a is equal to :
[27-Jun-2022-Shift-1]

Options:

- A. -6
- B. -2
- C. 2
- D. 6

Answer: A

Solution:

$$\lim_{x \rightarrow 7} \frac{18 - [1 - x]}{[x - 3a]} \text{ exist \& } a \in \mathbb{I},$$

$$= \lim_{x \rightarrow 7} \frac{17 - [-x]}{[x] - 3a} \text{ exist}$$

$$\text{RHL} = \lim_{x \rightarrow 7^+} \frac{17 - [-x]}{[x] - 3a} = \frac{25}{7 - 3a} \left[a \neq \frac{7}{3} \right]$$

$$\text{LHL} = \lim_{x \rightarrow 7^-} \frac{17 - [-x]}{[x] - 3a} = \frac{24}{6 - 3a} [a \neq 2]$$

For limit to exist

$$\text{LHL} = \text{RHL}$$

$$\frac{25}{7 - 3a} = \frac{24}{6 - 3a}$$

$$\Rightarrow \frac{25}{7 - 3a} = \frac{8}{2 - a}$$

$$\therefore a = -6$$

Question 41

Let $[t]$ denote the greatest integer $\leq t$ and $\{t\}$ denote the fractional part of t . The integral value of α for which the left hand limit of the function

$$f(x) = [1 + x] + \frac{\alpha^{2[x] + \{x\}} + [x] - 1}{2[x] + \{x\}} \text{ at } x = 0 \text{ is equal to } \alpha - \frac{4}{3}, \text{ is } \underline{\hspace{2cm}}$$

[27-Jun-2022-Shift-2]

Answer: 3

Solution:

Solution:

$$f(x) = [1 + x] + \frac{\alpha^{2[x] + \{x\}} + [x] - 1}{2[x] + \{x\}}$$

$$\lim_{x \rightarrow 0^-} f(x) = \alpha - \frac{4}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0^-} 1 + [x] + \frac{\alpha^{x + [x]} + [x] - 1}{x + [x]} = \alpha - \frac{4}{3}$$

$$\Rightarrow \lim_{h \rightarrow 0^-} 1 - 1 + \frac{\alpha^{-h - 1} - 1 - 1}{-h - 1} = \alpha - \frac{4}{3}$$

$$\therefore \frac{\alpha^{-1} - 2}{-1} = \alpha - \frac{4}{3}$$

$$\Rightarrow 3\alpha^2 - 10\alpha + 3 = 0$$

$$\therefore \alpha = 3 \text{ or } \frac{1}{3}$$

$$\therefore \alpha \text{ in integer, hence } \alpha = 3$$

Question 42

The value of

$$\lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{r^2 + 3r + 3} \right) \right\} \text{ is equal to :}$$

[28-Jun-2022-Shift-2]

Options:

- A. 1
- B. 2
- C. 3
- D. 6

Answer: C

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{n-1} \tan^{-1} \left(\frac{1}{r^2 + 3r + 3} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{n-1} \tan^{-1} \left(\frac{(r+2) - (r+1)}{1 + (r+2)(r+1)} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{n-1} \tan^{-1} (r+1) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \{ \tan^{-1}(n+2) - \tan^{-1} 2 \} \\ &= 6 \tan \left\{ \frac{\pi}{2} - \cot^{-1} \left(\frac{1}{2} \right) \right\} \\ &= 6 \tan \left(\tan^{-1} \left(\frac{1}{2} \right) \right) \\ &= 3 \end{aligned}$$

Question43

If $\lim_{x \rightarrow 1} \frac{\sin(3x^2 - 4x + 1) - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2$, then the value of (a – b) is equal to _____

[28-Jun-2022-Shift-2]

Answer: 11

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{\left(\frac{\sin(3x^2 - 4x + 1)}{3x^2 - 4x + 1} \right) (3x^2 - 4x + 1) - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2 \\ & \Rightarrow \lim_{x \rightarrow 1} \frac{3x^2 - 4x + 1 - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2 \\ & \Rightarrow \lim_{x \rightarrow 1} \frac{2(x-1)^2}{2x^3 - 7x^2 + ax + b} = -2 \end{aligned}$$

So $f(x) = 2x^3 - 7x^2 + ax + b = 0$ has $x = 1$ as repeated root, therefore $f(1) = 0$ and $f'(1) = 0$ gives $a + b + 5$ and $a = 8$

So, $a - b = 11$

Question44

The value of $\lim_{x \rightarrow 1} \frac{(x^2 - 1)\sin^2(\pi x)}{x^4 - 2x^3 + 2x - 1}$ is equal to:

[29-Jun-2022-Shift-2]

Options:

A. $\frac{\pi^2}{6}$

B. $\frac{\pi^2}{3}$

C. $\frac{\pi^2}{2}$

D. π^2

Answer: D

Solution:

Solution:

$$S = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \dots \quad (1)$$

$$\frac{1}{6}S = \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \dots \quad (2)$$

$$S - \frac{1}{6}S = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \dots$$

$$\Rightarrow \frac{5S}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \dots \quad (3)$$

Now, multiplying both sides by $\frac{1}{6}$, we get

$$\Rightarrow \frac{5S}{36} = \frac{1}{6} + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + \dots$$

Subtract equation (4) from equation (3), we get

$$\frac{25S}{36} = 1 + \frac{3}{6} + \frac{3}{6^2} + \frac{3}{6^3} + \dots$$

$$\Rightarrow \frac{25S}{36} = 1 + \frac{\frac{3}{5}}{1 - \frac{1}{6}}$$

$$= 1 + \frac{3}{6} \times \frac{6}{5}$$

$$= 1 + \frac{3}{5} = \frac{8}{5}$$

$$\Rightarrow S = \frac{8}{5} \times \frac{36}{25} = \frac{288}{125}$$

Question45

If $y = \tan^{-1}(\sec x^3 - \tan x^3)$, $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then

[24-Jun-2022-Shift-2]

Options:

A. $xy'' + 2y' = 0$

B. $x^2y'' - 6y + \frac{3\pi}{2} = 0$

C. $x^2y'' - 6y + 3\pi = 0$

D. $xy'' - 4y' = 0$

Answer: B

Solution:

Solution:

Let $x^3 = \theta \Rightarrow \frac{\theta}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$

$\therefore y = \tan^{-1}(\sec \theta - \tan \theta)$

$= \tan^{-1}\left(\frac{1 - \sin \theta}{\cos \theta} \right)$

$\therefore y = \frac{\pi}{4} - \frac{\theta}{2}$

$y = \frac{\pi}{4} - \frac{x^3}{2}$

$\therefore y' = \frac{-3x^2}{2}$

$y'' = -3x$

$\therefore x^2y'' - 6y + \frac{3\pi}{2} = 0$

Question 46

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x^3 + x - 5$. If $g(x)$ is a function such that $f(g(x)) = x$, $\forall x \in \mathbb{R}$, then $g'(63)$ is equal to [25-Jun-2022-Shift-1]

Options:

A. $\frac{1}{49}$

B. $\frac{3}{49}$

C. $\frac{43}{49}$

D. $\frac{91}{49}$

Answer: A

Solution:

Solution:

$$f(x) = 3x^2 + 1$$

$f(x)$ is bijective function

and $f(g(x)) = x \Rightarrow g(x)$ is inverse of $f(x)$ $g(f(x)) = x$

$$g'(f(x)) \cdot f'(x) = 1$$

$$g'(f(x)) = \frac{1}{3x^2 + 1}$$

Put $x = 4$ we get

$$g'(63) = \frac{1}{49}$$

Question 47

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $f(x + y) = 2^x f(y) + 4^y f(x)$, $\forall x, y \in \mathbb{R}$. If $f(2) = 3$, then

14. $\frac{f(4)}{f(2)}$ is equal to ____

[26-Jun-2022-Shift-2]

Answer: 248

Solution:

$$\therefore f(x + y) = 2^x f(y) + 4^y f(x) \dots (1)$$

$$\text{Now, } f(y + x) = 2^y f(x) + 4^x f(y) \dots (2)$$

$$\therefore 2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$$

$$(4^y - 2^y) f(x) = (4^x - 2^x) f(y)$$

$$\frac{f(x)}{4^x - 2^x} = \frac{f(y)}{4^y - 2^y} = k$$

$$\therefore f(x) = k(4^x - 2^x)$$

$$\because f(2) = 3 \text{ then } k = \frac{1}{4}$$

$$\therefore f(x) = \frac{4^x - 2^x}{4}$$

$$\therefore f'(x) = \frac{4^x \ln 4 - 2^x \ln 2}{4}$$

$$f'(x) = \frac{(2 \cdot 4^x - 2^x) \ln 2}{4}$$

$$\therefore \frac{f'(4)}{f'(2)} = \frac{2.256 - 16}{2.16 - 4}$$

$$\therefore 14 \frac{f'(4)}{f'(2)} = 248$$

Question48

If $\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5$, $|y| < 2$, then :

