



Integrals

Question 1

$6 \int_0^\pi |(\sin 3x + \sin 2x + \sin x)| dx$ is equal to ____ .

JEE Main 2026 (Online) 21st January Morning Shift

Answer: 17

Solution:

$$\text{Let, } I = \int_0^\pi |\sin 3x + \sin 2x + \sin x| dx$$

We know, $\sin 3x + \sin x = 2 \sin 2x \cos x$

$$\therefore \sin 3x + \sin 2x + \sin x = 2 \sin 2x \cos x + \sin 2x = \sin 2x(2 \cos x + 1)$$

$$I = \int_0^\pi |\sin 2x(2 \cos x + 1)| dx$$

Critical points : $\forall x \in [0, \pi]$

$$\sin 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi$$

$$\Rightarrow 2 \cos x + 1 = 0 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$$

$$2 \cos x + 1 > 0 \quad \forall x \in [0, 2\pi/3]$$

$$2 \cos x + 1 \leq 0 \quad \forall x \in [2\pi/3, \pi]$$

Interval	sign of $\sin 2x$	sign of $2 \cos x + 1$	sign of $(\sin 2x)(2 \cos x + 1)$
$(0, \pi/2)$	+	+	+
$(\pi/2, 2\pi/3)$	-	+	-
$(2\pi/3, \pi)$	-	-	+

$$I = \int_0^{\pi/2} \sin 2x(2 \cos x + 1) dx - \int_{\pi/2}^{2\pi/3} \sin 2x(2 \cos x + 1) dx + \int_{2\pi/3}^\pi \sin 2x(2 \cos x + 1) dx$$

Using, $\sin 2x = 2 \sin x \cos x$

$$I = \int_0^{\pi/2} 2 \sin x (2 \cos^2 x + \cos x) dx - \int_{\pi/2}^{2\pi/3} 2 \sin x (2 \cos^2 x + \cos x) dx + \int_{2\pi/3}^\pi 2 \sin x (2 \cos^2 x + \cos x) dx$$

Let $t = \cos x$, $\sin x dx = -dt$, limits get change

$$I = \int_1^0 -2(2t^2 + t) dt + \int_0^{-1/2} 2(2t^2 + t) dt - \int_{-1/2}^{-1} 2(2t^2 + t) dt$$

$$\Rightarrow I = 2 \int_0^1 (2t^2 + t) dt - 2 \int_{-1/2}^0 (2t^2 + t) dt + 2 \int_{-1}^{-1/2} (2t^2 + t) dt$$

$$\Rightarrow I = 2 \left[\left(\frac{2t^3}{3} + \frac{t^2}{2} \right) \Big|_0^1 - \left(\frac{2t^3}{3} + \frac{t^2}{2} \right) \Big|_{-1/2}^0 + \left(\frac{2t^3}{3} + \frac{t^2}{2} \right) \Big|_{-1}^{-1/2} \right]$$

$$\text{Ist part} = \left[\frac{2}{3}(-1/2)^3 + \frac{1}{2}(-1/2)^2 - \left\{ \frac{2}{3}(-1)^3 + \frac{1}{2}(-1)^2 \right\} \right] = [-1/12 + 1/8 + 2/3 - 1/2] = \frac{5}{24}$$

$$\text{IInd part} = \left[\frac{2}{3}(0) + 0 - \left\{ \frac{2}{3}(-1/2)^3 + \frac{1}{2}(-1/2)^2 \right\} \right] = [1/12 - 1/8] = -\frac{1}{24}$$

$$\text{IIIRD part} = \left[\frac{2}{3}(1)^3 + \frac{1}{2}(1)^2 - \{0 + 0\} \right] = \frac{7}{6} = \frac{28}{24}$$

$$\Rightarrow I = 2 \left[\frac{5}{24} - (-1/24) + \frac{28}{24} \right] = 2 \times \frac{34}{24} = \frac{17}{6}$$

$$\Rightarrow 6I = 17$$

Question2

If $\int_0^1 4 \cot^{-1}(1 - 2x + 4x^2) dx = a \tan^{-1}(2) - b \log_e(5)$, where $a, b \in \mathbb{N}$, then $(2a + b)$ is equal to

_____.

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Answer: 9

Solution:

We are given

$$\int_0^1 4 \cot^{-1}(1 - 2x + 4x^2) dx = a \tan^{-1}(2) - b \ln(5), \quad a, b \in \mathbb{N}$$

Step 1: Simplify the integrand

$$1 - 2x + 4x^2 = 4x^2 - 2x + 1 > 0 \quad \forall x \in [0, 1]$$

For positive arguments,

$$\cot^{-1}(u) = \tan^{-1}\left(\frac{1}{u}\right)$$

so

$$\int_0^1 4 \cot^{-1}(4x^2 - 2x + 1) dx = 4 \int_0^1 \tan^{-1}\left(\frac{1}{4x^2 - 2x + 1}\right) dx$$

Using the identity $\tan^{-1}(1/u) = \frac{\pi}{2} - \tan^{-1}(u)$ (for $u > 0$):

$$= 4 \left[\frac{\pi}{2} - \int_0^1 \tan^{-1}(4x^2 - 2x + 1) dx \right]$$

Step 2: Evaluate the remaining integral

Using integration by parts and standard integrals of rational functions, one obtains:

$$\int_0^1 \tan^{-1}(4x^2 - 2x + 1) dx = \tan^{-1}(3) - \frac{1}{4} \ln 5$$

Substitute back:

$$\begin{aligned} I &= 4 \left[\frac{\pi}{2} - \tan^{-1}(3) + \frac{1}{4} \ln 5 \right] \\ &= 4 \tan^{-1}\left(\frac{1}{3}\right) + \ln 5 \end{aligned}$$

Using the identity:

$$\tan^{-1}(2) = 2 \tan^{-1}\left(\frac{1}{3}\right)$$

we get:

$$I = 4 \tan^{-1}(2) - \ln 5$$

Step 3: Identify a and b

Comparing with

$$a \tan^{-1}(2) - b \ln(5)$$

we have:

$$a = 4, \quad b = 1$$

Final Answer

$$2a + b = 2(4) + 1 = \boxed{9}$$

Question3

Let $[\cdot]$ denote the greatest integer function and $f(x) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n \left[\frac{k^2}{3^x} \right]$. Then $12 \sum_{j=1}^{\infty} f(j)$ is equal to _____.

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Answer: 2

Solution:

$$f(x) = \lim_{n \rightarrow \infty} \left(\frac{1}{n^3} \cdot \sum_{k=1}^n \left[\frac{k^2}{3^x} \right] \right) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \sum_{k=1}^n \left(\frac{k^2}{3^x} \right) - \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \sum_{k=1}^n \left\{ \frac{k^2}{3^x} \right\}$$

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6 \times 3^x} - 0 \Rightarrow f(x) = \frac{1}{3^{x+1}}$$

$$12 \sum_{j=1}^{\infty} f(j) = 12 \left(\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots \right) = 12 \left(\frac{\frac{1}{9}}{1 - \frac{1}{3}} \right) = \frac{12}{9-3} = 2$$

Question4

Let $[.]$ be the greatest integer function. If $\alpha = \int_0^{64} (x^{1/3} - [x^{1/3}]) dx$, then $\frac{1}{\pi} \int_0^{\alpha\pi} \left(\frac{\sin^2 \theta}{\sin^6 \theta + \cos^6 \theta} \right) d\theta$ is equal to _____.

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Answer: 36

Solution:

$$\int_0^{64} X^{\frac{1}{3}} dx = \frac{3}{4} \cdot \left[X^{\frac{4}{3}} \right]_0^{64} = 192$$

$$\int_0^{64} [X^{1/3}] dx = \int_0^1 [X^{1/3}] dx + \int_1^8 [X^{1/3}] dx + \int_8^{27} [X^{1/3}] dx + \int_{27}^{64} [X^{1/3}] dx = 156$$

$$\text{So } \alpha = 192 - 156 = 36$$

$$\text{Now } E = \frac{1}{\pi} \int_0^{36\pi} \frac{\sin^2 \theta}{\sin^6 \theta + \cos^6 \theta} d\theta = \frac{36}{\pi} \cdot 2 \int_0^{\pi} \frac{\sin^2 \theta}{\sin^6 \theta + \cos^6 \theta} d\theta$$

$$\Rightarrow E = \frac{36}{\pi} \int_0^{\pi/2} \frac{\sin^2 \theta}{\sin^6 \theta + \cos^6 \theta} d\theta$$

$$\text{Let } J = \int_0^{\pi/2} \frac{\sin^2 \theta}{\sin^6 \theta + \cos^6 \theta} d\theta \dots\dots\dots$$

$$\text{Applying King } J = \int_0^{\pi/2} \frac{\cos^2 \theta}{\sin^6 \theta + \cos^6 \theta} d\theta \dots\dots\dots$$

$$\text{Now } 2J = \int_0^{\pi/2} \frac{1}{\sin^6 \theta + \cos^6 \theta} d\theta (\text{add}(1) \& (2)) = \int_0^{\pi/2} \frac{\sec^6 \theta}{\tan^6 \theta + 1} d\theta$$

$$\text{Put } \tan \theta = \lambda$$

$$\int_0^\infty \frac{(1 + \lambda^2)^2}{\lambda^4 - \lambda^2 + 1} d\lambda$$

$$\int_0^\infty \frac{1 + \frac{1}{\lambda^2}}{\lambda^2 - 1 + \frac{1}{\lambda^2}} d\lambda = \pi$$

$$\Rightarrow J = \frac{\pi}{2}$$

$$E = \frac{36}{\pi} \cdot 2J = 36$$

Question5

The number of elements in the set $S = \{x : x \in [0, 100] \text{ and } \int_0^x t^2 \sin(x-t) dt = x^2\}$ is _____

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Answer: 16

Solution:

$$\int_0^x t^2 \sin(x-t) dt = x^2$$

$$\text{use property } \int_0^a f(t) dt = \int_0^a f(a-t) dt$$

$$\int_0^x (x-t)^2 \sin(x-(x-t)) dt = \int_0^x (x-t)^2 \sin t dt = x^2$$

$$\int_0^x (x^2 - 2xt + t^2) \sin t dt = x^2$$

$$x^2 \int_0^x \sin t dt - 2x \int_0^x t \sin t dt + \int_0^x t^2 \sin t dt = x^2$$

differentiating with respect to x (Leibniz rule):

$$2x \int_0^x \sin t dt + x^2 \sin x - 2 \int_0^x t \sin t dt - 2x(x \sin x) + x^2 \sin x = 2x$$

$$2x \int_0^x \sin t dt - 2 \int_0^x t \sin t dt = 2x$$

differentiating again:

$$2 \int_0^x \sin t dt + 2x \sin x - 2x \sin x = 2$$

$$2[-\cos t]_0^x = 2$$

$$-(\cos x - \cos 0) = 1$$

$$-\cos x + 1 = 1 \Rightarrow \cos x = 0$$

finding number of elements in $S = \{x : x \in [0, 100] \text{ and } \cos x = 0\}$

$$x = (2n+1) \frac{\pi}{2}$$

$$0 \leq (2n+1) \frac{\pi}{2} \leq 100 \Rightarrow 0 \leq 2n+1 \leq \frac{200}{\pi} \approx 63.66$$

$$2n \leq 62.66 \Rightarrow n \leq 31.33$$

for $n \in \{0, 1, 2, \dots, 31\}$, there are 32 values.

the number of elements in set S is 32 .

Question6

Let a differentiable function f satisfy the equation $\int_0^{36} f\left(\frac{tx}{36}\right) dt = 4\alpha f(x)$. If $y = f(x)$ is a standard parabola passing through the points $(2, 1)$ and $(-4, \beta)$, then β^α is equal to ____ .

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Answer: 64

Solution:

$$\text{Given, } \int_0^{36} f\left(\frac{tx}{36}\right) dt = 4\alpha f(x)$$

$$\text{Let } \frac{tx}{36} = u \Rightarrow \frac{x}{36} dt = du$$

$$\int_0^x f(u) \frac{36}{x} du = 4\alpha f(x)$$

Integration is w.r.t u , so treat x as a constant

$$\int_0^x f(u) du = \frac{4x}{36} \alpha f(x) = \frac{x}{9} \alpha f(x)$$

Now, differentiate both sides w.r.t x , use Leibniz theorem in LHS:

$$1 \cdot f(x) = \frac{\alpha}{9} [1 \cdot f(x) + x f'(x)]$$

$$\Rightarrow 9f(x) = \alpha f(x) + \alpha x f'(x)$$

$$\Rightarrow \alpha x f'(x) = f(x)(9 - \alpha)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{9 - \alpha}{\alpha x}$$

Integrating both sides

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{9 - \alpha}{\alpha} \left(\frac{1}{x}\right) dx$$

$$\Rightarrow \ln f(x) = \frac{9 - \alpha}{\alpha} \ln x + c$$

$$\Rightarrow f(x) = e^{\frac{9 - \alpha}{\alpha} \ln x} \cdot e^c$$

$$\Rightarrow f(x) = x^{\frac{9 - \alpha}{\alpha}} \cdot K \quad \{K = e^c\}$$

It is given that $f(x)$ is a standard parabola passing through $(2, 1)$ and $(-4, \beta)$.

So $f(x) = Kx^2$ type.

$$\frac{9 - \alpha}{\alpha} = 2 \Rightarrow 9 - \alpha = 2\alpha \Rightarrow 3\alpha = 9$$

$$\Rightarrow \alpha = 3$$

$$f(x) = Kx^2 \text{ passing through } (2, 1)$$

$$f(2) = 1$$

$$1 = K(2)^2 \Rightarrow K = \frac{1}{4}$$

$$f(x) = \frac{1}{4}x^2$$

$$f(x) \text{ is also passing through } (-4, \beta), f(-4) = \beta$$

$$\beta = \frac{1}{4}(-4)^2 = \frac{16}{4} = 4$$

$$\text{then } \beta^\alpha = 4^3 = 64$$

Question7

If $f(x)$ satisfies the relation $f(x) = e^x + \int_0^1 (y + xe^x)f(y)dy$, then $e + f(0)$ is equal to ____.

JEE Main 2026 (Online) 24th January Evening Shift

Answer: 2

Solution:

$$f(x) = e^x + \int_0^1 (y + xe^x)f(y)dy$$

$$f(x) = e^x + \int_0^1 yf(y)dy + \int_0^1 xe^x f(y)dy$$

Integration is w.r.t y , so variable x is taken as a constant.

$$f(x) = e^x + \int_0^1 yf(y)dy + xe^x \int_0^1 f(y)dy.$$

Let $A = \int_0^1 yf(y)dy$ and $B = \int_0^1 f(y)dy$, both are constants.

So $f(x)$ becomes,

$$f(x) = e^x + A + Bxe^x \quad \dots (1)$$

$$\text{Or, } f(y) = e^y + A + Bye^y \quad \dots (2)$$

Solve, $B = \int_0^1 f(y)dy$.

$$B = \int_0^1 [e^y + A + Bye^y]dy, \{ \text{from (2) } f(y) = e^y + A + Bye^y \}.$$

$$B = \int_0^1 [e^y + A + Bye^y]dy, \{ \text{from (2) } f(y) = e^y + A + Bye^y \}.$$

Using integration by parts, $\int ye^y dy = ye^y - \int e^y = e^y(y-1)$.

$$B = [e^y + Ay + Be^y(y-1)]_0^1$$

$$B = [e^1 + A \cdot 1 + 0 - e^0 - 0 - B \cdot e^0(-1)].$$

$$B = e + A - 1 + B \implies A = 1 - e.$$

Finding $f(0)$ using equation (1).

$$f(0) = 1 + A + 0.$$

$$f(0) = 1 + 1 - e, \{ \text{substitute } A = 1 - e \}.$$

$$f(0) + e = 2$$

Question8

The value of $\sum_{r=1}^{20} \left(\left| \sqrt{\pi \left(\int_0^r x |\sin \pi x| dx \right)} \right| \right)$ is

JEE Main 2026 (Online) 28th January Morning Shift

Answer: 210

Solution:

$$\text{Let } J_r = \int_0^r x \cdot |\sin \pi x| dx$$

So we need to calculate

$$I = \sum_{r=1}^{20} \sqrt{\pi J_r}$$

So,

$$I = \sqrt{\pi} (J_1 + J_2 + J_3 + \dots + J_{20})$$

Let $\pi x = t \Rightarrow \pi dx = dt$

$$J_r = \int_0^{\pi r} \frac{t}{\pi} |\sin t| \frac{dt}{\pi} = \frac{1}{\pi^2} \int_0^{\pi r} t |\sin t| dt$$

Period of $|\sin t|$ is π , so break the interval

$$\int_0^{\pi r} t |\sin t| dt = \int_0^{\pi} t |\sin t| dt + \int_{\pi}^{2\pi} t |\sin t| dt + \int_{2\pi}^{3\pi} t |\sin t| dt + \dots + \int_{(r-1)\pi}^{r\pi} t |\sin t| dt$$

Now

$|\sin t| = \sin t$ for Ist and II Quadrant

$|\sin t| = -\sin t$ for III and IV Quadrant

$$\int_0^{\pi r} t |\sin t| dt = \int_0^{\pi} t \sin t dt - \int_{\pi}^{2\pi} t \sin t dt + \int_{2\pi}^{3\pi} t \sin t dt - \int_{3\pi}^{4\pi} t \sin t dt \dots (-1)^{r-1} \int_{(r-1)\pi}^{r\pi} t \sin t dt$$

Solving $\int t \cdot \sin t$ using by parts

$$\int u \cdot v dx = u \cdot \int v dx - \int (u' \int v dx) dx$$

$$\Rightarrow \int t \cdot \sin t = (-t \cos t) - \int 1 \cdot (-\cos t) = -t \cos t + \sin t$$

Substituting the limits

$$\Rightarrow \int_0^{\pi r} t \cdot |\sin t| dt = [-t \cos t + \sin t]_0^{\pi} - [-t \cos t + \sin t]_{\pi}^{2\pi} + [-t \cos t + \sin t]_{2\pi}^{3\pi} - [-t \cos t + \sin t]_{3\pi}^{4\pi} \dots \int_{(r-1)\pi}^{r\pi} t \sin t (-1)^{r-1} dt$$

$$= [\pi + 0 - 0] - [-2\pi + 0 - \{\pi + 0\}] + [3\pi + 0 - \{-2\pi\} + 0] - [-4\pi - \{-3\pi + 0\}] + \dots \int_{(r-1)\pi}^{r\pi} t \sin t (-1)^{r-1} dt$$

$$\Rightarrow \int_0^{\pi r} t |\sin t| dt = [\pi + 0] + [2\pi + \pi] + [3\pi + 2\pi] + [4\pi + 3\pi] \dots [r\pi + (r-1)\pi]$$

$$= \pi + 3\pi + 5\pi + 7\pi + \dots (2r-1)\pi$$

$$= \pi(1 + 3 + 5 + 7 + \dots (2r-1))$$

Sum of first r odd numbers is r^2

$$\int_0^{\pi r} t |\sin t| dt = \pi r^2$$

$$J_r = \frac{1}{\pi^2} \pi r^2 = \frac{r^2}{\pi}$$

so,

$$I = \sum_{r=1}^{20} \sqrt{\pi \cdot J_r} = \sum_{r=1}^{20} \sqrt{\pi \cdot \frac{r^2}{\pi}}$$

$$\Rightarrow I = \sum_{r=1}^{20} r = 1 + 2 + 3 + \dots 20$$

$$\Rightarrow I = \frac{20 \times 21}{2} = 210$$

Question9

Let f be a differentiable function satisfying $f(x) = 1 - 2x + \int_0^x e^{(x-t)} f(t) dt$, $x \in \mathbb{R}$ and let

$$g(x) = \int_0^x (f(t) + 2)^{15} (t - 4)^6 (t + 12)^{17} dt, x \in \mathbb{R}.$$

If p and q are respectively the points of local minima and local maxima of g , then the value of $|p + q|$ is equal to _____.

JEE Main 2026 (Online) 28th January Evening Shift

Answer: 9

Solution:

The given equation is:

$$f(x) = 1 - 2x + \int_0^x e^{(x-t)} f(t) dt$$

Since $e^{(x-t)} = e^x \cdot e^{-t}$, we can move e^x outside the integral because it is constant relative to t

$$f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t) dt$$

Differentiate both sides with respect to x using the Leibniz Rule and the product rule:

$$f'(x) = -2 + [e^x \int_0^x e^{-t} f(t) dt + e^x (e^{-x} f(x))]$$

Now, we use substitution from our original equation. Notice that $e^x \int_0^x e^{-t} f(t) dt = f(x) - (1 - 2x)$. Substituting this back into the derivative expression:

$$f'(x) = -2 + [f(x) - 1 + 2x] + f(x)$$

$$\Rightarrow f'(x) = 2f(x) + 2x - 3$$

Combine the terms to get the linear differential equation in the form of $\frac{df(x)}{dx} + P(x)f(x) = Q(x)$

$$\frac{df(x)}{dx} - 2f(x) = 2x - 3$$

Solve for $f(x)$

$$\text{Integrating Factor (IF): } e^{\int -2dx} = e^{-2x}$$

$$\text{General Solution: } f(x) \cdot e^{-2x} = \int (2x - 3)e^{-2x} dx$$

By integrating the right side (Integration by Parts), we get $f(x) = 1 - x + Ce^{2x}$.

Using $f(0) = 1$ (from the original equation), we find $C = 0$.

$$\text{Result: } f(x) = 1 - x.$$

Find the Extrema of $g(x)$

$$g(x) = \int_0^x (f(t) + 2)^{15} (t - 4)^6 (t + 12)^{17} dt$$

Differentiate both sides with respect to x using the Leibniz Rule

$$g'(x) = (f(x) + 2)^{15} (x - 4)^6 (x + 12)^{17}$$

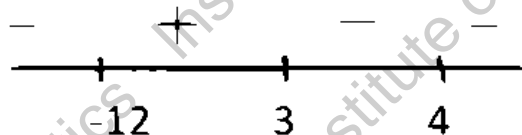
Substitute $f(x) = 1 - x$:

$$g'(x) = (1 - x + 2)^{15} (x - 4)^6 (x + 12)^{17}$$

$$g'(x) = (3 - x)^{15} (x - 4)^6 (x + 12)^{17}$$

Critical points: $x = 3, 4, -12$.

Plot on number line to find local maxima and minima.



Local Minima (p): At $x = -12$, $g'(x)$ changes sign from negative to positive. Thus, $p = -12$.

Local Maxima (q): At $x = 3$, $g'(x)$ changes sign from positive to negative. Thus, $q = 3$.

$x = 4$: No sign change occurs (even power 6), so it is not a local extremum.

Final Answer

$$|p + q| = |-12 + 3| = |-9| = 9$$

The value of $|p + q|$ is 9.

Question 10

If $\alpha = \int_0^{2\sqrt{3}} \log_2(x^2 + 4) dx + \int_2^4 \sqrt{2x - 4} dx$, then α^2 is equal to _____.

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Answer: 192

Solution:

As we know that

$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx = bf(b) - af(a)$$

$$\begin{aligned} \therefore \alpha &= (2\sqrt{3})(4) - (0)(2) \\ &= 8\sqrt{3} \\ \Rightarrow \alpha^2 &= 192 \end{aligned}$$

Question 11

If $\int_{\pi/6}^{\pi/4} \left(\cot\left(x - \frac{\pi}{3}\right) \cot\left(x + \frac{\pi}{3}\right) + 1 \right) dx = \alpha \log_e(\sqrt{3} - 1)$, then $9\alpha^2$ is equal to _____.

JEE Main 2026 (Online) 8th April Evening Shift

Answer: 12

Solution:

$$I = \int_{\pi/6}^{\pi/4} \left[\cot\left(x - \frac{\pi}{3}\right) \cdot \cot\left(x + \frac{\pi}{3}\right) + 1 \right] dx$$

$$\begin{aligned}
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left| \frac{\cos\left(x - \frac{\pi}{3}\right) \cdot \cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)}{\sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)} \right| dx \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left| \frac{\cos\left\{\left(x + \frac{\pi}{3}\right) - \left(x - \frac{\pi}{3}\right)\right\}}{\sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)} \right| dx \\
&= \cos\left(\frac{2\pi}{3}\right) \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{1}{\sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)} \right] dx \\
&\Rightarrow I = 2 \cos\left(\frac{2\pi}{3}\right) \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{1}{2 \sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)} \right] dx \\
&= 2 \cos\left(\frac{2\pi}{3}\right) \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{1}{\left\{\cos\left(x - \frac{\pi}{3}\right) \cdot \cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)\right\} - \left\{\cos\left(x - \frac{\pi}{3}\right) \cdot \cos\left(x + \frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x + \frac{\pi}{3}\right)\right\}} \right] dx \\
&= 2 \cos\left(\frac{2\pi}{3}\right) \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{1}{\cos\left(\frac{2\pi}{3}\right) - \cos(2x)} \right] dx \\
&\Rightarrow I = - \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{1}{\frac{1}{2} - \cos(2x)} \right] dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2}{1 - 2 \cos(2x)} \right] dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2}{1 + 2(1 - 2 \sin^2 x)} \right] dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2}{3 - 4 \sin^2 x} \right] dx \\
&\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2 \sec^2 x}{3 \sec^2 x - 4 \tan^2 x} \right] dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2 \sec^2 x}{3(1 + \tan^2 x) - 4 \tan^2 x} \right] dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left[\frac{2 \sec^2 x}{3 - \tan^2 x} \right] dx
\end{aligned}$$

Substitute $t = \tan x \Rightarrow dt = (\sec^2 x) dx$

$$\begin{aligned}
&\Rightarrow I = \int_{\frac{1}{\sqrt{3}}}^1 \left[\frac{2}{3 - t^2} \right] dt = \left(\frac{1}{\sqrt{3}} \right) \cdot \int_{\frac{1}{\sqrt{3}}}^1 \left[\frac{2\sqrt{3}}{(\sqrt{3}+t)(\sqrt{3}-t)} \right] dt \\
&= \left(\frac{1}{\sqrt{3}} \right) \cdot \int_{\frac{1}{\sqrt{3}}}^1 \left[\frac{(\sqrt{3}+t) + (\sqrt{3}-t)}{(\sqrt{3}+t)(\sqrt{3}-t)} \right] dt = \left(\frac{1}{\sqrt{3}} \right) \cdot \int_{\frac{1}{\sqrt{3}}}^1 \left[\frac{1}{\sqrt{3}-t} + \frac{1}{\sqrt{3}+t} \right] dt \\
&\Rightarrow I = \left(\frac{1}{\sqrt{3}} \right) \cdot [\ln(\sqrt{3}+t) - \ln(\sqrt{3}-t)]_{\frac{1}{\sqrt{3}}}^1 = \left(\frac{1}{\sqrt{3}} \right) \cdot \left[\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) \right]_{\frac{1}{\sqrt{3}}}^1 \\
&= \left(\frac{1}{\sqrt{3}} \right) \cdot \left[\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \ln 2 \right] = \left(\frac{1}{\sqrt{3}} \right) \cdot \left[\ln\left\{ \frac{2}{(\sqrt{3}-1)^2} \right\} - \ln 2 \right] \\
I &= \left(\frac{1}{\sqrt{3}} \right) \cdot \ln\left\{ \frac{2}{2(\sqrt{3}-1)^2} \right\} = \left(\frac{1}{\sqrt{3}} \right) \cdot \ln\left\{ \frac{1}{(\sqrt{3}-1)^2} \right\} \\
&= \left(\frac{1}{\sqrt{3}} \right) \cdot \ln\left\{ (\sqrt{3}-1)^{-2} \right\} = \left(-\frac{2}{\sqrt{3}} \right) \cdot \ln(\sqrt{3}-1) \dots\dots(i) \\
\therefore \alpha &= -\frac{2}{\sqrt{3}} \Rightarrow 9\alpha^2 = 12
\end{aligned}$$

Question 12

The value of $\int_{-\pi/6}^{\pi/6} \left(\frac{\pi + 4x^{11}}{1 - \sin(|x| + \pi/6)} \right) dx$ is equal to:

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Options:

A.

8π

B.

4π

C.

2π

D.

6π

Answer: B

Solution:

$$\begin{aligned} & \int_0^{\frac{\pi}{6}} \frac{\pi + 4x^{11}}{1 - \sin\left(|x| + \frac{\pi}{6}\right)} + \frac{\pi - 4x^{11}}{1 - \sin\left(|x| + \frac{\pi}{6}\right)} dx \\ &= 2\pi \int_0^{\frac{\pi}{6}} \frac{1}{1 - \sin\left(x + \frac{\pi}{6}\right)} dx \\ &= 2\pi \int_0^{\frac{\pi}{6}} \frac{1 + \sin\left(x + \frac{\pi}{6}\right)}{\cos^2\left(x + \frac{\pi}{6}\right)} dx \\ &= 2\pi \int_0^{\frac{\pi}{6}} \sec^2\left(x + \frac{\pi}{6}\right) + \tan\left(x + \frac{\pi}{6}\right) \sec\left(x + \frac{\pi}{6}\right) dx \\ &\Rightarrow 2\pi \left[\tan\left(x + \frac{\pi}{6}\right) + \sec\left(x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{6}} \\ &\Rightarrow 4\pi \end{aligned}$$

Question13

Let $f : [1, \infty) \rightarrow \mathbb{R}$ be a differentiable function. If $6 \int_1^x f(t) dt = 3xf(x) + x^3 - 4$ for all $x \geq 1$, then the value of $f(2) - f(3)$ is :

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Options:

A.

4

B.

3

C.

-4

D.

-3

Answer: B

Solution:

$$6 \int_1^x f(t) dt = 3xf(x) + x^3 - 4$$

By using leibniz rule

$$6f(x) = 3f(x) + 3x f'(x) + 3x^2$$

$$\frac{f(x) - x f'(x)}{x^2} = 1$$

$$\frac{d}{dx} \left(\frac{f(x)}{x} \right) = -1 f(x) = -x^2 + kx$$

$$1 = -1 + k \text{ (since } f(1) = 1)$$

$$f(x) = -x^2 + 2x$$

$$f(2) - f(3) = 3$$

Question 14

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{[x]+4} \right) dx$, where $[\cdot]$ denotes the greatest integer function, is

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Options:

A.

$$\frac{1}{60}(21\pi - 1)$$

B.

$$\frac{1}{60}(\pi - 7)$$

C.

$$\frac{7}{60}(\pi - 3)$$

D.

$$\frac{7}{60}(3\pi - 1)$$

Answer: D

Solution:

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{-1} \frac{1}{2} dx + \int_{-1}^0 \frac{1}{3} dx + \int_0^1 \frac{1}{4} dx + \int_1^{\frac{\pi}{2}} \frac{1}{5} dx \\ &= \frac{21\pi - 7}{60} = \frac{7}{60}(3\pi - 1) \end{aligned}$$

Question 15

The value of the integral $\int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \frac{dx}{1 + \sqrt[3]{\tan 2x}}$ is :

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Options:

A.

$$\frac{\pi}{3}$$

B.

$$\frac{\pi}{18}$$

C.

$$\frac{\pi}{6}$$

D.

$$\frac{\pi}{12}$$

Answer: D

Solution:

$$I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \frac{1}{1 + (\tan 2x)^{1/3}} dx \quad \dots (1)$$

$$\Rightarrow I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \frac{1}{1 + \tan \left(2 \left(\frac{5\pi}{24} + \frac{\pi}{24} - 4 \right) \right)^{1/3}} dx$$

$$I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \frac{1}{1 + \tan \left(\frac{\pi}{2} - 2x \right)^{1/3}} dx \quad \dots (2)$$

From equation (1) and (2)

$$2I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \frac{(\tan(2x))^{1/3} + 1}{1 + (\tan 2x)^{1/3}} dx$$

$$= (x) \Big|_{\frac{\pi}{24}}^{\frac{5\pi}{24}} \Rightarrow I = \frac{1}{2} \times \frac{4\pi}{24} = \frac{\pi}{12}$$

Question 16

Let f be a polynomial function such that $f(x^2 + 1) = x^4 + 5x^2 + 2$, for all $x \in \mathbb{R}$.

Then $\int_0^3 f(x) dx$ is equal to

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Options:

A. $\frac{33}{2}$

B.

C. $\frac{5}{3}$

D.

E. $\frac{27}{2}$

F.