[27-Jun-2022-Shift-1]

Options:

A. $x^2y'' + xy' - 25y = 0$

B. $x^2y'' - xy' - 25y = 0$

C. $x^2y'' - xy' + 25y = 0$

D. $x^2y'' + xy' + 25y = 0$

Answer: D

Solution:

Solution:

$$\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5 \quad |y| < 2$$

Differentiating on both side

$$-\frac{1}{\sqrt{1 - \left(\frac{y}{2}\right)^2}} \times \frac{y'}{2} = \frac{5}{x} \times \frac{1}{5}$$

$$\frac{-xy'}{2} = 5 \sqrt{1 - \left(\frac{y}{2}\right)^2}$$

Square on both side

$$\frac{x^2y'^2}{4} = 25 \left(\frac{4 - y^2}{4} \right)$$

Diff on both side

$$2xy'^2 + 2y'y''x^2 = -25 \times 2yy'$$

$$xy' + y''x^2 + 25y = 0$$

Question49

Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in \mathbb{R}$. Then the sum of the squares of all the values of a , for which $2f'(10) - f'(5) + 100 = 0$, is
[27-Jun-2022-Shift-2]

Options:

- A. 117
- B. 106
- C. 125
- D. 136

Answer: C

Solution:

Solution:

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}, a \in \mathbb{R}$$

$$f(x) = a(a^2 + ax) + 1(a^2x + ax^2) \\ = a(x + a)^2$$

$$f'(x) = 2a(x + a)$$

$$\text{Now, } 2f'(10) - f'(5) + 100 = 0$$

$$\Rightarrow 2 \cdot 2a(10 + a) - 2a(5 + a) + 100 = 0$$

$$\Rightarrow 2a(a + 15) + 100 = 0$$

$$\Rightarrow a^2 + 15a + 50 = 0$$

$$\Rightarrow a = -10, -5$$

$$\therefore \text{Sum of squares of values of } a = 125$$

Question 50

If $y(x) = (x^x)^x$, $x > 0$, then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to _____

[27-Jun-2022-Shift-2]

Answer: 16

Solution:

Solution:

$$\therefore y(x) = (x^x)^x$$

$$\therefore y = x^{x^2}$$

$$\therefore \frac{dy}{dx} = x^2 \cdot x^{x^2-1} + x^{x^2} \ln x \cdot 2x$$

$$\therefore \frac{dx}{dy} = \frac{1}{x^{x^2+1}(1 + 2 \ln x)} \dots (i)$$

$$\text{Now, } \frac{d^2x}{dy^2} = \frac{d}{dx} \left((x^{x^2+1}(1 + 2 \ln x))^{-1} \right) \cdot \frac{dx}{dy}$$

$$= \frac{-x(x^{x^2+1}(1 + 2 \ln x))^{-2} \cdot x^{x^2}(1 + 2 \ln x)(x^2 + 2x^2 \ln x + 3)}{x^{x^2}(1 + 2 \ln x)}$$

$$= \frac{-x^{x^2}(1 + 2 \ln x)(x^3 + 3 + 2x^2 \ln x)}{(x^{x^2}(1 + 2 \ln x))^3}$$

$$\frac{d^2x}{dy^2} \text{ at } x = 1 = -4$$

$$\therefore \frac{d^2x}{dy^2(atx=1)} + 20 = 16$$

Question 51

If $\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$, then $8(\alpha + \beta)$ is equal to:

[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. -8
- C. -4
- D. 8

Answer: C

Solution:

Solution:

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞ .]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - n - 1} + n\alpha + \beta)(\sqrt{n^2 - n - 1} - (n\alpha + \beta))}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$$

Here power of "n" in the numerator is 2 and power of "n" in the denominator is 1.

To get the value of limit equal to zero power of "n" should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$$\therefore 1 - \alpha^2 = 0$$

$$\Rightarrow \alpha = \pm 1$$

But α should be < 0

$$\therefore \alpha = +1 \text{ not possible}$$

$$\therefore \alpha = -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta^2)}{n \left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n} \right]} = 0$$

Divide numerator and denominator by n then we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta^2)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta)}{1 - \alpha} = 0$$

$$\Rightarrow -(1 + 2\alpha\beta) = 0$$

$$\Rightarrow 1 + 2\alpha\beta = 0$$

$$\Rightarrow 2\alpha\beta = -1$$

$$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$$

$$\therefore 8(\alpha + \beta)$$

$$= 8\left(-1 + \frac{1}{2}\right)$$

$$= 8 \times -\frac{1}{2}$$

$$= -4$$

Question 52

If $\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$, then $8(\alpha + \beta)$ is equal to:

[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. -8
- C. -4
- D. 8

Answer: C

Solution:

Solution:

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞ .]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - n - 1} + n\alpha + \beta)(\sqrt{n^2 - n - 1} - (n\alpha + \beta))}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$$

Here power of "n" in the numerator is 2 and power of "n" in the denominator is 1.

To get the value of limit equal to zero power of "n" should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$$\therefore 1 - \alpha^2 = 0$$

$$\Rightarrow \alpha = \pm 1$$

But α should be < 0

$$\therefore \alpha = +1 \text{ not possible}$$

$$\therefore \alpha = -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta^2)}{n \left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n} \right]} = 0$$

Divide numerator and denominator by n then we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta^2)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$$

$$\Rightarrow \frac{-(1+2\alpha\beta)}{1-\alpha} = 0$$

$$\Rightarrow -(1+2\alpha\beta) = 0$$

$$\Rightarrow 1+2\alpha\beta = 0$$

$$\Rightarrow 2\alpha\beta = -1$$

$$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$$

$$\therefore 8(\alpha + \beta)$$

$$= 8\left(-1 + \frac{1}{2}\right)$$

$$= 8 \times -\frac{1}{2}$$

$$= -4$$

Question 53

If $\lim_{n \rightarrow \infty} \frac{(n+1)^{k-1}}{n^{k+1}} [(nk+1) + (nk+2) + \dots + (nk+n)]$, then the integral value of k is $= 33 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^{k+1}} \cdot [1^k + 2^k + 3^k + \dots + n^k]$ equal to
[25-Jul-2022-Shift-1]

Answer: 5

Solution:

Solution:

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{k-1} \frac{1}{n} \sum_{r=1}^n \left(k + \frac{r}{n} \right) = 33 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{r}{n} \right)^k$$

$$\Rightarrow \int_0^1 (k+x) dx = 33 \int_0^1 x^k dx$$

$$\Rightarrow \frac{2k+1}{2} = \frac{33}{k+1}$$

$$\Rightarrow k = 5$$

Question 54

$\lim_{x \rightarrow \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x}$ is equal to

[25-Jul-2022-Shift-2]

Options:

- A. 14
- B. 7
- C. $14\sqrt{2}$
- D. $7\sqrt{2}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x} \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-7(\cos x + \sin x)^6 (-\sin x + \cos x)}{-2\sqrt{2} \cos 2x} \text{ using L - H Rule} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{56(\cos x - \sin x)}{2\sqrt{2} \cos 2x} \left(\frac{0}{0} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-56(\sin x + \cos x)}{-4\sqrt{2} \sin 2x} \text{ using L - H Rule} \\ &= 7\sqrt{2} \cdot \sqrt{2} = 14 \end{aligned}$$

Question55

If the function

$$f(x) = \begin{cases} \frac{\log_e(1-x+x^2) + \log_e(1+x+x^2)}{\sec x - \cos x}, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\} \\ k, & x = 0. \end{cases}$$

is continuous at $x = 0$, then k is equal to:
[26-Jul-2022-Shift-1]

Options:

- A. 1
- B. -1
- C. e
- D. 0

Answer: A

Solution:

$$f(x) = \begin{cases} \frac{\log_e(1-x+x^2) + \log_e(1+x+x^2)}{\sec x - \cos x} & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\} \\ k & x = 0. \end{cases}$$

for continuity at $x = 0$

$$\lim_{x \rightarrow 0} f(x) = k$$

$$\therefore k = \lim_{x \rightarrow 0} \frac{\log_e(x^4 + x^2 + 1)}{\sec x - \cos x} \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\cos x \log_e(x^4 + x^2 + 1)}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\log_e(x^4 + x^2 + 1)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1 + x^2 + x^4)}{x^2 + x^4} \cdot \frac{x^2 + x^4}{x^2}$$

$$= 1$$

Question56

Let $\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x(e^{3x} - 1)}$ for some $\alpha \in \mathbb{R}$. Then the value of $\alpha + \beta$ is :