G. $\frac{41}{3}$

Answer: A

Solution:

We are given the functional equation:

$$f(x^2 + 1) = x^4 + 5x^2 + 2$$

Let $t = x^2 + 1$. This implies that $x^2 = t - 1$. Now, substitute this into the right side of the equation:

$$f(t) = (t - 1)^2 + 5(t - 1) + 2$$

$$f(t) = (t^2 - 2t + 1) + 5t - 5 + 2$$

$$f(t) = t^2 + 3t - 2$$

Replacing t back with x , the polynomial function is:

$$f(x) = x^2 + 3x - 2$$

Now, we calculate the definite integral from 0 to 3 :

$$\begin{aligned}\int_0^3 f(x) dx &= \int_0^3 (x^2 + 3x - 2) dx \\&= \left[\frac{x^3}{3} + \frac{3x^2}{2} - 2x \right]_0^3 \\&= \left(\frac{3^3}{3} + \frac{3(3^2)}{2} - 2(3) \right) - (0) \\&= \left(\frac{27}{3} + \frac{27}{2} - 6 \right) = 9 + \frac{27}{2} - 6 = 3 + \frac{27}{2} \\&= \frac{6 + 27}{2} = \frac{33}{2}\end{aligned}$$

Question 17

Let $[.]$ denote the greatest integer function. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{12(3+[x])}{3+[\sin x]+[\cos x]} \right) dx$$

is equal to :

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Options:

A.

$$12\pi + 5$$

B.

$$11\pi + 2$$

C.

$$15\pi + 4$$

D.

$$13\pi + 1$$

Answer: B

Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{12(3+[x])}{3+[\sin x]+[\cos x]} \right) dx$$

Breakdown the Interval Since $\frac{\pi}{2} \approx 1.57$ and $-\frac{\pi}{2} \approx -1.57$, we break the integral at the integer points $x = -1, 0, 1$.

Interval	$[x]$	$[\sin x]$	$[\cos x]$	Integrand $f(x)$
$[-\frac{\pi}{2}, -1)$	-2	-1	0	$\frac{12(3-2)}{3-1+0} = \frac{12}{2} = 6$
$[-1, 0)$	-1	-1	0	$\frac{12(3-1)}{3-1+0} = \frac{24}{2} = 12$
$[0, 1)$	0	0	0	$\frac{12(3+0)}{3+0+0} = \frac{36}{3} = 12$
$[1, \frac{\pi}{2}]$	1	0	0	$\frac{12(3+1)}{3+0+0} = \frac{48}{3} = 16$

Notes on values :

- 1. For $x \in (0, 1)$, $0 < \cos x < 1$, so $[\cos x] = 0$. (At $x = 0$, $[\cos x] = 1$, but a single point doesn't change the integral).
- 2. For $x \in [1, \frac{\pi}{2})$, $0 < \sin x < 1$, so $[\sin x] = 0$.

To find the total value, we integrate the constant values found above over their respective subintervals:

$$I = \int_{-\pi/2}^{-1} 6dx + \int_{-1}^1 12dx + \int_1^{\pi/2} 16dx$$

$$\Rightarrow I = 6[x]_{-\pi/2}^{-1} + 12[x]_{-1}^1 + 16[x]_1^{\pi/2}$$

$$\Rightarrow I = 6(-1 - (-\frac{\pi}{2})) + 12(1 - (-1)) + 16(\frac{\pi}{2} - 1)$$

$$\Rightarrow I = -6 + 3\pi + 24 + 8\pi - 16I = 11\pi + 2$$

Question 18

Let $[.]$ denote the greatest integer function. Then the value of $\int_0^3 \left(\frac{e^x + e^{-x}}{[x]!} \right) dx$ is :

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Options:

A.

$$e^2 + e^3 - \frac{1}{e^2} - \frac{1}{e^3}$$

B.

$$\frac{1}{2} (e^2 + e^3 - \frac{1}{e^2} - \frac{1}{e^3})$$

C.

$$e^2 + e^3 - \frac{1}{2e^2} - \frac{1}{2e^3}$$

D.

$$\frac{1}{2} (e^2 + e^3) - \frac{1}{e^2} - \frac{1}{e^3}$$

Answer: B

Solution:

$$\begin{aligned} & \int_0^1 \frac{e^x + e^{-x}}{1} dx + \int_1^2 \frac{e^x + e^{-x}}{1} dx + \int_2^3 \frac{e^x + e^{-x}}{2} dx \\ &= \int_0^2 e^x dx + \int_0^2 e^{-x} dx + \frac{1}{2} \int_2^3 (e^x + e^{-x}) dx \\ &= (e^2 - 1) + \left(\frac{e^2 - 1}{-1} \right) + \frac{1}{2} ((e^3 - e^2) - (e^{-3} - e^{-2})) \\ &= \frac{e^3 + e^2}{2} + 1 - 1 - \frac{e^{-2}}{2} - \frac{e^{-3}}{2} \\ &= \frac{e^3 + e^2 - e^{-2} - e^{-3}}{2} \end{aligned}$$

Question 19

The value of $\int_0^{20\pi} (\sin^4 x + \cos^4 x) dx$ is equal to:

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Options:

A.

$$\frac{15\pi}{2}$$

B.

$$25\pi$$

C.

$$15\pi$$

D.

$$\frac{25\pi}{2}$$

Answer: C

Solution:

$$I = \int_0^{20\pi} (1 - 2 \sin^2 x \cos^2 x) \cdot dx$$

$$= 20\pi - \frac{1}{2} \int_0^{20\pi} \sin^2 2x \, dx$$

$$= 20\pi - \frac{1}{2} (40) \int_0^{\pi/2} \sin^2 2x \, dx$$

$$= 20\pi - \frac{20}{2} \left(x - \frac{\sin 2x}{2} \right)_0^{\pi/2}$$

$$= 20\pi - 10 \left(\frac{\pi}{2} \right) = 15\pi$$

Question20

Let $\int_{-2}^2 (|\sin x| + [x \sin x]) dx = 2(3 - \cos 2) + \beta$, where $[\cdot]$ is the greatest integer function. Then $\beta \sin \left(\frac{\beta}{2} \right)$ equals:

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Options:

A.

$$1$$

B.

$$2$$

C.

4

D.

8

Answer: B

Solution:

Let the given integral be denoted by I . We can split I into two separate integrals:

$$I = \int_{-2}^2 (|\sin x| + [x \sin x]) dx = \int_{-2}^2 |\sin x| dx + \int_{-2}^2 [x \sin x] dx$$

$$\text{Let } I_1 = \int_{-2}^2 |\sin x| dx \text{ and } I_2 = \int_{-2}^2 [x \sin x] dx.$$

Step 1: Evaluate I_1

Based on the property of definite integrals, if $f(x)$ is an even function (i.e., $f(-x) = f(x)$), then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

Since $|\sin(-x)| = |-\sin x| = |\sin x|$, the function is even. Thus,

$$I_1 = 2 \int_0^2 |\sin x| dx$$

In the interval $[0, 2]$, 2 radians is less than π (≈ 3.14), so $\sin x > 0$. Hence, $|\sin x| = \sin x$.

$$I_1 = 2 \int_0^2 \sin x dx$$

$$I_1 = 2[-\cos x]_0^2 = 2(-\cos 2 - (-\cos 0))$$

$$I_1 = 2(1 - \cos 2) = 2 - 2 \cos 2$$

Step 2: Evaluate I_2

Let $f(x) = x \sin x$. We check if it is even or odd:

$$f(-x) = (-x) \sin(-x) = (-x)(-\sin x) = x \sin x = f(x).$$

Since $f(x)$ is an even function, $[x \sin x]$ is also an even function.

$$I_2 = 2 \int_0^2 [x \sin x] dx$$

Now, let us analyze the behavior of $f(x) = x \sin x$ in the interval $x \in [0, 2]$:

- When $x = 0$, $f(0) = 0$.
- When $x = 2$, $f(2) = 2 \sin 2$.

Notice that $\pi/2 < 2 < 2\pi/3$, meaning $\sin 2$ is between $\sin(\pi/2) = 1$ and $\sin(2\pi/3) = \frac{\sqrt{3}}{2} \approx 0.866$.

So, $1.732 < 2 \sin 2 < 2$. Hence, the maximum value of $f(x)$ at $x = 2$ is between 1 and 2.

Let's verify if $f(x)$ is strictly increasing in $[0, 2]$. We differentiate $f(x)$ with respect to x :

$$f'(x) = \sin x + x \cos x$$

For $x \in (0, \pi/2]$, both $\sin x > 0$ and $\cos x \geq 0$, so $f'(x) > 0$.

For $x \in (\pi/2, 2]$, $\cos x < 0$. We can rewrite $f'(x)$ as:

$$f'(x) = \cos x (\tan x + x)$$

Since $\tan x$ is negative in the second quadrant and 2 radians is smaller than $\tan^{-1}(-2)$, the expression $(\tan x + x)$ evaluates to a negative value. Therefore, $\cos x$ (negative) $\times$ $(\tan x + x)$ (negative) will yield $f'(x) > 0$.

Since $f(x)$ strictly increases from 0 to a value between 1 and 2, there is exactly one real number $\alpha \in (0, 2)$ at which $f(\alpha) = 1$. So,

$$\alpha \sin \alpha = 1$$

For $x \in [0, \alpha)$, the function values are $0 \leq x \sin x < 1$, which gives $[x \sin x] = 0$.

For $x \in [\alpha, 2]$, the function values are $1 \leq x \sin x < 2$, which gives $[x \sin x] = 1$.

Now, split the integral I_2 at $x = \alpha$:

$$I_2 = 2 \left(\int_0^\alpha 0 dx + \int_\alpha^2 1 dx \right)$$

$$I_2 = 2[x]_\alpha^2 = 2(2 - \alpha) = 4 - 2\alpha$$

Step 3: Finding the value of $\beta \sin\left(\frac{\beta}{2}\right)$

Combining I_1 and I_2 , we get the total integral:

$$I = I_1 + I_2 = (2 - 2 \cos 2) + (4 - 2\alpha) = 6 - 2 \cos 2 - 2\alpha$$

$$I = 2(3 - \cos 2) - 2\alpha$$

According to the question, the total integral evaluates to:

$$I = 2(3 - \cos 2) + \beta$$

Comparing our result with the given equation, we find:

$$\beta = -2\alpha$$

Finally, substituting $\beta = -2\alpha$ into the expression $\beta \sin\left(\frac{\beta}{2}\right)$:

$$\beta \sin\left(\frac{\beta}{2}\right) = -2\alpha \cdot \sin\left(\frac{-2\alpha}{2}\right)$$

$$\beta \sin\left(\frac{\beta}{2}\right) = -2\alpha \cdot \sin(-\alpha) = -2\alpha(-\sin \alpha)$$

$$\beta \sin\left(\frac{\beta}{2}\right) = 2(\alpha \sin \alpha)$$

From our earlier deduction in Step 2, we established that $\alpha \sin \alpha = 1$. Therefore:

$$\beta \sin\left(\frac{\beta}{2}\right) = 2(1) = 2$$

Option B is the correct answer.

Question 21

Let f be a real polynomial of degree n such that $f(x) = f'(x)f''(x)$, for all $x \in \mathbb{R}$. If $f(0) = 0$, then $36 \left(f'(2) + f''(2) + \int_0^2 f(x) dx \right)$ is equal to:

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Options:

A.

42

B.

46

C.

56

D.

66

Answer: C

Solution:

Let the degree of the real polynomial f be n .

Since $f(x) = f'(x)f''(x)$ for all x , we compare degrees:

$$\begin{aligned}\deg(f) &= \deg(f') + \deg(f'') \\ n &= (n-1) + (n-2) = 2n-3 \\ \Rightarrow n &= 3\end{aligned}$$

So f must be a cubic polynomial. Write

$$f(x) = ax^3 + bx^2 + cx + d$$

Given $f(0) = 0$, we get $d = 0$. Hence

$$f(x) = ax^3 + bx^2 + cx$$

Now find derivatives:

$$\begin{aligned}f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b\end{aligned}$$

Use the given condition $f(x) = f'(x)f''(x)$:

$$(ax^3 + bx^2 + cx) = (3ax^2 + 2bx + c)(6ax + 2b)$$

Now compare coefficients of like powers of x .

For x^3 :

$$\begin{aligned}a &= 18a^2 \\ \Rightarrow a &= \frac{1}{18} \quad (\text{since } a \neq 0 \text{ for a cubic})\end{aligned}$$

For x^2 :

$$\begin{aligned}b &= 6ab + 12ab \\ b &= 18ab\end{aligned}$$

With $a = \frac{1}{18}$, this becomes $b = b$, so it gives no new condition on b .

For x :

$$\begin{aligned}c &= 4b^2 + 6ac \\ c &= 4b^2 + \frac{c}{3} \quad (\because 6a = \frac{1}{3}) \\ \Rightarrow \frac{2c}{3} &= 4b^2 \Rightarrow c = 6b^2\end{aligned}$$

For the constant term:

$$0 = 2bc \Rightarrow bc = 0$$

But $c = 6b^2$, so $bc = b(6b^2) = 6b^3 = 0 \Rightarrow b = 0$, and then $c = 0$.

So the polynomial is

$$f(x) = \frac{1}{18}x^3$$

Now compute the required value:

$$\begin{aligned}f'(x) &= \frac{1}{6}x^2 \Rightarrow f'(2) = \frac{1}{6} \cdot 4 = \frac{2}{3} \\ f''(x) &= \frac{1}{3}x \Rightarrow f''(2) = \frac{1}{3} \cdot 2 = \frac{2}{3} \\ \int_0^2 f(x) dx &= \int_0^2 \frac{1}{18}x^3 dx = \frac{1}{18} \left[\frac{x^4}{4} \right]_0^2 = \frac{1}{18} \cdot \frac{16}{4} = \frac{2}{9}\end{aligned}$$

Add them:

$$\begin{aligned}f'(2) + f''(2) + \int_0^2 f(x) dx \\ = \frac{2}{3} + \frac{2}{3} + \frac{2}{9} = \frac{14}{9}\end{aligned}$$

Finally,

$$36 \left(f'(2) + f''(2) + \int_0^2 f(x) dx \right) = 36 \cdot \frac{14}{9} = 56$$

Question 22

The integral $\int_0^1 \cot^{-1}(1+x+x^2) dx$ is equal to :

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Options:

A.

$$2 \tan^{-1} 2 + \frac{1}{2} \log_e \left(\frac{5}{4} \right) + \frac{\pi}{2}$$

B.

$$2 \tan^{-1} 2 + \frac{1}{2} \log_e \left(\frac{5}{4} \right) - \frac{\pi}{2}$$

C.

$$2 \tan^{-1} 2 - \frac{1}{2} \log_e \left(\frac{5}{4} \right) + \frac{\pi}{2}$$

D.

$$2 \tan^{-1} 2 - \frac{1}{2} \log_e \left(\frac{5}{4} \right) - \frac{\pi}{2}$$

Answer: D

Solution:

Let

$$I = \int_0^1 \cot^{-1}(1+x+x^2) dx$$

We will use the standard identity

$$\cot^{-1} t = \tan^{-1} \left(\frac{1}{t} \right) \quad \text{for } t > 0$$

Since $1+x+x^2 > 0$ on $[0, 1]$, we can write

$$I = \int_0^1 \tan^{-1} \left(\frac{1}{1+x+x^2} \right) dx$$

Now use the formula

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \left(\frac{a-b}{1+ab} \right)$$

We try to express

$$\frac{1}{1+x+x^2}$$

in a suitable form. Observe that

$$\tan^{-1}(x+1) - \tan^{-1} x = \tan^{-1} \left(\frac{(x+1)-x}{1+x(x+1)} \right) = \tan^{-1} \left(\frac{1}{1+x+x^2} \right)$$

Hence,

$$\cot^{-1}(1+x+x^2) = \tan^{-1}(x+1) - \tan^{-1} x$$

So,

$$I = \int_0^1 [\tan^{-1}(x+1) - \tan^{-1} x] dx$$

Therefore,

$$I = \int_0^1 \tan^{-1}(x+1) dx - \int_0^1 \tan^{-1} x dx$$

In the first integral, put $u = x + 1$. Then when $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. So

$$\int_0^1 \tan^{-1}(x+1) dx = \int_1^2 \tan^{-1} u du$$

Thus,

$$I = \int_1^2 \tan^{-1} u du - \int_0^1 \tan^{-1} x dx = \int_1^2 \tan^{-1} x dx - \int_0^1 \tan^{-1} x dx$$

Now use

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2)$$

So,

$$\int_1^2 \tan^{-1} x dx = \left[x \tan^{-1} x - \frac{1}{2} \log(1+x^2) \right]_1^2$$

and

$$\int_0^1 \tan^{-1} x dx = \left[x \tan^{-1} x - \frac{1}{2} \log(1+x^2) \right]_0^1$$

Hence,

$$I = \left[x \tan^{-1} x - \frac{1}{2} \log(1+x^2) \right]_1^2 - \left[x \tan^{-1} x - \frac{1}{2} \log(1+x^2) \right]_0^1$$

This becomes

$$I = \left(2 \tan^{-1} 2 - \frac{1}{2} \log 5 \right) - \left(\tan^{-1} 1 - \frac{1}{2} \log 2 \right) - \left(\tan^{-1} 1 - \frac{1}{2} \log 2 \right)$$

since at $x = 0$, the value is 0.

$$\text{Now } \tan^{-1} 1 = \frac{\pi}{4}, \text{ so}$$

$$I = 2 \tan^{-1} 2 - \frac{1}{2} \log 5 - 2 \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right)$$

$$I = 2 \tan^{-1} 2 - \frac{1}{2} \log 5 - \frac{\pi}{2} + \log 2$$

Now combine the logarithms:

$$\log 2 - \frac{1}{2} \log 5 = \frac{1}{2} \log 4 - \frac{1}{2} \log 5 = \frac{1}{2} \log \left(\frac{4}{5} \right) = -\frac{1}{2} \log \left(\frac{5}{4} \right)$$

Therefore,

$$I = 2 \tan^{-1} 2 - \frac{1}{2} \log \left(\frac{5}{4} \right) - \frac{\pi}{2}$$

So the correct answer is

$$\boxed{2 \tan^{-1} 2 - \frac{1}{2} \log_e \left(\frac{5}{4} \right) - \frac{\pi}{2}}$$

Hence, the correct option is:

Option D

Question 23

If $\alpha = 1$ and $\beta = 1 + i\sqrt{2}$, where $i = \sqrt{-1}$ are two roots of the equation

$x^3 + ax^2 + bx + c = 0, a, b, c \in \mathbb{R}$, then $\int_{-1}^1 (x^3 + ax^2 + bx + c) dx$ is equal to:

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Options:

A.

B.

-4

C.

-8

D.

-10

Answer: C

Solution:

Since the coefficients a, b, c are real, complex roots occur in conjugate pairs.

Given two roots are:

$$\alpha = 1, \quad \beta = 1 + i\sqrt{2}$$

So the third root must be:

$$1 - i\sqrt{2}$$

Hence the cubic polynomial is:

$$x^3 + ax^2 + bx + c = (x - 1)(x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$$

Now first multiply the conjugate factors:

$$(x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$$

This is of the form:

$$(A - iB)(A + iB) = A^2 + B^2$$

with

$$A = x - 1, \quad B = \sqrt{2}$$

So,

$$(x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2})) = (x - 1)^2 + 2$$

Therefore,

$$x^3 + ax^2 + bx + c = (x - 1)((x - 1)^2 + 2)$$

Let $y = x - 1$. Then:

$$(x - 1)((x - 1)^2 + 2) = (x - 1)^3 + 2(x - 1)$$

Now expand:

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$

So,

$$\begin{aligned} x^3 + ax^2 + bx + c &= x^3 - 3x^2 + 3x - 1 + 2x - 2 \\ &= x^3 - 3x^2 + 5x - 3 \end{aligned}$$

Now evaluate:

$$\int_{-1}^1 (x^3 + ax^2 + bx + c) dx = \int_{-1}^1 (x^3 - 3x^2 + 5x - 3) dx$$

Using symmetry:

- x^3 is an odd function, so

$$\int_{-1}^1 x^3 dx = 0$$

- $5x$ is also odd, so

$$\int_{-1}^1 5x dx = 0$$

Thus,

$$\int_{-1}^1 (x^3 - 3x^2 + 5x - 3) dx = \int_{-1}^1 (-3x^2 - 3) dx$$

Now,

$$\int_{-1}^1 x^2 dx = \left[\frac{x^3}{3} \right]_{-1}^1 = \frac{1}{3} - \left(-\frac{1}{3} \right) = \frac{2}{3}$$

Therefore,

$$\int_{-1}^1 (-3x^2) dx = -3 \cdot \frac{2}{3} = -2$$

Also,

$$\int_{-1}^1 (-3) dx = -3(2) = -6$$

So total value:

$$-2 - 6 = -8$$

Hence,

$$\int_{-1}^1 (x^3 + ax^2 + bx + c) dx = -8$$

The correct option is:

So, Option C is correct.

Question 24

The value of the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{4 - \operatorname{cosec}^2 x}{\cos^4 x} \right) dx$ is :

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Options:

A.

$$\frac{11}{\sqrt{3}}$$

B.

$$\frac{16}{\sqrt{3}}$$

C.

$$\frac{32}{3\sqrt{3}}$$

D.

$$\frac{64}{3\sqrt{3}}$$

Answer: C

Solution:

$$\begin{aligned} \text{Put } x &= 2t \Rightarrow dx = 2dt \\ I &= \int_0^\infty \frac{\ln 2t}{4t^2 + 4} (2dt) = \frac{1}{2} \int_0^\infty \frac{\ln 2 + \ln t}{t^2 + 1} dt \\ &= \frac{1}{2} \int_0^\infty \frac{\ln 2}{t^2 + 1} dt + \frac{1}{2} \int_0^\infty \frac{\ln t}{t^2 + 1} dt \end{aligned}$$

$$= \left[\frac{\ln 2}{2} \tan^{-1} t \right]_0^{\infty} + I_1$$

$$= \frac{\ln 2}{2} \cdot \frac{\pi}{2} + I_1$$

$$I_2 = \frac{1}{2} \int_0^{\infty} \frac{\ln t}{t^2+1} dt$$

$$t = \frac{1}{u} \Rightarrow dt = -\frac{du}{u^2}$$

$$= \frac{1}{2} \int_{\infty}^0 \frac{\ln(1/u)}{\frac{1}{u^2}+1} \left(-\frac{du}{u^2} \right)$$

$$\text{Add} \Rightarrow I_1 = 0$$

$$I = \frac{\pi \ln 2}{4}$$

Question 25

The value of the integral $\int_0^{\infty} \frac{\log_e(x)}{x^2+4} dx$ is:

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Options:

A.

$$\frac{\pi \log_e(2)}{2}$$

B.

$$\frac{\pi \log_e(2)}{4}$$

C.

$$1 + \pi \log_e(2)$$

D.

$$2 + \pi \log_e(2)$$

Answer: B

Solution:

Let

$$I = \int_0^{\infty} \frac{\ln x}{x^2+4} dx$$

We will use a standard substitution, as done in NCERT-type definite integral problems.

Take

$$x = 2t$$

Then

$$dx = 2 dt$$

and when $x = 0$, $t = 0$, and when $x \rightarrow \infty$, $t \rightarrow \infty$.

So,

$$I = \int_0^{\infty} \frac{\ln(2t)}{(2t)^2+4} \cdot 2 dt$$

Simplify the denominator:

$$(2t)^2 + 4 = 4t^2 + 4 = 4(1 + t^2)$$

Hence,

$$I = \int_0^\infty \frac{\ln(2t)}{4(1+t^2)} \cdot 2 dt = \frac{1}{2} \int_0^\infty \frac{\ln(2t)}{1+t^2} dt$$

Now use

$$\ln(2t) = \ln 2 + \ln t$$

So,

$$I = \frac{1}{2} \int_0^\infty \frac{\ln 2}{1+t^2} dt + \frac{1}{2} \int_0^\infty \frac{\ln t}{1+t^2} dt$$

That is,

$$I = \frac{\ln 2}{2} \int_0^\infty \frac{dt}{1+t^2} + \frac{1}{2} \int_0^\infty \frac{\ln t}{1+t^2} dt$$

Now,

$$\int_0^\infty \frac{dt}{1+t^2} = [\tan^{-1} t]_0^\infty = \frac{\pi}{2}$$

Also, let

$$J = \int_0^\infty \frac{\ln t}{1+t^2} dt$$

Using the substitution $t = \frac{1}{u}$, we get

$$dt = -\frac{1}{u^2} du$$

Then

$$J = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2} \left(-\frac{1}{u^2}\right) du$$

This simplifies to

$$J = \int_0^\infty \frac{-\ln u}{1+u^2} du = -J$$

Therefore,

$$J = 0$$

So,

$$I = \frac{\ln 2}{2} \cdot \frac{\pi}{2} = \frac{\pi \ln 2}{4}$$

Hence, the value of the integral is

$$\boxed{\frac{\pi \ln 2}{4}}$$

So, the correct option is:

Question 26

Let $(2^{1-a} + 2^{1+a}), f(a), (3^a + 3^{-a})$ be in A.P. and α be the minimum value of $f(a)$. Then the value of the integral $\int_{\log_e(\alpha-1)}^{\log_e(\alpha)} \frac{dx}{(e^{2x}-e^{-2x})}$ is :

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Options:

A.

$$\frac{1}{2} \log_e \left(\frac{4}{3} \right)$$

B.

$$\frac{1}{4} \log_e \left(\frac{4}{3} \right)$$

C.

$$\frac{1}{2} \log_e \left(\frac{8}{5} \right)$$

D.

$$\frac{1}{4} \log_e \left(\frac{8}{5} \right)$$

Answer: B

Solution:

$$f(a) = \frac{1}{2} (2(2^a + 2^{-a}) + (3^a + 3^{-a}))$$

$$a = \frac{1}{2} (2 \times 2 + 2) = 3$$

$$I = \int_{\ln(3-1)}^{\ln 3} \frac{e^{2x}}{e^{4x} - 1} dx$$

$$\text{put } e^{2x} = t$$

$$I = \frac{1}{2} \int_4^9 \frac{dt}{t^2 - 1}$$

$$I = \frac{1}{4} \left(\ln \left(\frac{t-1}{t+1} \right) \right)_4^9$$

$$I = \frac{1}{4} \ln \left(\frac{4}{3} \right)$$

Question 27

The value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{32 \cos^4 x}{1 + e^{\sin x}} \right) dx$ is :

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Options:

A.

$$4\pi + 2$$

B.

$$3\pi + 8$$

C.

$$3\pi + 4$$

D.

$$4\pi + 3$$

Answer: B

Solution:

Let

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{32 \cos^4 x}{1 + e^{\sin x}} dx$$

We will use the standard property:

If

$$I = \int_{-a}^a f(x) dx,$$

then

$$I = \int_{-a}^a f(-x) dx$$

and hence

$$2I = \int_{-a}^a (f(x) + f(-x)) dx$$

Now here,

$$f(x) = \frac{32 \cos^4 x}{1 + e^{\sin x}}$$

So,

$$f(-x) = \frac{32 \cos^4(-x)}{1 + e^{\sin(-x)}}$$

Using $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$,

$$f(-x) = \frac{32 \cos^4 x}{1 + e^{-\sin x}}$$

Therefore,

$$f(x) + f(-x) = 32 \cos^4 x \left(\frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}} \right)$$

Now simplify the bracket:

$$\frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}}$$

Let $t = e^{\sin x}$. Then $e^{-\sin x} = \frac{1}{t}$. So,

$$\frac{1}{1+t} + \frac{1}{1+\frac{1}{t}} = \frac{1}{1+t} + \frac{t}{1+t} = 1$$

Hence,

$$f(x) + f(-x) = 32 \cos^4 x$$

So,

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 32 \cos^4 x dx$$

Thus,

$$I = 16 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^4 x dx$$

Since $\cos^4 x$ is an even function,

$$I = 32 \int_0^{\frac{\pi}{4}} \cos^4 x dx$$

Now use the identity

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2} \right)^2$$

which gives

$$\cos^4 x = \frac{3 + 4 \cos 2x + \cos 4x}{8}$$

Therefore,

$$I = 32 \int_0^{\frac{\pi}{4}} \frac{3 + 4 \cos 2x + \cos 4x}{8} dx$$

$$I = 4 \int_0^{\frac{\pi}{4}} (3 + 4 \cos 2x + \cos 4x) dx$$

Now integrate termwise:

$$\int 3 dx = 3x, \quad \int 4 \cos 2x dx = 2 \sin 2x, \quad \int \cos 4x dx = \frac{\sin 4x}{4}$$

So,

$$I = 4 \left[3x + 2 \sin 2x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{4}}$$

At $x = \frac{\pi}{4}$,

$$3x = \frac{3\pi}{4}, \quad \sin 2x = \sin \frac{\pi}{2} = 1, \quad \sin 4x = \sin \pi = 0$$

At $x = 0$,

$$3x = 0, \quad \sin 0 = 0$$

Hence,

$$I = 4 \left(\frac{3\pi}{4} + 2 \right)$$

$$I = 3\pi + 8$$

So the correct answer is:

$$\boxed{3\pi + 8}$$

Hence, Option B is correct.

Question 28

The value of the integral $\int_{-1}^1 \left(\frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} \right) dx$ is equal to:

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Options:

A.

$$3 \log_e 2$$

B.

$$2 \log_e 2$$

C.

$$5 \log_e 3$$

D.

$$3 \log_e 3$$

Answer: B

Solution:

$$I = \int_{-1}^1 \left(\frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} \right) dx$$

We will use the property of modulus carefully.

Since $|x|$ changes at $x = 0$, split the integral into two parts:

$$I = \int_{-1}^0 \left(\frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} \right) dx + \int_0^1 \left(\frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} \right) dx$$

For $-1 \leq x < 0$

Here, $|x| = -x$.

So,

$$x^3 + |x| + 1 = x^3 - x + 1$$

and

$$x^2 + 2|x| + 1 = x^2 - 2x + 1 = (x - 1)^2$$

Thus,

$$I_1 = \int_{-1}^0 \frac{x^3-x+1}{(x-1)^2} dx$$

For $0 \leq x \leq 1$

Here, $|x| = x$.

So,

$$x^3 + |x| + 1 = x^3 + x + 1$$

and

$$x^2 + 2|x| + 1 = x^2 + 2x + 1 = (x+1)^2$$

Thus,

$$I_2 = \int_0^1 \frac{x^3+x+1}{(x+1)^2} dx$$

Hence,

$$I = I_1 + I_2$$

Now simplify both integrands.

Simplifying I_1

We divide:

$$\frac{x^3-x+1}{(x-1)^2} = \frac{x^3-x+1}{x^2-2x+1}$$

On division,

$$\frac{x^3-x+1}{x^2-2x+1} = x + 2 + \frac{2x-1}{(x-1)^2}$$

Also,

$$\frac{2x-1}{(x-1)^2} = \frac{2(x-1)+1}{(x-1)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

So,

$$I_1 = \int_{-1}^0 \left(x + 2 + \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

Integrating term by term,

$$\int x dx = \frac{x^2}{2}, \quad \int 2 dx = 2x$$

$$\int \frac{2}{x-1} dx = 2 \ln|x-1|$$

$$\int \frac{1}{(x-1)^2} dx = \int (x-1)^{-2} dx = -\frac{1}{x-1}$$

Therefore,

$$I_1 = \left[\frac{x^2}{2} + 2x + 2 \ln|x-1| - \frac{1}{x-1} \right]_{-1}^0$$

At $x = 0$:

$$\frac{0}{2} + 0 + 2 \ln 1 - \frac{1}{-1} = 1$$

At $x = -1$:

$$\frac{1}{2} - 2 + 2 \ln 2 - \frac{1}{-2} = \frac{1}{2} - 2 + 2 \ln 2 + \frac{1}{2} = -1 + 2 \ln 2$$

So,

$$I_1 = 1 - (-1 + 2 \ln 2) = 2 - 2 \ln 2$$

Simplifying I_2

Now,

$$\frac{x^3+x+1}{(x+1)^2} = \frac{x^3+x+1}{x^2+2x+1}$$

On division,

$$\frac{x^3+x+1}{(x+1)^2} = x - 2 + \frac{4x+3}{(x+1)^2}$$

Also,

$$\frac{4x+3}{(x+1)^2} = \frac{4(x+1)-1}{(x+1)^2} = \frac{4}{x+1} - \frac{1}{(x+1)^2}$$

So,

$$I_2 = \int_0^1 \left(x - 2 + \frac{4}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

Integrating term by term,

$$I_2 = \left[\frac{x^2}{2} - 2x + 4 \ln|x+1| + \frac{1}{x+1} \right]_0^1$$

At $x = 1$:

$$\frac{1}{2} - 2 + 4 \ln 2 + \frac{1}{2} = -1 + 4 \ln 2$$

At $x = 0$:

$$0 - 0 + 4 \ln 1 + 1 = 1$$

So,

$$I_2 = (-1 + 4 \ln 2) - 1 = -2 + 4 \ln 2$$

Adding I_1 and I_2

$$I = (2 - 2 \ln 2) + (-2 + 4 \ln 2) = 2 \ln 2$$

Therefore,

$$\int_{-1}^1 \left(\frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} \right) dx = 2 \log_e 2$$

So, the correct option is:

Option B

Question 29

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & \alpha \\ 0 & 1 & -1 \end{bmatrix}$ be a singular matrix. Let $f(x) = \int_0^x (t^2 + 2t + 3) dt$, $x \in [1, \alpha]$. If M and m are respectively the maximum and the minimum values of f in $[1, \alpha]$, then $3(M - m)$ is equal to :

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Options:

A.

64

B.

68

C.