[26-Jul-2022-Shift-2]

Options:

A. $\frac{14}{5}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{7}{2}$

Answer: C

Solution:

Solution:

$$\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x (e^{3x} - 1)}, \alpha \in \mathbb{R}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\alpha}{3} - \left(\frac{e^{3x} - 1}{3x} \right)}{\alpha x \left(\frac{e^{3x} - 1}{3x} \right)}$$

So, $\alpha = 3$ (to make independent form)

$$\beta = \lim_{x \rightarrow 0} \frac{1 - \left(\frac{e^{3x} - 1}{3x} \right)}{3x} = \frac{1 - \left(3x + \frac{9x^2}{2} + \dots \right)}{3x}$$

$$= \frac{-\left(\frac{9}{2}x^2 + \frac{(3x)^3}{3!} + \dots \right)}{9x^2} = \frac{-1}{2}$$

$$\therefore \alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$$

Question 57

$$\lim_{x \rightarrow 0} \left(\frac{(x + 2 \cos x)^3 + 2(x + 2 \cos x)^2 + 3 \sin(x + 2 \cos x)}{(x + 2)^3 + 2(x + 2)^2 + 3 \sin(x + 2)} \right)^{\frac{100}{x}} \text{ is equal to } \underline{\hspace{2cm}}.$$

[28-Jul-2022-Shift-1]

Answer: 1

Solution:

$$\text{Let } x + 2 \cos x = a$$

$$x + 2 = b$$

$$\text{as } x \rightarrow 0, a \rightarrow 2 \text{ and } b \rightarrow 2$$

$$\lim_{x \rightarrow 0} \left(\frac{a^3 + 2a^2 + 3 \sin a}{b^3 + 2b^2 + 3 \sin b} \right)^{\frac{100}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{100}{x} \cdot \frac{(a^3 - b^3) + 2(a^2 - b^2) + 3(\sin a - \sin b)}{b^3 + 2b^2 + 3 \sin b}$$

$$\therefore \lim_{x \rightarrow 0} \frac{a - b}{x} = \lim_{x \rightarrow 0} \frac{2(\cos x - 1)}{x} = 0$$

$$= e^0$$

$$= 1$$

Question 58

If $\lim_{x \rightarrow 0} \frac{e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$, where $\alpha, \beta, \gamma \in \mathbb{R}$, then which of the following is

NOT correct?

[29-Jul-2022-Shift-1]

Options:

A. $\alpha^2 + \beta^2 + \gamma^2 = 6$

B. $\alpha\beta + \beta\gamma + \gamma\alpha + 1 = 0$

C. $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 0$

D. $\alpha^2 - \beta^2 + \gamma^2 = 4$

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$$

$\Rightarrow \alpha + \beta = 0$ (to make indeterminant form) (i)

Now,

$$\lim_{x \rightarrow 0} \frac{\alpha e^x - \beta e^{-x} + \gamma \cos x}{3x^2} = \frac{2}{3} \text{ (Using L-H Rule)}$$

$\Rightarrow \alpha - \beta + \gamma = 0$ (to make indeterminant form) (ii)

Now,

$$\lim_{x \rightarrow 0} \frac{\alpha e^x + \beta e^{-x} - \gamma \sin x}{6x} = \frac{2}{3} \text{ (Using L-H Rule)}$$

$$\Rightarrow \frac{\alpha - \beta + \gamma}{6} = \frac{2}{3}$$

$\Rightarrow \alpha - \beta + \gamma = 4$ (iii)

$\Rightarrow \gamma = -2$

and (i) + (ii)

$$2\alpha = -\gamma$$

$\Rightarrow \alpha = 1$ and $\beta = -1$

$$\text{and } \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 1 - 4 - 2 + 3 = -2$$

Question59

The value of $\log_e 2 \frac{d}{dx} (\log_{\cos x} \operatorname{cosec} x)$ at $x = \frac{\pi}{4}$ is
[26-Jul-2022-Shift-2]

Options:

A. $-2\sqrt{2}$

B. $2\sqrt{2}$

C. -4

D. 4

Answer: D

Solution:

Solution:

$$\text{Let } f(x) = \log_{\cos x} \operatorname{cosec} x$$

$$= \frac{\log \operatorname{cosec} x}{\log \cos x}$$

$$\Rightarrow f'(x) = \frac{\log \cos x \cdot \sin x \cdot \left(-\operatorname{cosec} x \cot x - \log \operatorname{cosec} x \cdot \frac{1}{\cos x} \cdot \sin x \right)}{(\log \cos x)^2}$$

$$\text{at } x = \frac{\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \frac{-\log\left(\frac{1}{\sqrt{2}}\right) + \log\sqrt{2}}{\left(\log\frac{1}{\sqrt{2}}\right)^2} = \frac{2}{\log\sqrt{2}}$$

$$\therefore \log_e 2 f'(x) \text{ at } x = \frac{\pi}{4} = 4$$

Question60

Let $x(t) = 2\sqrt{2} \cos t \sqrt{\sin 2t}$ and

$y(t) = 2\sqrt{2} \sin t \sqrt{\sin 2t}$, $t \in \left(0, \frac{\pi}{2}\right)$

Then $\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$ at $t = \frac{\pi}{4}$ is equal to:

[28-Jul-2022-Shift-2]

Options:

A. $\frac{-2\sqrt{2}}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

D. $\frac{-2}{3}$

Answer: D

Solution:

Solution:

$$x = 2\sqrt{2} \cos t \sqrt{\sin 2t}, y = 2\sqrt{2} \sin t \sqrt{\sin 2t}$$

$$\therefore \frac{dx}{dt} = \frac{2\sqrt{2} \cos 3t}{\sqrt{\sin 2t}}, \frac{dy}{dt} = \frac{2\sqrt{2} \sin 3t}{\sqrt{\sin 2t}}$$

$$\therefore \frac{dy}{dx} = \tan 3t, \left(\text{at } t = \frac{\pi}{4}, \frac{dy}{dx} = -1 \right)$$

$$\text{and } \frac{d^2y}{dx^2} = 3 \sec^2 3t \cdot \frac{dt}{dx} = \frac{3 \sec^2 3t \cdot \sqrt{\sin 2t}}{2\sqrt{2} \cos 3t}$$

$$\left(\text{At } t = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -3 \right)$$

$$\therefore \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = \frac{2}{-3} = \frac{-2}{3}$$

Question61

For the curve $C : (x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$, the value of $3y' - y^3y''$, at the point (α, α) , $\alpha > 0$, on C , is equal to _____.

[27-Jul-2022-Shift-2]

Answer: 16

Solution:

Solution:

$$\because C : (x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0 \text{ for point } (\alpha, \alpha)$$

$$\alpha^2 + \alpha^2 - 3 + (\alpha^2 - \alpha^2 - 1)^5 = 0$$

$$\therefore \alpha = \sqrt{2}$$

$$\text{On differentiating } (x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0 \text{ we get}$$

$$x + yy' + 5(x^2 - y^2 - 1)^4(x - yy') = 0$$

$$\text{When } x = y = \sqrt{2} \text{ then } y' = \frac{3}{2}$$

$$\text{Again on differentiating eq. (i) we get :}$$

$$1 + (y')^2 + yy'' + 20(x^2 - y^2 - 1)(2x - 2yy')(x - yy') + 5(x^2 - y^2 - 1)^4(1 - y'^2 - yy'') = 0$$

$$\text{For } x = y = \sqrt{2} \text{ and } y' = \frac{3}{2} \text{ we get } y'' = -\frac{23}{4\sqrt{2}}$$

$$\therefore 3y' - y^3y'' = 3 \cdot \frac{3}{2} - (\sqrt{2})^3 \cdot \left(-\frac{23}{4\sqrt{2}}\right) = 16$$

Question62

If $\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)}$ exists and is equal to b , then the value of $a - 2b$ is

[2021, 25 Feb. Shift-11]

Answer: 5

Solution:

$$\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)} = L \text{ (say) } \left[\frac{0}{0} \text{ form} \right]$$