72

D.

76

Answer: B

Solution:

Since A is a singular matrix, its determinant must be zero.

We have

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & \alpha \\ 0 & 1 & -1 \end{bmatrix}$$

So,

$$|A| = 0$$

Now let us calculate the determinant by expanding along the first row:

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 1 & \alpha \\ 0 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & \alpha \\ 0 & -1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 1[(1)(-1) - (\alpha)(0)] - 3[(2)(-1) - (\alpha)(0)] - 1[(2)(1) - (1)(0)] \\ &= (-1 - \alpha) - 3(-2) - 2 \\ &= -1 - \alpha + 6 - 2 \\ &= 3 - \alpha \end{aligned}$$

Since $|A| = 0$,

$$3 - \alpha = 0 \Rightarrow \alpha = 3$$

Now,

$$f(x) = \int_0^x (t^2 + 2t + 3) dt$$

Integrating,

$$f(x) = \left[\frac{t^3}{3} + t^2 + 3t \right]_0^x = \frac{x^3}{3} + x^2 + 3x$$

We need maximum and minimum values of $f(x)$ in $[1, \alpha] = [1, 3]$.

Now check whether $f(x)$ is increasing or decreasing.

$$f'(x) = x^2 + 2x + 3$$

$$f'(x) = x^2 + 2x + 3 = (x+1)^2 + 2 > 0 \quad \text{for all } x$$

So $f(x)$ is strictly increasing on $[1, 3]$.

Hence,

- minimum value at $x = 1$
- maximum value at $x = 3$

Now,

$$m = f(1) = \frac{1}{3} + 1 + 3 = \frac{13}{3}$$

and

$$M = f(3) = \frac{27}{3} + 9 + 9 = 9 + 9 + 9 = 27$$

Therefore,

$$M - m = 27 - \frac{13}{3} = \frac{81-13}{3} = \frac{68}{3}$$

So,

$$3(M - m) = 3 \cdot \frac{68}{3} = 68$$

Hence the correct answer is:

68

So, the correct option is **Option B**.

Question 30

Let $f(x) = \begin{cases} \frac{1}{3}, & x \leq \pi/2 \\ \frac{b(1-\sin x)}{(\pi-2x)^2}, & x > \pi/2 \end{cases}$. If f is continuous at $x = \pi/2$, then the value of

$$\int_0^{\pi/2} |x^2 + 2x - 3| dx \text{ is :}$$

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Options:

A.

5

B.

2

C.

3

D.

4

Answer: D

Solution:

The function $f(x)$ is continuous at $x = \frac{\pi}{2}$ where

$$f(x) = \begin{cases} \frac{1}{3} & ; x \leq \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & ; x > \frac{\pi}{2} \end{cases} \dots (i).$$

We observe the LHL from equation (i) :

$$\lim_{x \rightarrow (\frac{\pi}{2})^-} f(x) = f\left(\frac{\pi}{2}\right) = \frac{1}{3} \dots (ii)$$

We must have the RHL from equation (i) to be $\frac{1}{3}$ for the function $f(x)$ to be continuous at $x = \frac{\pi}{2}$:

$$\lim_{x \rightarrow (\frac{\pi}{2})^+} f(x) = \frac{1}{3} \Rightarrow \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{b(1-\sin x)}{(\pi-2x)^2} = \frac{1}{3} \dots (iii)$$

$$\Rightarrow \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{b(1-\sin x)}{(\pi-2x)^2} = \frac{0}{0} \text{ form} \leftarrow \text{Apply L'hôpital's rule}$$

$$\Rightarrow \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{b \cos x}{4(\pi-2x)} = \frac{0}{0} \text{ form} \leftarrow \text{Apply L'hôpital's rule once more}$$

$$\Rightarrow \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{b \sin x}{8} = \frac{b}{8} \dots (iv)$$

We can mathematically demonstrate conditional continuity of $f(x)$ at $x = \frac{\pi}{2}$ from equation (iii) and (iv) :

$$\frac{b}{8} = \frac{1}{3} \Rightarrow b = \frac{8}{3} \dots (v)$$

$$\text{Now, } I = \int_0^{\pi/2} |x^2 + 2x - 3| dx$$

$$= \int_0^{\pi/2} |x^2 + 2x - 3| dx$$

$$\begin{aligned}
 &= \int_0^2 |x^2 + 2x - 3| dx \\
 &= \int_0^2 |(x+3)(x-1)| dx \\
 &\begin{cases} (x+3)(x-1) \geq 0; x \in (-\infty, -3] \\ (x+3)(x-1) \leq 0; x \in [-3, 1] \\ (x+3)(x-1) \geq 0; x \in [1, \infty) \end{cases} \\
 &= \int_0^1 (x+3)(1-x) dx + \int_1^2 (x+3)(x-1) dx \\
 &= \int_0^1 (3-2x-x^2) dx + \int_1^2 (x^2+2x-3) dx \\
 &= \left[3x - x^2 - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x \right]_1^2 \\
 &= \left(3 - 1 - \frac{1}{3} \right) + \left(\frac{8}{3} + 4 - 6 \right) - \left(\frac{1}{3} + 1 - 3 \right) \\
 &= 4
 \end{aligned}$$

Question 31

The value of the integral $\int_0^2 \frac{\sqrt{x(x^2+x+1)}}{(\sqrt{x+1})(\sqrt{x^4+x^2+1})} dx$ is equal to:

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Options:

A.

$$\frac{1}{3} \log_e (3 - 2\sqrt{2})$$

B.

$$\frac{2}{3} \log_e (4 + \sqrt{2})$$

C.

$$\frac{2}{3} \log_e (3 + 2\sqrt{2})$$

D.

$$\frac{1}{3} \log_e (1 + 6\sqrt{2})$$

Answer: C

Solution:

$$I = \int_0^2 \left[\frac{\sqrt{x(x^2+x+1)}}{(\sqrt{x+1}) \cdot (\sqrt{x^4+x^2+1})} \right] dx$$

$$\begin{aligned}
&= \int_0^2 \left[\frac{\sqrt{x(x^2+x+1)}}{(\sqrt{x+1}) \cdot (\sqrt{x^4+2x^2+1-x^2})} \right] dx \\
&= \int_0^2 \left[\frac{\sqrt{x(x^2+x+1)}}{(\sqrt{x+1}) \cdot (\sqrt{(x^2+1)^2-x^2})} \right] dx \\
&= \int_0^2 \left[\frac{\sqrt{x(x^2+x+1)}}{(\sqrt{x+1}) \cdot \left\{ \sqrt{(x^2+1+x)(x^2+1-x)} \right\}} \right] dx \\
&= \int_0^2 \left[\frac{(\sqrt{x}) \cdot (\sqrt{x^2+x+1})}{(\sqrt{x+1}) \cdot (\sqrt{x^2+1+x}) \cdot (\sqrt{x^2+1-x})} \right] dx \\
&= \int_0^2 \left[\frac{\sqrt{x}}{(\sqrt{x+1}) \cdot (\sqrt{x^2-x+1})} \right] dx \\
&= \int_0^2 \left[\frac{\sqrt{x}}{\sqrt{(x+1)(x^2-x+1)}} \right] dx \\
&= \int_0^2 \left[\frac{\sqrt{x}}{\sqrt{x^3+1}} \right] dx
\end{aligned}$$

$$\text{Let } x = t^2 \Rightarrow dx = 2t dt$$

$$I = \int_0^{\sqrt{2}} \left[\frac{2t^2}{\sqrt{t^6+1}} \right] dt = \left(\frac{2}{3} \right) \int_0^{\sqrt{2}} \left[\frac{3t^2}{\sqrt{t^6+1}} \right] dt$$

$$\text{Let } t^3 = u \Rightarrow 3t^2 dt = du$$

$$I = \left(\frac{2}{3} \right)^2 \cdot \int_0^{2\sqrt{2}} \left[\frac{du}{\sqrt{u^2+1}} \right]$$

Using standard integration formulae :

$$\begin{aligned}
\int \left(\frac{1}{\sqrt{x^2+a^2}} \right) dx &= \ln |x + \sqrt{x^2+a^2}| + C \\
&= \left(\frac{2}{3} \right) \cdot \left[\ln |u + \sqrt{u^2+1}| \right]_0^{2\sqrt{2}} \\
&= \left(\frac{2}{3} \right) \cdot [\ln(2\sqrt{2}+3) - \ln 1] = \left(\frac{2}{3} \right) \cdot [\ln(3+2\sqrt{2})]
\end{aligned}$$

Question 32

If $\int (\sin x)^{-\frac{11}{2}} (\cos x)^{-\frac{5}{2}} dx = -\frac{p_1}{q_1} (\cot x)^{\frac{9}{2}} - \frac{p_2}{q_2} (\cot x)^{\frac{5}{2}} - \frac{p_3}{q_3} (\cot x)^{\frac{1}{2}} + \frac{p_4}{q_4} (\cot x)^{-\frac{3}{2}} + C$, where p_i and q_i are positive integers with $\gcd(p_i, q_i) = 1$ for $i = 1, 2, 3, 4$ and C is the constant of integration, then $\frac{15p_1p_2p_3p_4}{q_1q_2q_3q_4}$ is equal to _____

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Answer: 16

Solution:

$$\begin{aligned}
\int (\sin x)^{-\frac{11}{2}} (\cos x)^{-\frac{5}{2}} dx &= \int \frac{1}{\tan^{11/2} x} \cdot \sec^8 x dx = \int \frac{(1 + \tan^2 x)^3 \sec^2 x}{\tan^{11/2} x} dx \\
\text{put } \tan x &= t \\
\frac{15p_1p_2p_3p_4}{q_1q_2q_3q_4} &= \frac{15(2 \times 6 \times 6 \times 2)}{9 \times 5 \times 1 \times 3} = 16
\end{aligned}$$

Question 33

Let $f(x) = \int \frac{(2-x^2) \cdot e^x}{(\sqrt{1+x})(1-x)^{3/2}} dx$. If $f(0) = 0$, then $f\left(\frac{1}{2}\right)$ is equal to:

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Options:

A.

$$\sqrt{2e} - 1$$

B.

$$\sqrt{2e} + 1$$

C.

$$\sqrt{3e} - 1$$

D.

$$\sqrt{3e} + 1$$

Answer: C

Solution:

$$f(x) = \int \frac{(2-x^2) \cdot e^x}{(\sqrt{1+x})(1-x)^{3/2}} dx$$

$$\Rightarrow f(x) = \int \frac{(2-x^2) e^x}{\sqrt{1+x} \sqrt{1-x} (1-x)} dx$$

$$f(x) = \int \frac{(2-x^2) \cdot e^x}{(\sqrt{1-x^2})(1-x)} dx$$

$$= \int \frac{[1 + (1-x^2)] e^x}{(\sqrt{1-x^2})(1-x)} dx$$

$$= \int e^x \left[\frac{1}{(\sqrt{1-x^2})(1-x)} + \frac{1+x}{\sqrt{1-x^2}} \right] dx$$

This integral of the form $\int e^x [g(x) + g'(x)] dx = e^x \cdot g(x) + C$

$$\text{Let } g(x) = \frac{1+x}{\sqrt{1-x^2}}$$

$$g'(x) = \frac{(1+x)' \cdot \sqrt{1-x^2} - (1+x) (\sqrt{1-x^2})'}{(\sqrt{1-x^2})^2}$$

$$= \frac{\sqrt{1-x^2} + \frac{(x^2+x)}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1-x^2+x^2+x}{\sqrt{1-x^2}(1-x^2)}$$

$$= \frac{(x+1)}{\sqrt{1-x^2}(1-x)(1+x)}$$

$$= \frac{1}{(\sqrt{1-x^2})(1-x)}$$

So value of integral

$$f(x) = \frac{e^x \cdot (1+x)}{\sqrt{1-x^2}} + C$$

Apply initial condition $f(0) = 0$ to find C

$$0 = \frac{e^0(1+0)}{\sqrt{1-0}} + C \Rightarrow C = -1$$

$$\Rightarrow f(x) = \frac{e^x(1+x)}{\sqrt{1-x^2}} - 1$$

$$\therefore f(1/2) = \frac{e^{1/2}(1+1/2)}{\sqrt{1-(1/2)^2}} - 1$$

$$= \frac{\sqrt{e} \cdot \frac{3}{2}}{\frac{\sqrt{3}}{2}} - 1 = \sqrt{3e} - 1$$

Question 34

Let $I(x) = \int \frac{3dx}{(4x+6)(\sqrt{4x^2+8x+3})}$ and $I(0) = \frac{\sqrt{3}}{4} + 20$. If

$I\left(\frac{1}{2}\right) = \frac{a\sqrt{2}}{b} + c$, where $a, b, c \in \mathbb{N}$, $\gcd(a, b) = 1$, then $a + b + c$ is equal to :

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Options:

A.

30

B.

29

C.

28

D.

31

Answer: D

Solution:

$$I(x) = \int \frac{3dx}{(4x+6)\sqrt{4x^2+8x+3}} = \int \frac{3dx}{2(2x+3)\sqrt{(2x+2)^2-1}}$$

$$\text{let } 2x+3 = \frac{1}{t} \Rightarrow 2dx = -\frac{1}{t^2}dt \Rightarrow dx = -\frac{1}{2t^2}dt$$

$$\text{let } 2x+2 = \frac{1}{t} - 1 = \frac{1-t}{t}$$

$$I(x) = \int \frac{3\left(-\frac{1}{2t^2}dt\right)}{2\left(\frac{1}{t}\right)\sqrt{\left(\frac{1-t}{t}\right)^2-1}} = \int \frac{-3dt}{4t\sqrt{\frac{1+t^2-2t-t^2}{t^2}}} = \int \frac{-3dt}{4\sqrt{1-2t}}$$

$$I(x) = -\frac{3}{4} \int (1-2t)^{-1/2} dt$$

$$I(x) = -\frac{3}{4} \left[\frac{(1-2t)^{1/2}}{(1/2)(-2)} \right] + C = \frac{3}{4} \sqrt{1-2t} + C$$

$$\text{Substitute } t = \frac{1}{2x+3}$$

$$I(x) = \frac{3}{4} \sqrt{1 - \frac{2}{2x+3}} + C = \frac{3}{4} \sqrt{\frac{2x+3-2}{2x+3}} + C = \frac{3}{4} \sqrt{\frac{2x+1}{2x+3}} + C$$

$$I(0) = \frac{\sqrt{3}}{4} + 20 \quad (\text{Given}), \text{ substitute } x = 0 \text{ to find value of } C.$$

$$\frac{3}{4} \sqrt{\frac{1}{3}} + C = \frac{\sqrt{3}}{4} + 20 \Rightarrow \frac{\sqrt{3}}{4} + C = \frac{\sqrt{3}}{4} + 20 \Rightarrow C = 20$$

$$I(x) = \frac{3}{4} \sqrt{\frac{2x+1}{2x+3}} + 20$$

$$x = \frac{1}{2} \text{ for } I\left(\frac{1}{2}\right)$$

$$I\left(\frac{1}{2}\right) = \frac{3}{4} \sqrt{\frac{1+1}{1+3}} + 20 = \frac{3}{4} \sqrt{\frac{2}{4}} + 20 = \frac{3\sqrt{2}}{8} + 20$$

Comparing with $\frac{a\sqrt{b}}{c} + c$:

$$a = 3, b = 8, c = 20$$

$$a + b + c = 3 + 8 + 20 = 31$$

Question 35

Let $f(t) = \int \left(\frac{1 - \sin(\log_e t)}{1 - \cos(\log_e t)} \right) dt, t > 1$.

If $f(e^{\pi/2}) = -e^{\pi/2}$ and $f(e^{\pi/4}) = \alpha e^{\pi/4}$, then α equals

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Options:

A.

$$1 + \sqrt{2}$$

B.

$$-1 - 2\sqrt{2}$$

C.

$$-1 - \sqrt{2}$$

D.

$$-1 + \sqrt{2}$$

Answer: C

Solution:

$$f(t) = \int \left(\frac{1 - \sin(\log_e t)}{1 - \cos(\log_e t)} \right) dt, t > 1$$

$$\text{Let } \log_e t = x,$$

$$t = e^x \Rightarrow dt = e^x dx$$

$$\begin{aligned} f(t) &= \int \left(\frac{1 - \sin x}{1 - \cos x} \right) e^x dx \\ &= \int \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) e^x dx \left\{ \begin{array}{l} \text{use } \sin 2\theta = 2 \sin \theta \cos \theta \\ 1 - \cos 2\theta = 2 \sin^2 \theta \end{array} \right\} \end{aligned}$$

$$f(t) = \int \left[\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right] e^x dx$$

This standard integral of type $\int e^x [g(x) + g'(x)] = e^x \cdot g(x) + C$.

$$\text{let } g(x) = -\cot \frac{x}{2},$$

$$g'(x) = -(-\operatorname{cosec}^2 \frac{x}{2}) \cdot \frac{1}{2} = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}.$$

$$\text{so } f(t) = e^x \cdot (-\cot \frac{x}{2}) + C.$$

substitute value of $x = \log_e t$ back.

$$f(t) = -t \cot\left(\frac{\log_e t}{2}\right) + C$$

use $f(e^{\pi/2}) = -e^{\pi/2}$ to find C :

$$-e^{\pi/2} = -e^{\pi/2} \cot\left(\frac{\log_e e^{\pi/2}}{2}\right) + C$$

$$-e^{\pi/2} = -e^{\pi/2} \cot\left(\frac{\pi}{4}\right) + C$$

$$-e^{\pi/2} = -e^{\pi/2} + C \Rightarrow C = 0$$

$$f(t) = -t \cot\left(\frac{\log_e t}{2}\right)$$

It is given that $f(e^{\pi/4}) = \alpha e^{\pi/4}$.

$$\alpha e^{\pi/4} = -e^{\pi/4} \cot\left(\frac{\log_e e^{\pi/4}}{2}\right)$$

$$\alpha = -\cot\left(\frac{\pi}{8}\right)$$

using result $1 + \cos 2\theta = 2 \cos^2 \theta$... (1)

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{2 \cos^2 \theta}{2 \sin \theta \cos \theta} = \cot \theta \quad \dots (2)$$

$$\alpha = -\cot\left(\frac{\pi}{8}\right) = -\left(\frac{1 + \cos \frac{\pi}{4}}{\sin \frac{\pi}{4}}\right)$$

$$\alpha = -\left(\frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = -(\sqrt{2} + 1).$$

Question 36

If $\int \left(\frac{1-5 \cos^2 x}{\sin^5 x \cos^2 x} \right) dx = f(x) + C$, where **C** is the constant of integration, then $f\left(\frac{\pi}{6}\right) - f\left(\frac{\pi}{4}\right)$ is equal to

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Options:

A.

$$\frac{1}{\sqrt{3}}(26 - \sqrt{3})$$

B.

$$\frac{4}{\sqrt{3}}(8 - \sqrt{6})$$

C.

$$\frac{1}{\sqrt{3}}(26 + \sqrt{3})$$

D.

$$\frac{2}{\sqrt{3}}(4 + \sqrt{6})$$

Answer: B

Solution:

The given integral is :

$$I = \int \left(\frac{1-5\cos^2 x}{\sin^5 x \cos^2 x} \right) dx$$

We can rewrite the integrand by splitting the terms:

$$I = \int \left(\frac{1}{\sin^5 x \cos^2 x} - \frac{5 \cos^2 x}{\sin^5 x \cos^2 x} \right) dx$$

$$I = \int \left(\frac{\sec^2 x}{\sin^5 x} - 5 \operatorname{cosec}^5 x \right) dx$$

Observe that the derivative of $f(x) = \frac{\tan x}{\sin^5 x}$ gives us this exact expression:

$$\begin{aligned} \frac{d}{dx} \left(\frac{\tan x}{\sin^5 x} \right) &= \frac{\sec^2 x \cdot \sin^5 x - \tan x \cdot (5 \sin^4 x \cos x)}{\sin^{10} x} \\ &= \frac{\sec^2 x \sin^5 x - 5 \sin^5 x}{\sin^{10} x} = \frac{\sec^2 x - 5}{\sin^5 x} = \frac{1 - 5 \cos^2 x}{\sin^5 x \cos^2 x} \end{aligned}$$

$$\text{Therefore, } f(x) = \frac{\tan x}{\sin^5 x} = \frac{\sin x / \cos x}{\sin^5 x} = \frac{1}{\sin^4 x \cos x}.$$

$$\text{Calculate } f\left(\frac{\pi}{6}\right) - f\left(\frac{\pi}{4}\right)$$

We substitute the values into the function $f(x) = \frac{1}{\sin^4 x \cos x}$:

• At $x = \frac{\pi}{6}$:

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ f\left(\frac{\pi}{6}\right) &= \frac{1}{(1/2)^4 \cdot (\sqrt{3}/2)} = \frac{1}{(1/16) \cdot (\sqrt{3}/2)} = \frac{32}{\sqrt{3}} \end{aligned}$$

• At $x = \frac{\pi}{4}$:

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{(1/\sqrt{2})^4 \cdot (1/\sqrt{2})} = \frac{1}{(1/4) \cdot (1/\sqrt{2})} = 4\sqrt{2}$$

$$\therefore f\left(\frac{\pi}{6}\right) - f\left(\frac{\pi}{4}\right) = \frac{32}{\sqrt{3}} - 4\sqrt{2}$$

Factor out $\frac{4}{\sqrt{3}}$ to match the options :

$$= \frac{4}{\sqrt{3}} (8 - \sqrt{2} \cdot \sqrt{3}) = \frac{4}{\sqrt{3}} (8 - \sqrt{6})$$

Question 37

Let $f(x) = \int \frac{dx}{x^{2/3} + 2x^{1/2}}$, be such that $f(0) = -26 + 24 \log_e(2)$. If $f(1) = a + b \log_e(3)$, where $a, b \in \mathbb{Z}$, then $a + b$ is equal to :

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Options:

A.

-5

B.

-11

C.

-18

D.

-26

Answer: B

Solution:

Given

$$f(x) = \int \frac{dx}{x^{2/3} + 2x^{1/2}},$$

$$f(0) = -26 + 24 \ln 2 \text{ and}$$

$$f(1) = a + b \ln 3, \quad a, b \in \mathbb{Z}$$

Use substitution Let $x = t^6 \Rightarrow dx = 6t^5 dt$

$$x^{2/3} = (t^6)^{2/3} = t^4$$

$$x^{1/2} = (t^6)^{1/2} = t^3$$

Substituting these into the integral :

$$f(x) = \int \frac{6t^5}{t^4 + 2t^3} dt = \int \frac{6t^5}{t^3(t+2)} dt = \int \frac{6t^2}{t+2} dt$$

manipulate

$$\frac{t^2}{t+2} = \frac{(t^2-4)+4}{t+2} = \frac{(t-2)(t+2)+4}{t+2} = (t-2) + \frac{4}{t+2}$$

Now, integrate term by term:

$$f(x) = 6 \int \left(t - 2 + \frac{4}{t+2} \right) dt$$

$$f(x) = 6 \left[\frac{t^2}{2} - 2t + 4 \ln |t+2| \right] + C$$

$$f(x) = 3t^2 - 12t + 24 \ln |t+2| + C$$

Back-substitute $t = x^{1/6}$:

$$f(x) = 3x^{1/3} - 12x^{1/6} + 24 \ln |x^{1/6} + 2| + C$$

Finding the Constant C

We are given the initial condition $f(0) = -26 + 24 \ln(2)$:

$$\begin{aligned} f(0) &= 3(0) - 12(0) + 24 \ln(0+2) + C \\ -26 + 24 \ln(2) &= 24 \ln(2) + C \Rightarrow C = -26 \end{aligned}$$

So, the complete function is:

$$f(x) = 3x^{1/3} - 12x^{1/6} + 24 \ln (x^{1/6} + 2) - 26$$

Solving for a and b

Now we evaluate $f(1)$:

$$f(1) = 3(1)^{1/3} - 12(1)^{1/6} + 24 \ln (1^{1/6} + 2) - 26$$

$$f(1) = 3 - 12 + 24 \ln(3) - 26$$

$$f(1) = -35 + 24 \ln(3)$$

Comparing this to the given form $f(1) = a + b \ln(3)$:

$$a = -35 \quad \text{and} \quad b = 24$$

Then

$$a + b = -35 + 24 = -11$$

Question 38

Let $f(x) = \int \left(\frac{16x+24}{x^2+2x-15} \right) dx$. If $f(4) = 14 \log_e(3)$ and $f(7) = \log_e(2^\alpha \cdot 3^\beta)$, $\alpha, \beta \in \mathbb{N}$, then $\alpha + \beta$ is equal to :

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Options:

A.

31

B.

37

C.

39

D.

41

Answer: C

Solution:

$$f(x) = \int \frac{8(2x+2)+8}{x^2+2x-15} dx$$

$$\Rightarrow f(x) = 8\ln|x^2+2x-15| + \ln\left|\frac{x-3}{x+5}\right| + C$$

$$\Rightarrow f(4) = 14\ln 3 + C$$

$$\Rightarrow 14\ln 3 = 14\ln 3 + C$$

$$\therefore C = 0$$

$$\text{Now, } f(7) = 8\ln 48 - \ln 3$$

$$= \ln\left(\frac{(48)^8}{3}\right)$$

$$\Rightarrow f(7) = \ln(2^{32} \times 3^7)$$

$$\alpha = 32, \beta = 7 \Rightarrow \alpha + \beta = 39$$

Question 39

Let $f : (0, \infty) \rightarrow \mathbf{R}$ be a twice differentiable function. If for some

$a \neq 0$, $\int_0^1 f(\lambda x) d\lambda = af(x)$, $f(1) = 1$ and $f(16) = \frac{1}{8}$, then $16 - f'\left(\frac{1}{16}\right)$ is equal to _____.

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Answer: 112

Solution:

$$\int_0^1 f(\lambda x) d\lambda = af(x)$$

$$\lambda x = t$$

$$d\lambda = \frac{1}{x} dt$$

$$\frac{1}{x} \int_0^x f(t) dt = af(x)$$

$$\int_0^x f(t) dt = axf(x)$$

$$f(x) = a(xf'(x) + f(x))$$

$$(1-a)f(x) = a \cdot xf'(x)$$

$$\frac{f'(x)}{f(x)} = \frac{(1-a)}{a} \frac{1}{x}$$

$$\ell \ln f(x) = \frac{1-a}{a} \ell \ln x + c$$

$$x=1, f(1)=1 \Rightarrow c=0$$

$$x=16, f(16) = \frac{1}{8}$$

$$\frac{1}{8} = (16)^{\frac{1-a}{a}} \Rightarrow -3 = \frac{4-4a}{a} \Rightarrow a=4$$

$$f(x) = x^{-\frac{3}{4}}$$

$$f'(x) = -\frac{3}{4} x^{-\frac{7}{4}}$$

$$\therefore 16 - f' \left(\frac{1}{16} \right)$$

$$= 16 - \left(-\frac{3}{4} (2^{-4})^{-7/4} \right)$$

$$= 16 + 96 = 112$$

Question40

If $\lim_{t \rightarrow 0} \left(\int_0^1 (3x+5)^t dx \right)^{\frac{1}{t}} = \frac{\alpha}{5e} \left(\frac{8}{5} \right)^{\frac{2}{3}}$, then α is equal to _____.

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Answer: 64

Solution:

1^∞ form

$$\text{Now } L = e^{t \rightarrow 0} \frac{1}{t} \left(\left(\frac{(3x+5)^{t+1}}{3(t+1)} \right) \Big|_0^1 - 1 \right)$$

$$= e^{t \rightarrow 0} \frac{8^{t+1} - 5^{t+1} - 3t - 3}{3t(t+1)}$$

$$= e \frac{8 \ln 8 - 5 \ln 5 - 3}{3}$$

$$= \left(\frac{8}{5} \right)^{2/3} \left(\frac{64}{5} \right) = \frac{\alpha}{5e} \left(\frac{8}{5} \right)^{2/3}$$

On comparing

$$\alpha = 64$$

Question41

If $24 \int_0^{\frac{\pi}{4}} \left[\sin \left| 4x - \frac{\pi}{12} \right| + [2 \sin x] \right] dx = 2\pi + \alpha$, where $[\cdot]$ denotes the greatest integer function, then α is equal to _____.

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Answer: 12

Solution:

$$\begin{aligned}
 &= 24 \int_0^{\frac{\pi}{48}} -\sin \left(4x - \frac{\pi}{12} \right) + \int_{\frac{\pi}{48}}^{\frac{\pi}{4}} \sin \left(4x - \frac{\pi}{12} \right) \\
 &+ \int_0^{\frac{\pi}{6}} [0] dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} [2 \sin x] dx \\
 &= 24 \left[\frac{(1 - \cos \frac{\pi}{12})}{4} - \frac{(-\cos \frac{\pi}{12} - 1)}{4} \right] + \frac{\pi}{4} - \frac{\pi}{6} \\
 &= 24 \left(\frac{1}{2} + \frac{\pi}{12} \right) = 2\pi + 12 \\
 \alpha &= 12
 \end{aligned}$$

Question 42

Let $[\cdot]$ denote the greatest integer function. If $\int_0^{e^3} \left[\frac{1}{e^x - 1} \right] dx = \alpha - \log_e 2$, then α^3 is equal to _____.

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Answer: 8

Solution:

To solve this, we start by evaluating the integral:

$$I = \int_0^{e^3} \left[\frac{1}{e^x - 1} \right] dx$$

The greatest integer function $[\cdot]$ returns the largest integer less than or equal to the input value. Here's how we can approach the problem:

Determine the function inside the integral:

$$\frac{1}{e^x - 1} = e^{1-x}.$$

Identifying the intervals:

When $e^{1-x} \geq 2$, which simplifies to $x \leq 1 - \ln 2$, we have $[e^{1-x}] = 2$.

When $1 \leq e^{1-x} < 2$, simplifying gives $1 - \ln 2 < x \leq 1$, and thus $[e^{1-x}] = 1$.

When $0 \leq e^{1-x} < 1$, which holds for $x > 1$, thus $[e^{1-x}] = 0$ from $x = 1$ to $x = e^3$.

Evaluate the integral on these intervals:

$$\int_0^{1-\ln 2} 2 dx = 2(1 - \ln 2)$$

$$\int_{1-\ln 2}^1 1 \, dx = 1 - (1 - \ln 2) = \ln 2$$

$$\int_1^{e^3} 0 \, dx = 0$$

Combine these results:

$$I = 2(1 - \ln 2) + \ln 2 + 0 = 2 - \ln 2$$

Thus, we are given that:

$$\alpha - \ln 2 = 2 - \ln 2$$

This implies that:

$$\alpha = 2$$

$$\text{Therefore, } \alpha^3 = 2^3 = 8.$$

Question 43

Let for $f(x) = 7 \tan^8 x + 7 \tan^6 x - 3 \tan^4 x - 3 \tan^2 x$, $I_1 = \int_0^{\pi/4} f(x) dx$ and $I_2 = \int_0^{\pi/4} x f(x) dx$. Then $7I_1 + 12I_2$ is equal to :

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Options:

A. 2π

B. 1

C. π

D. 2

Answer: B

Solution:

$$\begin{aligned} f(x) &= 7 \tan^8 x + 7 \tan^6 x - 3 \tan^4 x - 3 \tan^2 x \\ &= 7 \tan^6 x (1 + \tan^2 x) - 3 \tan^2 x (1 + \tan^2 x) \\ &= (7 \tan^6 x - 3 \tan^2 x) (1 + \tan^2 x) \\ &= (7 \tan^6 x - 3 \tan^2 x) \sec^2 x \end{aligned}$$

$$\begin{aligned} I_1 &= \int_0^{\pi/4} f(x) dx = \int_0^{\pi/4} (7 \tan^6 x - 3 \tan^2 x) \sec^2 x dx \\ &= \left(\frac{7 \tan^7 x}{7} - \frac{3 \tan^3 x}{3} \right) \Bigg|_0^{\pi/4} = 1 - 1 = 0 \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^{\pi/4} x f(x) dx = \int_0^{\pi/4} x (7 \tan^6 x - 3 \tan^2 x) \sec^2 x dx \\ &= x (\tan^7 x - \tan^3 x) \Big|_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot (\tan^7 x - \tan^3 x) dx \\ &= 0 - \int_0^{\pi/4} \tan^3 x (\tan^2 x - 1) (\tan^2 x + 1) dx \\ &= \int_0^{\pi/4} (\tan^3 x - \tan^5 x) \sec^2 x dx = \frac{\tan^4 x}{4} - \frac{\tan^6 x}{6} \Bigg|_0^{\pi/4} \\ &= \frac{1}{12} \end{aligned}$$

$$\text{Hence } 7I_1 + 12I_2 = 1$$

Question 44

The value of $\int_{e^2}^{e^4} \frac{1}{x} \left(\frac{e^{((\log_e x)^2 + 1)^{-1}}}{e^{((\log_e x)^2 + 1)^{-1}} + e^{((6 - \log_e x)^2 + 1)^{-1}}} \right) dx$ is

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Options:

- A. 1
- B. $\log_e 2$
- C. e^2
- D. 2

Answer: A

Solution:

$$\text{Let } \ln x = t \Rightarrow \frac{dx}{x} = dt$$

$$I = \int_2^4 \frac{e^{\frac{1}{1+t^2}}}{e^{\frac{1}{1+t^2}} + e^{\frac{1}{1+(6-t)^2}}} dt$$

$$I = \int_2^4 \frac{\frac{1}{e^{\frac{1}{1+(6-t)^2}}}}{\frac{1}{e^{\frac{1}{1+(6-t)^2}} + e^{\frac{1}{1+t^2}}}} dt$$

$$2I = \int_2^4 dt = (t)_2^4 = 4 - 2 = 2$$

$$I = 1$$

Question 45

If $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$, then $\int_0^{2I} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ equals :

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Options:

- A. $\frac{\pi^2}{12}$
- B. $\frac{\pi^2}{4}$
- C. $\frac{\pi^2}{16}$
- D. $\frac{\pi^2}{8}$

Answer: C

Solution:

For I

Apply king (P – 5) and add

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

$$I_2 = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Apply king and add

$$I_2 = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \sec^2 x dx}{\tan^4 x + 1}$$

put $\tan^2 x = t$

$$\frac{\pi}{8} \int_0^{\infty} \frac{dt}{t^2 + 1}$$

$$= \frac{\pi}{8} \cdot \frac{\pi}{2} = \frac{\pi^2}{16}$$

Question 46

If $I(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$, $m, n > 0$, then $I(9, 14) + I(10, 13)$ is

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Options:

A. $I(9, 1)$

B. $I(1, 13)$

C. $I(19, 27)$

D. $I(9, 13)$

Answer: D

Solution:

$$I(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Let } x = \sin^2 \theta \quad dx = 2 \sin \theta \cos \theta d\theta$$

$$I(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta$$

$$I(9, 14) + I(10, 13) = 2 \int_0^{\pi/2} (\sin \theta)^{17} (\cos \theta)^{27} d\theta$$

$$+ 2 \int_0^{\pi/2} (\sin \theta)^{19} (\cos \theta)^{25} d\theta$$

$$= 2 \int_0^{\pi/2} (\sin \theta)^{17} (\cos \theta)^{25} d\theta$$

$$= I(9, 13)$$

Question 47

If $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{96x^2 \cos^2 x}{(1+e^x)} dx = \pi(\alpha\pi^2 + \beta)$, $\alpha, \beta \in \mathbb{Z}$, then $(\alpha + \beta)^2$ equals

JEE Main 2025 (Online) 28th January Morning Shift

Options:

A. 196

B. 100

C. 64

D. 144

Answer: B

Solution:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{96x^2 \cos^2 x}{(1+e^x)} dx \quad (\text{Apply King Property})$$

$$\int_0^{\frac{\pi}{2}} 96x^2 \cos^2 x = 48 \int_0^{\frac{\pi}{2}} x^2 (1 + \cos 2x) dx$$

$$48 \left[\left(\frac{x^3}{3} \right)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x^2 \cos 2x dx \right]$$

$$\Rightarrow \text{On solving } \pi (2\pi^2 - 12)$$

$$\Rightarrow \alpha = 2, \beta = -12$$

$$\Rightarrow (\alpha + \beta)^2 = 100$$

Question 48

Let f be a real valued continuous function defined on the positive real axis such that

$$g(x) = \int_0^x t f(t) dt. \text{ If } g(x^3) = x^6 + x^7, \text{ then value of } \sum_{r=1}^{15} f(r^3) \text{ is :}$$

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Options:

A.

270

B.

340

C.

310

D.

320

Answer: C

Solution:

$$g(x) = x^2 + x^{\frac{7}{3}}$$

$$g'(x) = 2x + \frac{7}{3}x^{\frac{4}{3}}$$

$$f(x) = \frac{g'(x)}{x}$$

$$f(x) = 2 + \frac{7}{3}x^{\frac{1}{3}}$$

$$f(r^3) = 2 + \frac{7r}{3}$$

$$\sum_{r=1}^{15} \left(1 + \frac{7}{3}r\right) = 310$$

Question49

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function such that $f(2) = 1$. If $F(x) = \int_2^x f(x) dx$ for all $x \in \mathbb{R}$, $\int_0^2 xF'(x)dx = 6$ and $\int_0^2 x^2 F''(x)dx = 40$, then $F'(2) + \int_0^2 F(x)dx$ is equal to :

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Options:

A.

13

B.

11

C.

9

D.

15

Answer: B

Solution:

$$\begin{aligned} \int_0^2 xF'(x)dx &= 6 \\ &= xF(x)\Big|_0^2 - \int_0^2 f(x)dx = 6 \\ &= 2F(2) - \int_0^2 xF(x)dx = 6[\because f(2) = 2F(2) = 2] \end{aligned}$$

$$\int_0^2 xF(x)dx = -2 \quad \dots (1)$$

$$\Rightarrow \int_0^2 F(x)dx = -2 \quad \dots (2)$$

Also

$$\begin{aligned} \int_0^2 x^2 F''(x)dx &= x^2 F'(x)\Big|_0^2 - 2 \int_0^2 xF'(x)dx = 40 \\ &= 4F'(2) - 2 \times 6 = 40 \end{aligned}$$

$$F'(2) = 13$$

$$\therefore F'(2) + \int_0^2 F(x)dx = 13 - 2 = 11$$

Question 50

The integral $80 \int_0^{\frac{\pi}{4}} \left(\frac{\sin \theta + \cos \theta}{9 + 16 \sin 2\theta} \right) d\theta$ is equal to :

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Options:

A.

$$3 \log 4$$

B.

$$4 \log 3$$

C.

$$6 \log \frac{4}{3}$$

D.

$$2 \log 3$$

Answer: B

Solution:

$$\begin{aligned} I &= 80 \int_0^{\frac{\pi}{4}} \left(\frac{\sin \theta + \cos \theta}{9 + 16(2 \sin \theta \cdot \cos \theta)} \right) d\theta \\ &= 80 \int_0^{\frac{\pi}{4}} \frac{\sin \theta + \cos \theta}{9 - 16(1 - 2 \sin \theta \cdot \cos \theta - 1)} d\theta \end{aligned}$$

$$= 80 \int_0^{\frac{\pi}{4}} \frac{\sin \theta + \cos \theta}{9 + 16 - 16(\sin \theta - \cos \theta)^2} d\theta$$

$$\text{Let } \sin \theta - \cos \theta = t \\ (\cos \theta + \sin \theta) d\theta = dt$$

$$= 80 \int_{-1}^0 \frac{dt}{25 - 16t^2}$$

$$= \frac{80}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{5}{2 \left(\frac{5}{4}\right)} \ln \left| \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right| \Bigg|_{-1}^0$$

$$= 2 \ln(1) + 4 \ln 3$$

$$= 4 \ln 3$$

Question 51

Let $f(x) = \int_0^x (t^2 - 9t + 20) dt, 1 \leq x \leq 5$. If the range of f is $[\alpha, \beta]$, then $4(\alpha + \beta)$ equals :

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Options:

A.

253

B.

157

C.

154

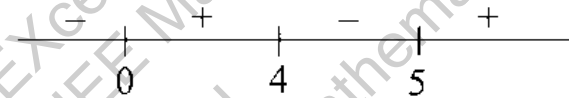
D.

125

Answer: B

Solution:

$$f'(x) = x^3 - 9x^2 + 20x = x(x-4)(x-5)$$



$$\therefore f(x) = \frac{x^4}{4} - \frac{9x^3}{3} + \frac{20x^2}{2}$$

$$f(1) = \frac{1}{4} - 3 + 10 = \frac{29}{4} = \alpha$$

$$f(4) = \frac{256}{4} - 3(64) + 10(16) = 32 = \beta$$

$$4(\alpha + \beta) = 4\left(\frac{29}{4} + 32\right) = 157$$

Question 52

$4 \int_0^1 \left(\frac{1}{\sqrt{3+x^2} + \sqrt{1+x^2}} \right) dx - 3 \log_e(\sqrt{3})$ is equal to :

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Options:

A. $2 - \sqrt{2} - \log_e(1 + \sqrt{2})$

B. $2 + \sqrt{2} + \log_e(1 + \sqrt{2})$

C. $2 + \sqrt{2} - \log_e(1 + \sqrt{2})$

D. $2 - \sqrt{2} + \log_e(1 + \sqrt{2})$

Answer: A

Solution:

$$\begin{aligned} I &= 4 \int_0^1 \frac{1}{\sqrt{3+x^2} + \sqrt{1+x^2}} dx \\ &= 2 \int_0^1 \frac{\sqrt{3+x^2} - \sqrt{1+x^2}}{(\sqrt{3+x^2} + \sqrt{1+x^2})(\sqrt{3+x^2} - \sqrt{1+x^2})} dx \\ &= 2 \left[\int_0^1 \sqrt{3+x^2} dx - \int_0^1 \sqrt{1+x^2} dx \right] \\ &= 2 \left[\left(\frac{1}{2} x \sqrt{x^2+3} + \frac{3}{2} \ln \left| \sqrt{3+x^2} + x \right| \right) - \left(\frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \ln \left| \sqrt{1+x^2} + x \right| \right) \right]_0^1 \end{aligned}$$

$$\begin{aligned}
 &= 2 \left[\left(1 + \frac{3}{2} \ln 3 - \frac{3}{2} \ln \sqrt{3} \right) - \left(\frac{\sqrt{2}}{2} + \frac{1}{2} \ln(\sqrt{2} + 1) \right) \right] \\
 &= 2 \left(1 + \frac{3}{4} \ln 3 - \frac{1}{\sqrt{2}} - \frac{1}{2} \ln(\sqrt{2} + 1) \right) \\
 &= 3 \ln \sqrt{3} + 2 - \sqrt{2} - \ln(\sqrt{2} + 1) \\
 I - 3 \ln \sqrt{3} &= 2 - \sqrt{2} - \ln(\sqrt{2} + 1)
 \end{aligned}$$

Question 53

Let (a, b) be the point of intersection of the curve $x^2 = 2y$ and the straight line $y - 2x - 6 = 0$ in the second quadrant. Then the integral $I = \int_a^b \frac{9x^2}{1+5^x} dx$ is equal to :

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Options:

A. 27

B. 18

C. 24

D. 21

Answer: C

Solution:

$$x^2 = 2y \text{ and } y - 2x - 6 = 0$$

$$\frac{x^2}{2} - 2x - 6 = 0$$

$$x^2 - 4x - 12 = 0$$

$$x^2 - 6x + 2x - 12 = 0$$

$$x(x - 6) + 2(x - 6) = 0$$

$$(x - 6)(x + 2) = 0$$

Point of intersection are $(6, 18)$ and $(-2, 2)$

$(-2, 2)$ is in second quadrant

$$a = -2, b = 2$$

$$I = \int_{-2}^2 \frac{9x^2}{1+5^x} dx \quad \dots (i)$$

$$I = \int_{-2}^2 \frac{9x^2}{1+5^{-x}} dx \quad \dots (ii)$$

Adding (i) and (ii)

$$2I = \int_{-2}^2 9x^2 dx$$

$$I = 9 \int_0^2 x^2 dx$$

$$I = 9 \left(\frac{x^3}{3} \right)_0^2 \Rightarrow I = 24$$

Question 54

Let $f : [1, \infty) \rightarrow [2, \infty)$ be a differentiable function. If $10 \int_1^1 f(t) dt = 5xf(x) - x^5 - 9$ for all $x \geq 1$, then the value of $f(3)$ is :

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Options:

A. 22

B. 26

C. 32

D. 18

Answer: C

Solution:

To solve the problem, we start with the given equation:

$$10 \int_1^x f(t) dt = 5xf(x) - x^5 - 9$$

By differentiating both sides with respect to x , we have:

$$\frac{d}{dx} (10 \int_1^x f(t) dt) = \frac{d}{dx} (5xf(x) - x^5 - 9)$$

The left side simplifies to $10f(x)$. For the right side, using the product rule and the power rule, we get:

$$10f(x) = 5f(x) + 5x \frac{d}{dx} f(x) - 5x^4$$

Rearranging terms, we obtain:

$$5f(x) = 5x \frac{d}{dx} f(x) - 5x^4$$

Let $y = f(x)$. Thus:

$$5y = 5x \frac{dy}{dx} - 5x^4$$

Dividing by 5, we have:

$$y = x \frac{dy}{dx} - x^4$$

Rewriting, we get:

$$\frac{dy}{dx} - \frac{y}{x} = x^3$$

This is a linear differential equation. The integrating factor (I.F.) is calculated as:

$$\text{I.F.} = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

Multiplying through by the integrating factor, we have:

$$y \cdot \frac{1}{x} = \int x^3 \cdot \frac{1}{x} dx = \int x^2 dx = \frac{x^3}{3} + C$$

Thus, we solve for y :

$$y = \frac{x^4}{3} + Cx$$

Substituting back, we need to find C using the condition at $x = 1$:

Since $\int_1^1 f(t) dt = 0$, we substitute:

$$10 \cdot 0 = 5 \cdot 1 \cdot f(1) - 1^5 - 9 \Rightarrow 0 = 5f(1) - 1 - 9$$

$$5f(1) = 10 \Rightarrow f(1) = 2$$

Now use $f(1) = 2$ in the function:

$$2 = \frac{1}{3} + C \cdot 1 \Rightarrow C = \frac{5}{3}$$

The function $f(x)$ is given by:

$$f(x) = \frac{x^4}{3} + \frac{5}{3}x$$

To find $f(3)$:

$$f(3) = \frac{3^4}{3} + \frac{5}{3} \cdot 3 = 27 + 5 = 32$$

Therefore, the value of $f(3)$ is 32.

Question 55

Let the domain of the function $f(x) = \log_2 \log_4 \log_6 (3 + 4x - x^2)$ be (a, b) . If $\int_0^{b-a} [x^2] dx = p - \sqrt{q} - \sqrt{r}$, $p, q, r \in \mathbb{N}$, $\gcd(p, q, r) = 1$, where $[\cdot]$ is the greatest integer function, then $p + q + r$ is equal to

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

- A. 10
- B. 11
- C. 9
- D. 8

Answer: A

Solution:

Step 1: Ensure the innermost function is greater than zero:

$$\log_6(3 + 4x - x^2) > 1$$

Step 2: Simplify the inequality from Step 1:

$$3 + 4x - x^2 > 6 \implies x^2 - 4x + 3 < 0$$

Step 3: Solve the quadratic inequality:

$$(x - 1)(x - 3) < 0$$

This inequality indicates that x must lie between the roots, giving the interval $x \in (1, 3)$.