Apply L - Hospital rule,

$$L = \lim_{x \rightarrow 0} \frac{a - 4e^{4x}}{a(e^{4x} - 1) + ax(4e^{4x})}$$

[Limit exist everywhere except $a = 4$]

Again, apply L-Hospital rule,

$$L = \lim_{x \rightarrow 0} \frac{-16e^{4x}}{a(4e^{4x}) + a(4e^{4x}) + ax(16e^{4x})}$$

$$= \frac{-16}{4a + 4a} = \frac{-2}{a}$$

$$= \frac{-2}{4} = \frac{-1}{2} \text{ (use } a = 4 \text{)}$$

Given, $L = b$

$$\Rightarrow \frac{-2}{a} = \frac{-1}{2} = b$$

Then, $a - 2b = 4 - 2\left(\frac{-1}{2}\right) = 4 + 1 = 5$

Question63

$\lim_{n \rightarrow \infty} 1 + \left(\frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n^2} \right)^n$ is equal to

[2021, 25 Feb. Shift-1]

Options:

A. $\frac{1}{2}$

B. 0

C. $\frac{1}{e}$

D. 1

Answer: D

Solution:

Solution:

Given, limit form is 1^∞ .

$$L = e^{\lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{n} \right)}$$

$$S = 1 + \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \dots$$

Clearly,

$$S < 1 + \left(\frac{1}{2} + \frac{1}{2} \right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) + \dots +$$

$$\left(\frac{1}{2^n} + \dots + \frac{1}{2^n} \right)_{2^n \text{ times}}$$

$$S < 1 + 1 + 1 + 1 + \dots + 1$$

$$S < n + 1$$

$$\therefore L = e^{\lim_{n \rightarrow \infty} \left(\frac{n+1}{2^{n+1}-1} \right)} \Rightarrow L = e^0$$

$$\therefore L = 1$$

Question64

$\lim_{x \rightarrow 0} \frac{\int_0^x (\sin \sqrt{t}) dt}{x^3}$ is equal to

[2021, 24 Feb. Shift-1]

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. $\frac{1}{15}$

D. 0

Answer: A

Solution:

Solution:

Given, $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3}$

$\therefore$ It is of the form $\frac{0}{0}$.

By differentiating numerator and denominator,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin \sqrt{x^2} \cdot 2x}{3x^2} &= \lim_{x \rightarrow 0} \frac{\sin x \cdot 2x}{3x^2} \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{2}{3} (1) = \frac{2}{3} \end{aligned}$$

Question65

$\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to

[2021, 24 Feb. Shift-I]

Answer: 1

Solution:

Solution:

Given, $\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$

$$= \tan \left(\lim_{n \rightarrow \infty} \sum_{r=1}^n [\tan^{-1}(r+1) - \tan^{-1}r] \right)$$

$$= \tan \left(\lim_{n \rightarrow \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right)$$

$$= \tan \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1$$

Hence, the required value is 1.

Question66

The value of $\lim_{h \rightarrow 0} 2 \left\{ \frac{\sqrt{3} \sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h(\sqrt{3} \cosh - \sin h)} \right\}$

[2021, 26 Feb. Shift-1]

Options:

- A. $\frac{4}{3}$
 B. $\frac{2}{\sqrt{3}}$
 C. $\frac{3}{4}$
 D. $\frac{2}{3}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{h \rightarrow 0} 2 \left\{ \frac{\sqrt{3} \sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h(\sqrt{3} \cosh - \sin h)} \right\} \\ &= \lim_{h \rightarrow 0} 2 \left\{ \frac{2 \left(\frac{\sqrt{3}}{2} \sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2} \cos\left(\frac{\pi}{6} + h\right) \right)}{2 \times \sqrt{3}h \left(\frac{\sqrt{3}}{2} \cosh - \frac{1}{2} \sin h \right)} \right\} \\ &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\cos\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h \left(\cos \frac{\pi}{6} \cosh - \sin \frac{\pi}{6} \sin h \right)} \right\} \\ &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\cos\left(\frac{\pi}{6} + h\right)}{6} \right\} \\ &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\sin\left(\frac{\pi}{6} + h - \frac{\pi}{6}\right)}{\sqrt{3}h \cos\left(h + \frac{\pi}{6}\right)} \right\} \\ &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{3}} \left\{ \frac{\sinh}{h \cos(h + \pi/6)} \right\} \\ &= \frac{2}{\sqrt{3}} \cdot \lim_{h \rightarrow 0} \frac{\sinh}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{\cos(h + \pi/6)} \\ &= \frac{2}{\sqrt{3}} \cdot (1) \cdot \frac{1}{\cos(\pi/6)} \\ &= \frac{2}{\sqrt{3}} \cdot 1 \cdot \frac{2}{\sqrt{3}} = \frac{4}{3} \end{aligned}$$

Question67

The value of $\lim_{n \rightarrow \infty} \frac{[r] + [2r] + \dots + [nr]}{n^2}$, where r is non-zero real number and $[r]$ denotes the greatest integer less than or equal to r , is equal to
 [2021, 17 March Shift-11]

Options:

A. $\frac{r}{2}$

B. r

C. $2r$

D. 0

Answer: A

Solution:

Solution:

As, we know that,

$$r \leq [r] < r + 1$$

$$2r \leq [2r] < 2r + 1$$

$$3r \leq [3r] < 3r + 1$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$nr \leq [nr] < nr + 1$$

$$\text{Adding } (r + 2r + 3r + \dots + nr) \leq [r] + [2r] + [3r]$$

$$+ [4r] + \dots + [nr] < (r + 1) + (2r + 1) + (3r + 1)$$

$$+ (4r + 1) + \dots + (nr + 1)$$

$$\Rightarrow r(1 + 2 + 3 + 4 + \dots + n) \leq [r] + [2r]$$

$$+ [3r] + \dots + [nr]$$

$$< (r + 2r + 3r + \dots + nr) + (1 + 1 + 1 + \dots + 1)_{n \text{ times}}$$

$$\Rightarrow r \cdot \frac{n(n+1)}{2} \leq [r] + [2r] + [3r] + \dots + [nr]$$

$$< \frac{r \cdot (n(n+1))}{2} + n$$

$$\Rightarrow \frac{r \cdot \left(\frac{n(n+1)}{2} \right)}{n^2}$$

$$\leq \frac{[r] + [2r] + [3r] + \dots + [nr]}{n^2}$$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{n(n+1) \cdot r}{2 \cdot n^2} = \lim_{n \rightarrow \infty} \frac{n \cdot n \left(1 + \frac{1}{n} \right) \cdot r}{2n^2}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{(1+0) \cdot r}{2} = \frac{r}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(n + \frac{n(n+1)}{2} \right) + n}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 \left\{ \left(1 + \frac{1}{n} \right) \cdot r + 2n + \frac{2}{n} \right\}} \dots (i)$$

$$= \frac{(1+0) \cdot r + 0}{2} = \frac{r}{2} \dots (ii)$$

$$\text{From Eqs. (i) and (ii), by Sandwich theorem, we conclude that, } \lim_{n \rightarrow \infty} \frac{[r] + [2r] + [3r] + \dots + [nr]}{n^2} = \frac{r}{2}$$

Sandwich Theorem

$$\Rightarrow \text{Let } g(x) \leq f(x) \leq h(x)$$

$$\text{and } \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = 1$$

$$\therefore \lim_{x \rightarrow a} f(x) = 1$$

Question68

$$\text{If } \lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2, \text{ then } a + b + c \text{ is equal to } \dots\dots\dots$$

[2021, 16 March Shift-1]

Answer: 4

Solution:

Solution:

$$\text{We have, } \lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$$

$$\Rightarrow a \left(1 + x + \frac{x^2}{2!} \dots \right) - b \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \dots \right)$$

$$+ c \left(1 - x + \frac{x^2}{2!} \dots \right)$$

$$\lim_{x \rightarrow 0} \frac{x \left(x - \frac{x^3}{3!} + \dots \right)}{(a - b + c) + (a - c)x}$$

$$+ \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) x^2 + \dots$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 - \frac{x^4}{6} + \dots}{x^2 - \frac{x^4}{6} + \dots}$$

Here, in numerator, all the coefficients of x^k , where $k < 2$ has to be zero, then only limit will exist.

$$a - b + c = 0$$

$$a - c = 0$$

$$\Rightarrow a = c$$

$$\Rightarrow b = 2a$$

After solving limit,

$$\frac{a + b + c}{2} = 2$$

$$\text{So, } a + 2a + a = 4$$

$$\Rightarrow a = 1$$

$$\therefore a = 1, b = 2 \text{ and } c = 1$$

$$a + b + c = 1 + 2 + 1 = 4$$

Question 69

If $\lim_{x \rightarrow 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$ is equal to L, then the value of $(6L + 1)$ is

[2021, 18 March Shift-1]

Options:

A. $\frac{1}{6}$

B. $\frac{1}{2}$

C. 6

D. 2

Answer: D

Solution:

Solution:

$$\text{Given, } L = \lim_{x \rightarrow 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$$

$\Rightarrow L$

$$\left(x + \frac{1^2 \cdot x^3}{3!} + \frac{1^2 \cdot 3^2 \cdot x^5}{5!} + \dots \right)$$

$$= \lim_{x \rightarrow 0} \frac{- \left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots \right)}{3x^3}$$

$\Rightarrow$ (using expansion of $\sin^{-1}x$ and $\tan^{-1}x$)

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{x^3}{3!} + \frac{9x^5}{5!} + \dots \right) - \left(-\frac{x^3}{3} + \frac{x^5}{5} \dots \right)}{3x^3}$$

$\Rightarrow L$

$$= \lim_{x \rightarrow 0} \frac{x^3 \left[\left(\frac{1}{6} + \frac{9x^2}{120} + \dots \right) + \left(\frac{1}{3} - \frac{x^2}{5} + \dots \right) \right]}{3x^3}$$

$$\Rightarrow L = \frac{\frac{1}{6} + \frac{1}{3}}{3} = \frac{\frac{1+2}{6}}{3} = \frac{1}{6}$$

$$\therefore 6L + 1 = 6 \times \frac{1}{6} + 1 = 2$$

Question70

The value of $\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \cdot \sin^{-1}(x - [x]^2)}{x - x^3}$ where $[x]$ denotes the greatest integer $\leq x$ is
[2021, 17 March Shift-1]

Options:

A. π

B. 0

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \cdot \sin^{-1}(x - [x]^2)}{x - x^3}$$

$x \rightarrow 0 + h$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(h - 0) \cdot \sin^{-1}(h - 0)}{h - h^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}h \cdot \sin^{-1}h}{h(1 - h)(1 + h)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sin^{-1}h}{h} \right) \left[\frac{\cos^{-1}h}{(1 - h)(1 + h)} \right] = 1 \cdot \frac{\pi}{2}$$

$$\text{RHL} = \frac{\pi}{2}$$

Question71

The value of the limit $\lim_{\theta \rightarrow 0} \frac{\tan(n\cos^2\theta)}{\sin(2n\sin^2\theta)}$ is equal to

[2021, 17 March Shift-II]

Options:

A. $-\frac{1}{2}$

B. $-\frac{1}{4}$

C. 0

D. $\frac{1}{4}$

Answer: A

Solution:

Solution:

Method (I)

$$\begin{aligned}\text{Let } L &= \lim_{\theta \rightarrow 0} \left(\frac{\tan(n\cos^2\theta)}{\sin(2n\sin^2\theta)} \right) \\&= \lim_{\theta \rightarrow 0} \left(\frac{\tan[\pi(1 - \sin^2\theta)]}{\sin(2n\sin^2\theta)} \right) \\&= \lim_{\theta \rightarrow 0} \left(\frac{\tan(\pi - n\sin^2\theta)}{\sin(2n\sin^2\theta)} \right) \\&= \lim_{\theta \rightarrow 0} \left(\frac{-\tan(n\sin^2\theta)}{\sin(2n\sin^2\theta)} \right) \\&= \lim_{\theta \rightarrow 0} \left[\frac{\frac{-\tan(n\sin^2\theta)}{(n\sin^2\theta)} \times (n\sin^2\theta)}{\frac{\sin(2n\sin^2\theta)}{(2n\sin^2\theta)} \times (2n\sin^2\theta)} \right] = \frac{-1}{2}\end{aligned}$$

Method (II)

$$\begin{aligned}\text{Let } L &= \lim_{\theta \rightarrow 0} \left[\frac{\tan(n\cos^2\theta)}{\sin(2n\sin^2\theta)} \right] \left(\text{Form } \frac{0}{0} \right) \\[\text{Using L-Hospital Rule}] \\L &= \lim_{\theta \rightarrow 0} \frac{\sec^2(n\cos^2\theta)(-2n\cos\theta \cdot \sin\theta)}{\cos(2n\sin^2\theta) \cdot (4n\sin\theta \cdot \cos\theta)} \\&= \frac{-1}{2} \times \frac{(-1)^2}{1} = \frac{-1}{2}\end{aligned}$$

Question 72

If $f(x) = \sin \left(\cos^{-1} \left(\frac{1 - 2^{2x}}{1 + 2^{2x}} \right) \right)$ and its first derivative with respect to x is

$-\frac{b}{a} \log_e 2$ when $x = 1$, where a and b are integers, then the minimum value of $|a^2 - b^2|$ is

[2021, 17 March Shift-1]

Answer: 481

Solution:

Solution:

$$f(x) = \sin \left(\cos^{-1} \left(\frac{1 - 2^{2x}}{1 + 2^{2x}} \right) \right)$$

Let 2^{2x} be $\tan^2 \theta$.

$$\therefore f(x) = \sin \left[\cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \right]$$

$$= \sin[\cos^{-1}(\cos 2\theta)] = \sin 2\theta$$

$$= \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \cdot 2^x}{1 + 2^{2x}}$$

$$f(x) = 2 \cdot \left(\frac{2^x}{1 + 2^{2x}} \right)$$

$$f'(x) = 2 \cdot \left[\frac{(1 + 2^{2x})(2^x \log 2) - 2^x(2^{2x} \log 2 \cdot 2)}{(1 + 2^{2x})^2} \right]$$

$$f'(1) = 2 \left(\frac{5 \cdot 2 \log 2 - 2 \cdot 8 \log 2}{5^2} \right)$$

$$= \left(-\frac{12}{25} \right) \log 2$$

$$= \frac{-b}{a} \log e 2$$

$$\Rightarrow a = 25 \text{ and } b = 12$$

$$\therefore |a^2 - b^2|_{\min} = |25^2 - 12^2| = 481$$

Question 73

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of $\lim_{x \rightarrow 2} \frac{x^2 f(2) - 4f(x)}{x - 2}$ is equal to

[2021, 27 July Shift-1]

Options:

A. 4

B. 8

C. 16

D. 12

Answer: D

Solution:

Solution:

$$f(2) = 4, f'(2) = 1$$

$$\text{Now, } \lim_{x \rightarrow 2} \frac{x^2 f(2) - 4f(x)}{x - 2}$$

Applying L-Hospital Rule as $\frac{0}{0}$ form on putting $x = 2$

$$\text{So, } \lim_{x \rightarrow 2} \frac{2x f(2) - 4f(x)}{1}$$

$$= 2 \cdot 2 \cdot f(2) - 4f'(2)$$

$$= 4 \cdot 4 - 4 \cdot 1 = 12$$

Question 74

If $\lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x} = 10$, $\alpha, \beta, \gamma \in \mathbb{R}$, then the value of $\alpha + \beta + \gamma$ is

[2021, 20 July Shift-2]

Answer: 3

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x} = 10$$

$$\text{Now, } \lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x}$$

$$\alpha x(1+x+x^2/2+\dots) - \beta \left(x - \frac{x^2}{2} + \frac{x^3}{3} \dots \right) - \gamma x^2(1-x+x^2/2+\dots)$$
$$= \lim_{x \rightarrow 0} \frac{x(\alpha + \beta) + x^2(\alpha + \beta/2 + \gamma) + x^3 \left(\frac{\alpha}{2} - \frac{\beta}{3} - \gamma \right) \dots}{x \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{x(\alpha + \beta) + x^2(\alpha + \beta/2 + \gamma) + x^3 \left(\frac{\alpha}{2} - \frac{\beta}{3} - \gamma \right) \dots}{x \sin^2 x}$$

For limit to exist, the numerator must have degree greater than or equal to denominator.

Degree of denominator = 3

$\therefore$ For limit to exist, $\alpha - \beta = 0 \dots$ (i)

and $\alpha + \frac{\beta}{2} + \gamma = 0 \dots$ (ii)

Also, for terms greater than degree '3', gives 0 as $x \rightarrow 0$

$\therefore \frac{\alpha}{2} - \frac{\beta}{3} - \gamma = 10 \dots$ (iii)

From Eq. (i), $\beta = \alpha$

From Eq. (ii),

$$\gamma = \left(\frac{\alpha}{2} + \alpha \right) = -\frac{3\alpha}{2}$$

Putting these in Eq. (iii),

$$\frac{\alpha}{2} - \frac{\alpha}{3} + \frac{3\alpha}{2} = 10 \Rightarrow \frac{3\alpha - 2\alpha + 9\alpha}{6} = 10$$

$$\Rightarrow \frac{10\alpha}{6} = 10 \Rightarrow \alpha = 6$$

Now, $\alpha = \beta \Rightarrow \beta = 6$

$$\text{Again, } \gamma = \frac{-3\alpha}{2} \Rightarrow \gamma = -9$$

$$\therefore \alpha + \beta + \gamma = 6 + 6 - 9 = 12 - 9 = 3$$

Question 75

The value of $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}} \right)$ is equal to

[2021, 27 July Shift-II]

Options:

A. 0

B. 4

C. -4

D. -1

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$$

Rationalise denominator three times, $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$

$$\left(\frac{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}}{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}} \right)$$

$$\left(\frac{\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x}}{\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x}} \right)$$

$$\left(\frac{\sqrt{1 - \sin x} + \sqrt{1 + \sin x}}{\sqrt{1 - \sin x} + \sqrt{1 + \sin x}} \right)$$

$$= \lim_{x \rightarrow 0} \left[\frac{x}{(1 - \sin x) - (1 + \sin x)} \right]$$

$$\left(\frac{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}}{(\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x})(\sqrt{1 - \sin x} + \sqrt{1 + \sin x})} \right)$$

$$= \lim_{x \rightarrow 0} \left[\frac{x}{-2 \sin x} \right] \left(\frac{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}}{(\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x})(\sqrt{1 - \sin x} + \sqrt{1 + \sin x})} \right)$$

$$= \left(-\frac{1}{2} \right) (2)(2)(2) \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= -4$$

Question 76

If the value of $\lim_{x \rightarrow 0} (2 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$ is equal to e^a , then a is equal

to

[2021, 20 July Shift-1]

Answer: 3

Solution:

Solution:

$$\lim_{x \rightarrow 0} (2 - \cos x \sqrt{\cos 2x}) \frac{x+2}{x^2} = 1^\infty$$

$$\Rightarrow \lim_{x \rightarrow 0} (1 + 1 - \cos x \sqrt{\cos 2x}) \frac{x+2}{x^2}$$

$$\Rightarrow e^{x \rightarrow 0} (1 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$$

$$\Rightarrow \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} -$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \right) \left(1 - 2x^2 + \frac{2}{3}x^4 \dots \right) \frac{1}{2}$$

We have to extract till the coefficient of x^2 as denominator is x^2 .

$$\text{So, } \left(1 - \frac{x^2}{2} \right) (1 - 2x^2) \frac{1}{2} = \left(1 - \frac{x^2}{2} \right) (1 - x^2)$$

$$= \left(1 - \frac{x^2}{2} - x^2 + \frac{x^4}{2} \right) = \left(1 - \frac{3x^2}{2} \right)$$

$$\text{So, } \lim_{x \rightarrow 0} (1 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$$

$$= \lim_{x \rightarrow 0} \left[1 - \left(\frac{3x^2}{2} \right) \right] \left(\frac{x+2}{x^2} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{3x^2}{2} \right) \left(\frac{x+2}{x^2} \right) = e^3$$

$$\therefore e^a = e^3 \Rightarrow a = 3$$

Question 77

If α_1, β are the distinct roots of $x^2 + bx + c = 0$, then $\lim_{x \rightarrow \beta} \frac{e^{2(x^2 + bx + c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2}$ is equal to
[2021, 27 Aug. Shift-1]

Options:

A. $b^2 + 4c$

B. $2(b^2 + 4c)$

C. $2(b^2 - 4c)$

D. $b^2 - 4c$

Answer: C

Solution:

Solution:

$\because \alpha, \beta$ are distinct roots of $x^2 + bx + c = 0$

$$\Rightarrow x^2 + bx + c = (x - \alpha)(x - \beta) = 0$$

$$\text{Now, } \lim_{x \rightarrow \beta} \frac{e^{2(x^2 + bx + c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2}$$

$$= \lim_{x \rightarrow \beta} \frac{e^{2(x - \alpha)(x - \beta)} - 1 - 2(x - \alpha)(x - \beta)}{(x - \beta)^2}$$

$$= \lim_{h \rightarrow 0} \frac{e^{2(\beta - \alpha + h)h} - 1 - 2(\beta - \alpha + h)h}{h^2}$$

$$1 + 2(\beta - \alpha + h)h + \frac{[2(\beta - \alpha + h)h]^2}{2!}$$

$$= \lim_{h \rightarrow 0} \frac{+ \dots - 1 - 2h(\beta - \alpha + h)}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{2(\beta - \alpha + h)^2 h^2 + \dots}{h^2}$$

$$= 2(\beta - \alpha)^2 = 2(b^2 - 4c)$$

Question 78

If $\lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) = b$, then the ordered pair (a, b) is
[2021, 27 Aug. Shift-2]

Options:

A. $\left(1, \frac{1}{2}\right)$

B. $\left(1, -\frac{1}{2}\right)$

C. $\left(-1, \frac{1}{2}\right)$

D. $\left(-1, -\frac{1}{2}\right)$

Answer: B

Solution:

Solution:

Given, $\lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) = b$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax)$$

$$\frac{(\sqrt{x^2 - x + 1}) + ax}{(\sqrt{x^2 - x + 1} + ax)} = b$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^2 - x + 1 - a^2 x^2}{\sqrt{x^2 - x + 1} + ax} = b$$

Limit exists only if $a^2 = 1$

$$\therefore \lim_{x \rightarrow \infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} + ax} = \pm$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{-1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a} = b \Rightarrow \frac{-1}{1 + a} = b$$

But $a \neq -1$
 $a = 1$

$$b = -\frac{1}{2} \quad (a, b) = \left(1, -\frac{1}{2}\right)$$

Question 79

$\lim_{x \rightarrow 2} \left(\sum_{n=1}^9 \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4} \right)$ is equal to
[2021, 26 Aug. Shift-II]

Options:

A. $\frac{9}{44}$

B. $\frac{5}{24}$

C. $\frac{1}{5}$

D. $\frac{7}{36}$

Answer: A

Solution:

Solution:

We have,

$$\begin{aligned} S &= \lim_{x \rightarrow 2} \sum_{n=1}^9 \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4} \\ &= \sum_{n=1}^9 \frac{2}{4(n^2 + 3n + 2)} \\ &= \frac{1}{2} \sum_{n=1}^9 \frac{(n+2) - (n+1)}{(n+1)(n+2)} \\ &= \frac{1}{2} \sum_{n=1}^9 \left[\frac{1}{n+1} - \frac{1}{n+2} \right] \\ &= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{10} - \frac{1}{11} \right) \right] \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{11} \right) = \frac{1}{2} \times \left(\frac{11-2}{2 \times 11} \right) = \frac{9}{44} \end{aligned}$$

Question80

$\lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4}$ is equal to

[2021, 31 Aug. Shift-1]

Options:

A. π^2

B. $2\pi^2$

C. $4\pi^2$

D. 4π

Answer: C

Solution:

Solution:

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2[\pi(1 - \cos^4 x)]}{[\pi(1 - \cos^4 x)]^2} \cdot \frac{\pi^2(1 - \cos^4 x)^2}{x^4} \\ &= \lim_{x \rightarrow 0} \pi^2 \frac{\sin^4 x(1 + \cos^2 x)^2}{x^4} \end{aligned}$$

$$= \lim_{x \rightarrow 0} \pi^2 (1 + \cos^2 x)^2 = 4\pi^2$$

Question81

If $\alpha = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ and $\beta = \lim_{x \rightarrow 0} (\cos x)^{\cot x}$ are the roots of the quation,

$ax^2 + bx - 4 = 0$, then the ordered pair (a, b) is
[2021, 31 Aug. Shift-II]

Options:

- A. (1, -3)
- B. (-1, 3)
- C. (-1, -3)
- D. (1, 3)

Answer: D

Solution:

Solution:

$$\begin{aligned} \alpha &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x (\tan x + 1)(\tan x - 1)}{\cos\left(x + \frac{\pi}{4}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\sin x}{\cos x} \cdot \left(\frac{\sin x - \cos x}{\cos x}\right) \left(\frac{\sin x + \cos x}{\cos x}\right)}{\frac{1}{\sqrt{2}}(\cos x - \sin x)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-\sqrt{2} \sin x (\sin x + \cos x)}{\cos 3x} \\ &= \frac{-\sqrt{2} \times \frac{1}{\sqrt{2}} \times \sqrt{2}}{\frac{1}{2\sqrt{2}}} = -4 \end{aligned}$$

$$\begin{aligned} \text{and } \beta &= \lim_{x \rightarrow 0} (\cos x)^{\cot x} \\ &= e^{\lim_{x \rightarrow 0} \frac{\tan x}{\cos x - 1}} \\ &= e^{x \rightarrow 0} - \frac{\sin x}{\sec x} = e^0 = 1 \end{aligned}$$

Equation whose roots are x and β , is

$$\begin{aligned} x^2 + 3x - 4 &= 0 \\ \therefore 1 \cdot b &= 3 \end{aligned}$$

Question82

If $y^{1/4} + y^{-1/4} = 2x$, and $(x^2 - 1) \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0$ then $|\alpha - \beta|$ is equal to

[2021, 27 Aug. Shift-1]

Answer: 17

Solution:

Solution:

$$\text{Given, } y^{\frac{1}{4}} + y^{-\frac{1}{4}} = 2x$$

$$\Rightarrow (y^{\frac{1}{4}} + y^{-\frac{1}{4}})^2 = (2x)^2$$

$$\Rightarrow (y^{\frac{1}{4}} + y^{-\frac{1}{4}})^2 = 4x^2$$

Differentiating w.r.t. x , we get

$$\frac{1}{4y} \left(y^{\frac{1}{4}} - y^{-\frac{1}{4}} \right) \frac{dy}{dx} = 2$$

$$\Rightarrow \left(y^{\frac{1}{4}} - y^{-\frac{1}{4}} \right) \frac{dy}{dx} = 8y \dots (i)$$

$$\text{Now, } y^{\frac{1}{4}} - y^{-\frac{1}{4}} = \frac{1}{y^{\frac{1}{4}} + y^{-\frac{1}{4}}} \left(y^{\frac{1}{4}} + y^{-\frac{1}{4}} \right)^2 - 4$$

$$= \frac{1}{y^{\frac{1}{4}} + y^{-\frac{1}{4}}} \left(y^{\frac{1}{4}} + y^{-\frac{1}{4}} \right)^2 - 4$$

$$\Rightarrow y^{\frac{1}{4}} - y^{-\frac{1}{4}} = 2\sqrt{x^2 - 1} \dots (ii)$$

$$\Rightarrow \left(\sqrt{x^2 - 1} \right) \frac{dy}{dx} = 4y$$

[using Eqs. (i) and (ii)] Squaring on both sides,

$$(x^2 - 1) \left(\frac{dy}{dx} \right)^2 = 16y^2$$

Again, differentiating w.r.t. x

$$(x^2 - 1) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx} \right)^2 = 32y \frac{dy}{dx}$$

On dividing by $\frac{2dy}{dx}$, we get

$$(x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 16y$$

$$\text{or } (x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 16y = 0$$

Comparing with

$$(x^2 - 1) \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0$$

$$\therefore \alpha = 1, \beta = -16$$

$$\therefore |\alpha - \beta| = |1 + 16| = 17$$

Question83

• If $y = y(x)$ is an implicit function of x such that $\log_e(x + y) = 4xy$, then

$\frac{d^2y}{dx^2}$ at $x = 0$ is equal to

[2021, 26 Aug. Shift-1]

Answer: 40

Solution:

Solution:

We have, $\ln(x + y) = 4xy$

$$\Rightarrow x + y = e^{4xy}$$

$$\Rightarrow 1 + \frac{dy}{dx} = \left(4x \frac{dy}{dx} + 4y \right) e^{4xy}$$

If $x = 0$, then $y = 1$

$$\text{At } (0, 1), \quad \frac{dy}{dx} = 3$$

$$\frac{d^2y}{dx^2} = e^{4xy} \left(4x \frac{dy}{dx} + 4y \right)^2$$

$$+ e^{4xy} \left(4x \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4 \frac{dy}{dx} \right)$$

At $x = 0$,

$$\frac{d^2y}{dx^2} = 16 + 24 = 40$$

Question 84

Let $f(x) = \cos \left(2 \tan^{-1} \sin \left(\cot^{-1} \sqrt{\frac{1-x}{x}} \right) \right)$ $0 < x < 1$. Then,

[2021, 26 Aug. Shift-1]

Options:

A. $(1-x)^2 f'(x) - 2(f(x))^2 = 0$

B. $(1+x)^2 f'(x) + 2(f(x))^2 = 0$

C. $(1-x)^2 f'(x) + 2(f(x))^2 = 0$

D. $(1+x)^2 f'(x) - 2f(x)^2 = 0$

Answer: C

Solution:

Solution:

$$f(x) = \cos \left(2 \tan^{-1} \sin \left(\cot^{-1} \sqrt{\frac{1-x}{x}} \right) \right)$$

$$\cot^{-1} \sqrt{\frac{1-x}{x}} = \sin^{-1} \sqrt{x}$$

$$\therefore f(x) = \cos(2 \tan^{-1} \sin \sin^{-1} \sqrt{x})$$

$$\text{or } f(x) = \cos(2 \tan^{-1} \sqrt{x})$$

$$= \cos \tan^{-1} \left(\frac{2\sqrt{x}}{1-x} \right)$$

$$f(x) = \cos \cos^{-1} \left(\frac{1-x}{1+x} \right)$$

$$f(x) = \frac{1-x}{1+x}$$

$$f'(x) = \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$f'(x)(1-x)^2 = -2 \left(\frac{1-x}{1+x} \right)^2$$

$$(1-x)^2 f'(x) + 2[f(x)]^2 = -2 \left(\frac{1-x}{1+x} \right)^2 + 2 \left(\frac{1-x}{1+x} \right)^2 = 0$$

$$+2\left(\frac{1-x}{1+x}\right)^2 = 0$$

Question85

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\pi}{4} \int^{\sec^2 x} f(x) dx}{x^2 - \frac{\pi^2}{16}}$ is equal to

[2021, 01 Sep. Shift-II]

Options:

- A. $f(2)$
- B. $2f(2)$
- C. $2f(\sqrt{2})$
- D. $4f(2)$

Answer: B

Solution:

Solution:

Using L-Hopital's rule

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\pi}{4} \cdot 2 \sec x \cdot \sec x \cdot \tan x \cdot f(\sec^2 x) - 0}{2x}$$

[using Leibnitz theorem]

$$= \frac{\frac{\pi}{4} \cdot 2(\sqrt{2})^2 \cdot (1)f(2)}{2 \cdot \frac{\pi}{4}} = 2f(2)$$

Question86

$\lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{\frac{x}{2}} - 3^{1-x}}$ is equal to _____.

[NA Jan. 7, 2020 (I)]

Answer: 36

Solution:

Solution:

Let $3^x = t^2$

$$\lim_{t \rightarrow 3} \frac{t^2 + \frac{27}{t^2} - 12}{\frac{1}{t} - \frac{3}{t^2}} = \lim_{t \rightarrow 3} \frac{t^4 - 12t^2 + 27}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{(t^2 - 3)(t + 3)(t - 3)}{t - 3}$$

$$= (3^2 - 3)(3 + 3) = 36$$

Question87

$$\lim_{x \rightarrow 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{1/x^2} \text{ is equal to:}$$

[Jan. 8, 2020 (I)]

Options:

A. $\frac{1}{e}$

B. $\frac{1}{e^2}$

C. e^2

D. e

Answer: B

Solution:

Solution:

$$\text{Let } R = \lim_{x \rightarrow 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{\frac{1}{x^2}} = e^{\lim_{x \rightarrow 0} \left\{ \frac{3x^2 + 2}{7x^2 + 2} - 1 \right\}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x^2} \left\{ \frac{-4x^2}{7x^2 + 2} \right\}} = e^{\frac{-4}{2}} = e^{-2} = \frac{1}{e^2}$$

Question88

$$\lim_{x \rightarrow 0} \int_0^x \frac{t \sin(10t) dt}{x} \text{ is equal to:}$$

[Jan. 8, 2020 (II)]

Options:

A. 0

B. $\frac{1}{10}$

C. $-\frac{1}{5}$

D. $-\frac{1}{10}$

Answer: A

Solution:

Solution:

Using L' Hospital rule,

$$\lim_{x \rightarrow 0} \frac{x \sin(10x)}{1} = 0$$

Question 89

If $\lim_{x \rightarrow 1} \frac{x + x^2 + x^3 + \dots + x^n - n}{x - 1} = 820, (n \in \mathbb{N})$ then the value of n is equal to

[NA Sep. 02, 2020 (I)]

Answer: 40

Solution:

Solution:

$$\lim_{x \rightarrow 1} \frac{x + x^2 + x^3 + \dots + x^n - n}{x - 1} = 820 \left(\frac{0}{0} \text{ case} \right)$$

$$\lim_{x \rightarrow 1} \frac{1 + 2x + 3x^2 + \dots + nx^{n-1} - 1}{1} = 820 \text{ (Using L' Hospital rule)}$$

$$\Rightarrow 1 + 2 + 3 + \dots + n = 820$$

$$\Rightarrow \frac{n(n+1)}{2} = 820 \Rightarrow n^2 + n - 1640 = 0$$

$$\Rightarrow n = 40, n \in \mathbb{N}$$

Question 90

$\lim_{x \rightarrow 0} \left(\tan \left(\frac{\pi}{4} + x \right) \right)^{1/x}$ is equal to :

[Sep. 02, 2020 (II)]

Options:

A. e

B. 2

C. 1

D. e^2

Answer: D

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 - \tan x} \right)^{1/x} \\ & \Rightarrow e^{\lim_{x \rightarrow 0} \frac{1}{x} [\tan(\frac{\pi}{4} + x) - 1]} = e^{\lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1 + \tan x}{1 - \tan x} - 1 \right)} \\ & \Rightarrow e^{\lim_{x \rightarrow 0} \left(\frac{2 \tan x}{1 - \tan x} \right) \frac{1}{x}} = e^{\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right) \left(\frac{2}{1 - \tan x} \right)} = e^2 \end{aligned}$$

Question91

If

$$\lim_{x \rightarrow 0} \left\{ \frac{1}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4} \right) \right\} = 2^{-k},$$

then the value of k is _____.

[NA Sep. 03, 2020 (I)]

Answer: 8

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\left(1 - \cos \frac{x^2}{2} \right) \left(1 - \cos \frac{x^2}{4} \right)}{x^4 \cdot x^4} = 2^{-k} \\ & \Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x^2}{4}}{\frac{x^4}{16} \times 16} \times \frac{2 \sin^2 \frac{x^2}{8}}{\frac{x^4}{64} \times 64} = 2^{-k} \\ & \Rightarrow \frac{4}{16 \times 64} = 2^{-8} = 2^{-k} \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ & \therefore k = 8 \end{aligned}$$

Question92

Let $[t]$ denote the greatest integer $\leq t$. If for some

$\lambda \in \mathbb{R} - \{0, 1\}$, $\lim_{x \rightarrow 0} \left| \frac{1 - x + |x|}{\lambda - x + [x]} \right| = L$, then L is equal to:

[Sep. 03, 2020 (I)]

Options:

A. 1

B. 2

C. $\frac{1}{2}$

D. 0

Answer: B

Solution:

Solution:

$$\text{Given } \lim_{x \rightarrow 0} \left| \frac{1-x+\lfloor x \rfloor}{\lambda-x+\lfloor x \rfloor} \right| = L$$

$$\text{Here, L.H.L.} = \lim_{h \rightarrow 0} \left| \frac{1+h+h}{\lambda+h-1} \right| = \left| \frac{1}{\lambda-1} \right|$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} \left| \frac{1-h+h}{\lambda+h+0} \right| = \left| \frac{1}{\lambda} \right|$$

Given that limit exists. Hence L.H.L. = R.H.L.

$$\Rightarrow |\lambda - 1| = |\lambda|$$

$$\Rightarrow \lambda = \frac{1}{2} \text{ and } L = \left| \frac{1}{\lambda} \right| = 2$$

Question93

$$\lim_{x \rightarrow a} \frac{(a+2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}} \text{ (a} \neq 0 \text{) is equal to}$$

[Sep. 03, 2020 (II)]

Options:

A. $\left(\frac{2}{3}\right)^{\frac{4}{3}}$

B. $\left(\frac{2}{3}\right)\left(\frac{2}{9}\right)^{\frac{1}{3}}$

C. $\left(\frac{2}{9}\right)^{\frac{4}{3}}$

D. $\left(\frac{2}{9}\right)\left(\frac{2}{3}\right)^{\frac{1}{3}}$

Answer: B

Solution:

Solution:

$$\lim_{x \rightarrow a} \frac{(a+2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}} \left[\frac{0}{0} \text{ case} \right]$$

Apply L'Hospital rule

$$\lim_{x \rightarrow a} \frac{\frac{1}{3}(a+2x)^{-2/3} \cdot 2 - \frac{1}{3} \cdot (3x)^{-2/3} \cdot 3}{\frac{1}{3}(3a+x)^{-2/3} \cdot 1 - \frac{1}{3}(4x)^{-2/3} \cdot 4}$$

$$= \frac{\frac{1}{3}(3a)^{-2/3} \cdot (2-3)}{\frac{1}{3}(4a)^{-2/3} \cdot (1-4)} = \frac{3^{-2/3} \cdot \frac{1}{3}}{4^{-2/3} \cdot \frac{1}{3}} = \frac{2^{4/3}}{9^{1/3}} \cdot \frac{1}{3} = \frac{2}{3} \cdot \left(\frac{2}{9}\right)^{1/3}$$

Question94

$\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to _

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Answer: 1

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tan \left(\sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r(r+1)} \right) \right) \\ &= \lim_{n \rightarrow \infty} \tan \left(\sum_{r=1}^n \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right) \right) \\ &= \tan \left(\lim_{n \rightarrow \infty} \sum_{r=1}^n [\tan^{-1}(r+1) - \tan^{-1}(r)] \right) \\ &= \tan \left(\lim_{n \rightarrow \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right) \\ &= \tan \left(\frac{\pi}{4} \right) = 1 \end{aligned}$$

Question95

Substitution & Rationalisation If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$, then $\lim_{x \rightarrow \alpha^+} \frac{\sqrt{1 - \cos(p(x))}}{x + \alpha - 4}$ is equal to:

[Sep. 05, 2020 (I)]

Options:

A. $\frac{3}{2}$

B. $\frac{3}{\sqrt{2}}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{1}{2}$

Answer: B

Solution:

Solution:

$$\begin{aligned} x^2 - x - 2 = 0 &\Rightarrow (x-2)(x+1) = 0 \\ \Rightarrow x = 2, -1 &\Rightarrow \alpha = 2 \end{aligned}$$

$$\begin{aligned}
 & \therefore \lim_{x \rightarrow 2^+} \frac{\sqrt{1 - \cos(x^2 - x - 2)}}{x - 2} \\
 &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2} \left| \sin \left(\frac{x^2 - x - 2}{2} \right) \right|}{x - 2} \\
 &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2} \sin(x^2 - x - 2) \cdot 2}{\left(\frac{x^2 - x - 2}{2} \right)} \times \frac{(x^2 - x - 2)}{2(x - 2)} \\
 &= \frac{1}{\sqrt{2}} \lim_{x \rightarrow 2^+} \left(\frac{\sin \left(\frac{x^2 - x - 2}{2} \right)}{\frac{x^2 - x - 2}{2}} \right) \times \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 1)}{(x - 2)} \\
 &= \frac{1}{\sqrt{2}} \times 1 \times 3 = \frac{3}{\sqrt{2}}
 \end{aligned}$$

Question96

$$\lim_{x \rightarrow 0} \frac{x(e^{\sqrt{1+x^2+x^4}-1}/x - 1)}{\sqrt{1+x^2+x^4}-1}$$

[Sep. 05, 2020 (II)]

Options:

- A. is equal to $\sqrt{e}$
- B. is equal to
- C. is equal to 0
- D. does not exist

Answer: B

Solution:

Solution:

$$\text{Let } L = \lim_{x \rightarrow 0} x \left(e^{\frac{\sqrt{1+x^2+x^4}-1}{x}} - 1 \right)$$

$$= \lim_{x \rightarrow 0} \frac{e^{\frac{\sqrt{1+x^2+x^4}-1}{x}} - 1}{\frac{\sqrt{1+x^2+x^4}-1}{x}}$$

$$\text{Put } \frac{\sqrt{1+x^2+x^4}-1}{x} = t \text{ when } x \rightarrow 0 \Rightarrow t \rightarrow 0$$

$$\therefore L = \lim_{t \rightarrow 0} \frac{e^t - 1}{t} = 1$$