With the domain of $f(x)$ identified as $(a, b) = (1, 3)$, we calculate the definite integral over $[0, b - a]$:

Calculate the Integral:

Given:

$$\int_0^{b-a} [x^2] dx$$

For $b - a = 2$, we compute:

$$\int_1^2 [x^2] dx = \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx$$

Computation of Each Integral Segment:

$$\int_1^{\sqrt{2}} 1 dx = \sqrt{2} - 1$$

$$\int_{\sqrt{2}}^{\sqrt{3}} 2 dx = 2(\sqrt{3} - \sqrt{2})$$

$$\int_{\sqrt{3}}^2 3 dx = 3(2 - \sqrt{3})$$

Summing these, we have:

$$5 - \sqrt{2} - \sqrt{3}$$

Conclusion:

The values for p, q , and r are $p = 5$, $q = 2$, and $r = 3$, with the greatest common divisor of these numbers being 1. Therefore, adding them together gives:

$$p + q + r = 5 + 2 + 3 = 10$$

Question 56

The integral $\int_0^{\pi} \frac{8x dx}{4 \cos^2 x + \sin^2 x}$ is equal to

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Options:

A. $2\pi^2$

B. $4\pi^2$

C. π^2

D. $\frac{3\pi^2}{2}$

Answer: A

Solution:

$$I = \int_0^{\pi} \frac{8x}{4 \cos^2 x + \sin^2 x} dx \dots (1)$$

$$I = \int_0^{\pi} \frac{8(\pi - x)}{4 \cos^2(\pi - x) + \sin^2(\pi - x)} dx$$

$$I = \int_0^{\pi} \frac{8(\pi - x)}{4 \cos^2 x + \sin^2 x} dx \dots (2)$$

Adding (1) and (2)

$$2I = 8\pi \int_0^{\pi} \frac{1}{4 \cos^2 x + \sin^2 x} dx$$

$$I = 4\pi \times 2 \int_0^{\pi} \frac{\sec^2 x}{4 \tan^2 x} dx$$

Put $\tan x = t$

$$\sec^2 x dx = dt$$

$$I = 8\pi \int_0^{\infty} \frac{dt}{4+t^2}$$

$$I = 8\pi \frac{1}{2} \left(\tan^{-1} \frac{t}{2} \right)_0^{\infty}$$

$$I = 4\pi \left(\frac{\pi}{2} \right)$$

$$I = 2\pi^2$$

Question 57

The value of $\int_{-1}^1 \frac{(1 + \sqrt{|x|-x})e^x + (\sqrt{|x|-x})e^{-x}}{e^x + e^{-x}} dx$ is equal to

JEE Main 2025 (Online) 4th April Morning Shift

Options:

A. $1 + \frac{2\sqrt{2}}{3}$

B. $1 - \frac{2\sqrt{2}}{3}$

C. $2 + \frac{2\sqrt{2}}{3}$

D. $3 - \frac{2\sqrt{2}}{3}$

Answer: A

Solution:

$$\begin{aligned} I &= \int_{-1}^1 \frac{(1 + \sqrt{|x| - x})e^x + (\sqrt{|x| - x})e^{-x}}{e^x + e^{-x}} dx \\ &= \int_0^1 \frac{(1 + \sqrt{|x| - x})e^x + (\sqrt{|x| - x})e^{-x}}{e^x + e^{-x}} \\ &\quad + \left(\frac{(1 + \sqrt{|x| + x})e^{-x} + (\sqrt{|x| + x})e^{-x}}{e^{-x} + e^x} \right) dx \\ &= \int_0^1 \frac{(1 + \sqrt{|x| - x} + \sqrt{|x| + x})(e^x + e^{-x})}{e^x + e^{-x}} dx \\ &= \int_0^1 (1 + \sqrt{|x| - x} + \sqrt{|x| + x}) dx \\ &= \int_0^1 (1 + \sqrt{2x}) dx = x \Big|_0^1 + \frac{\sqrt{2}x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 \\ &= 1 + \frac{2\sqrt{2}}{3} \end{aligned}$$

Question 58

Let $f(x) + 2f\left(\frac{1}{x}\right) = x^2 + 5$ and $2g(x) - 3g\left(\frac{1}{2}\right) = x, x > 0$. If $\alpha = \int_1^2 f(x) dx$, and $\beta = \int_1^2 g(x) dx$, then the value of $9\alpha + \beta$ is :

JEE Main 2025 (Online) 4th April Evening Shift

Options:

A. 0

B. 10

C. 1

D. 11

Answer: D

Solution:

$$f(x) + 2f\left(\frac{1}{x}\right) = x^2 + 5$$

$$2f\left(\frac{1}{x}\right) + 4f(x) = 2\left(\frac{1}{x^2} + 5\right)$$

$$3f(x) = \frac{2}{x^2} - x^2 + 5$$

$$f(x) = \frac{1}{3}\left(\frac{2}{x^2} - x^2 + 5\right)$$

$$2g(x) - 3g\left(\frac{1}{x}\right) = x$$

$$2g\left(\frac{1}{x}\right) - 3g(x) = \frac{1}{x}$$

$$\text{Or } 4g(x) - 6g\left(\frac{1}{x}\right) = 2x$$

$$6g\left(\frac{1}{x}\right) - 9g(x) = \frac{3}{x}$$

$$-5g(x) = 2x + \frac{3}{x}$$

$$\text{Or } g(x) = -\frac{1}{5}\left(2x + \frac{3}{x}\right)$$

$$\int_1^2 f(x) dx = \int_1^2 \frac{1}{3}\left(\frac{2}{x^2} - x^2 + 5\right) dx$$

$$= \frac{1}{3}\left[-\frac{2}{x} - \frac{x^3}{3} + 5x\right]_1^2$$

$$= \frac{1}{3}\left[\left(-\frac{2}{2} - \frac{8}{3} + 10\right) - \left(-2 - \frac{1}{3} + 5\right)\right]$$

$$= \frac{1}{3}\left[-1 - \frac{8}{3} + 10 + 2 + \frac{1}{3} - 5\right]$$

$$\alpha = \frac{11}{9}$$

$$\text{Now, } 2g(x) = x + 3g\left(\frac{1}{x}\right)$$

$$2g\left(\frac{1}{x}\right) = \frac{1}{x} + 3g\left(\frac{1}{x}\right)$$

$$g\left(\frac{1}{x}\right) = -\frac{1}{x}$$

$$\therefore \beta = \int_1^2 g(x) dx$$

$$= \frac{1}{2} \int_1^2 \left(x + 3g\left(\frac{1}{x}\right)\right) dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} + 3g\left(\frac{1}{x}\right)x\right]_1^2$$

$$= 0$$

$$\therefore 9\alpha + \beta = 11$$

Question 59

The integral $\int_0^\pi \frac{(x+3)\sin x}{1+3\cos^2 x} dx$ is equal to

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Options:

A. $\frac{\pi}{\sqrt{3}}(\pi + 1)$

B. $\frac{\pi}{3\sqrt{3}}(\pi + 6)$

C. $\frac{\pi}{\sqrt{3}}(\pi + 2)$

D. $\frac{\pi}{2\sqrt{3}}(\pi + 4)$

Answer: B

Solution:

$$I = \int_0^\pi \frac{(x+3)\sin x}{1+3\cos^2 x} dx \quad \dots (i)$$

$$= \int_0^\pi \frac{(\pi-x+3)\sin(\pi-x)}{1+3\cos^2(\pi-x)} dx$$

$$= \int_0^\pi \frac{(\pi+3)\sin x - x\sin x}{1+3\cos^2 x} dx \quad \dots (ii)$$

Add (i) & (ii)

$$2I = \int_0^\pi \frac{(\pi+6)\sin x}{1+3\cos^2 x} dx$$

$$\text{Let } \cos x = t \Rightarrow -\sin x dx = dt$$

$$(\pi+6) \int_1^{-1} \frac{-dt}{(1+3t^2)} = \left(\frac{\pi+6}{3}\right) \int_1^{-1} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$\Rightarrow 2I = \left(\frac{\pi+6}{3}\right) \cdot \frac{1}{\left(\frac{1}{\sqrt{3}}\right)} \cdot \tan^{-1} \frac{t}{\frac{1}{\sqrt{3}}} \Big|_{-1}^1$$

$$\Rightarrow 2I = \left(\frac{\pi+6}{3}\right) \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3}) \right]$$

$$= \left(\frac{\pi+6}{3}\right) \cdot \left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right)$$

$$= \left(\frac{\pi+6}{3}\right) \cdot \frac{2\pi}{3}$$

$$\Rightarrow I = \frac{\pi(\pi+6)}{3\sqrt{3}}$$

Question 60

Let $f(x)$ be a positive function and $I_1 = \int_{-\frac{1}{2}}^1 2x f(2x(1-2x)) dx$ and $I_2 = \int_{-1}^2 f(x(1-x)) dx$.

Then the value of $\frac{I_2}{I_1}$ is equal to _____

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Options:

A.

12

B.

9

C.

6

D.

4

Answer: D

Solution:

$$\begin{aligned} I_1 &= \int_{-\frac{1}{2}}^1 2x f(2x(1-2x)) dx \\ \Rightarrow 2x &= t \Rightarrow 2dx = dt \Rightarrow I_1 = \frac{1}{2} \int_{-1}^2 t f(t(1-t)) dt \\ \Rightarrow 2I_1 &= \int_{-1}^2 (1-t) f(1-t)(1-(1-t)) dt \\ \Rightarrow 2I_1 &= \int_{-1}^2 f(t(1-t)) dt - \int_{-1}^2 t f(t(1-t)) dt \\ \Rightarrow 2I_1 &= I_2 - 2I_1 \\ \Rightarrow 4I_1 &= I_2 \\ \Rightarrow \frac{I_2}{I_1} &= 4 \end{aligned}$$

Question 61

The integral $\int_{-1}^{\frac{3}{2}} (|\pi^2 x \sin(\pi x)|) dx$ is equal to:

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Options:

A.

$$2 + 3\pi$$

B.

$$4 + \pi$$

C.

$$1 + 3\pi$$

D.

$$3 + 2\pi$$

Answer: C

Solution:

$$\begin{aligned} \text{Let, } I &= \pi^2 \int_{-1}^{3/2} |x \sin \pi x| dx \\ &= \pi^2 \left\{ \int_{-1}^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx \right\} \\ &= \pi^2 \left\{ 2 \int_0^1 x \sin \pi x dx - \int_{-1}^{3/2} x \sin \pi x dx \right\} \end{aligned}$$

Consider

$$\begin{aligned} & \int x \sin \pi x dx \\ &= x \cdot \frac{1}{\pi} \cos \pi x + \int 1 \cdot \frac{1}{\pi} \cos \pi x dx \\ &= -\frac{x}{\pi} \cos \pi x + \frac{\sin \pi x}{\pi^2} \\ I &= \pi^2 \left\{ 2 \left(-\frac{x}{\pi} \cos \pi x + \frac{\sin \pi x}{\pi^2} \right) \right\}_0^1 - \left(-\frac{x}{\pi} \cos \pi x + \frac{\sin \pi x}{\pi^2} \right) \Big|_1^{3/2} \\ &= \pi^2 \left\{ \frac{2}{\pi} - \left(-\frac{1}{\pi^2} - \frac{1}{\pi} \right) \right\} \\ &= \pi^2 \left\{ \frac{3}{\pi} + \frac{1}{\pi^2} \right\} \\ &= 3\pi + 1 \end{aligned}$$

Question62

If $\int e^x \left(\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2} \right) dx = g(x) + C$, where C is the constant of integration, then $g\left(\frac{1}{2}\right)$ equals :

JEE Main 2025 (Online) 22nd January Evening Shift

Options:

- A. $\frac{\pi}{6} \sqrt{\frac{e}{3}}$
- B. $\frac{\pi}{6} \sqrt{\frac{e}{2}}$
- C. $\frac{\pi}{4} \sqrt{\frac{e}{3}}$
- D. $\frac{\pi}{4} \sqrt{\frac{e}{2}}$

Answer: A

Solution:

$$\begin{aligned} \therefore \frac{d}{dx} \left(\frac{x \sin^{-1} x}{\sqrt{1-x^2}} \right) &= \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2} \\ \Rightarrow \int e^x \left(\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2} \right) dx \\ &= e^x \cdot \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + c = g(x) + C \end{aligned}$$

$$\begin{aligned} \text{Note : assuming } g(x) &= \frac{xe^x \sin^{-1} x}{\sqrt{1-x^2}} \\ g(1/2) &= \frac{e^{1/2}}{2} \cdot \frac{\frac{\pi}{6} \times 2}{\sqrt{3}} = \frac{\pi}{6} \sqrt{\frac{e}{3}} \end{aligned}$$

Comment : In this question we will not get a unique function $g(x)$, but in order to match the answer we will have to assume $g(x) = \frac{xe^x \sin^{-1} x}{\sqrt{1-x^2}}$.

Question63

Let $I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$. If $I(37) - I(24) = \frac{1}{4} \left(\frac{1}{b^{\frac{1}{13}}} - \frac{1}{c^{\frac{1}{13}}} \right)$, $b, c \in \mathcal{N}$, then $3(b+c)$ is equal to

JEE Main 2025 (Online) 23rd January Morning Shift

Options:

- A. 39
- B. 22
- C. 40
- D. 26

Answer: A

Solution:

$$I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$$

$$\text{Put } \frac{x-11}{x+15} = t \Rightarrow \frac{26}{(x+15)^2} dx = dt$$

$$I(x) = \frac{1}{26} \int \frac{dt}{t^{11/13}} = \frac{1}{26} \cdot \frac{t^{2/13}}{2/13}$$

$$I(x) = \frac{1}{4} \left(\frac{x-11}{x+15} \right)^{2/13} + C$$

$$I(37) - I(24) = \frac{1}{4} \left(\frac{26}{52} \right)^{2/13} - \frac{1}{4} \left(\frac{13}{39} \right)^{2/13}$$

$$= \frac{1}{4} \left(\frac{1}{2^{2/13}} - \frac{1}{3^{2/13}} \right)$$

$$= \frac{1}{4} \left(\frac{1}{4^{1/13}} - \frac{1}{9^{1/13}} \right)$$

$$\therefore b = 4, c = 9$$

$$3(b+c) = 39$$

Question 64

Let $\int x^3 \sin x \, dx = g(x) + C$, where C is the constant of integration. If $8 \left(g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right) \right) = \alpha\pi^3 + \beta\pi^2 + \gamma$, $\alpha, \beta, \gamma \in \mathbb{Z}$, then $\alpha + \beta - \gamma$ equals :

JEE Main 2025 (Online) 23rd January Evening Shift

Options:

- A. 47
- B. 55
- C. 62
- D. 48

Answer: B

Solution:

$$\begin{aligned}\int x^3 \sin x dx &= -x^3 \cos x + \int 3x^2 \cos x dx \\ &= -x^3 \cos x + 3x^2 \sin x - \int 6x \sin x dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c\end{aligned}$$

$$\text{So } g(x) = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$$

$$g\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{4} - 6$$

$$g'(x) = x^3 \sin x$$

$$g'\left(\frac{\pi}{2}\right) = \frac{\pi^3}{8}$$

$$8\left(g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\right) = \pi^3 + 6\pi^2 - 48$$

$$\text{So } \alpha + \beta - \gamma = 55$$

Question 65

If $f(x) = \int \frac{1}{x^{1/4}(1+x^{1/4})} dx$, $f(0) = -6$, then $f(1)$ is equal to :

JEE Main 2025 (Online) 28th January Evening Shift

Options:

A. $4(\log_e 2 - 2)$

B. $\log_{e^2} 2 + 2$

C. $2 - \log_e 2$

D. $4(\log_e 2 + 2)$

Answer: A

Solution:

Substitution: Start by substituting $x = t^4$ which implies $dx = 4t^3 dt$.

Integral Transformation:

$$\int \frac{1}{x^{1/4}(1+x^{1/4})} dx \Rightarrow \int \frac{4t^3 dt}{t(1+t)} = \int \frac{4t}{1+t} dt$$

Rewriting the Integral: The expression $\frac{4t}{1+t}$ can be decomposed as:

$$\frac{4t}{1+t} = 4 \left(\frac{t^2 - 1 + 1}{1+t} \right) = 4 \left((t-1) + \frac{1}{t+1} \right)$$

Separate and Integrate:

$$4 \int (t-1) dt + 4 \int \frac{1}{t+1} dt$$

Solving the Integrals:

The integral of $t-1$ is $\frac{(t-1)^2}{2}$.

The integral of $\frac{1}{t+1}$ is $\ln(t+1)$.

Therefore:

$$f(x) = 4 \left\{ \frac{(t-1)^2}{2} + \ln(t+1) \right\} + C$$

Substitute $t = x^{1/4}$: Replace t back with $x^{1/4}$.

$$f(x) = 2(x^{1/4} - 1)^2 + 4\ln(1 + x^{1/4}) + C$$

Using the Condition $f(0) = -6$:

$$f(0) = 2(0-1)^2 + 4\ln(1+0) + C = 2 \times 1 - 0 + C = -6$$

Solving gives $C = -8$.

Find $f(1)$:

$$f(1) = 2(1^{1/4} - 1)^2 + 4\ln(1 + 1^{1/4}) - 8$$

Simplifies to:

$$f(1) = 0 + 4\ln 2 - 8 = 4(\ln 2 - 2)$$

Question66

Let $f(x) = \int x^3 \sqrt{3-x^2} dx$. If $5f(\sqrt{2}) = -4$, then $f(1)$ is equal to

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

A. $-\frac{6\sqrt{2}}{5}$

B. $-\frac{8\sqrt{2}}{5}$

C. $-\frac{2\sqrt{2}}{5}$

D. $-\frac{4\sqrt{2}}{5}$

Answer: A

Solution:

$$\int x^3 \sqrt{3-x^2} dx$$

$$3-x^2 = t^2$$

$$-2x dx = 2t dt$$

$$= -\int t^2 (3-t^2) dt = \int t^4 - 3t^2 dt$$

$$= \frac{t^5}{5} - t^3 + c$$

$$f(x) = \frac{(3-x^2)^{\frac{5}{2}}}{5} - (3-x^2)^{\frac{3}{2}} + c$$

$$\therefore f(\sqrt{2}) = -\frac{4}{5}$$

$$\Rightarrow -\frac{4}{5} = \frac{1}{5} - 1 + c \Rightarrow c = 0$$

$$\therefore f(x) = \frac{(3-x^2)^{\frac{5}{2}}}{5} - (3-x^2)^{\frac{3}{2}}$$

$$\text{Now } f(1) = \frac{2^{\frac{5}{2}}}{5} - 2^{\frac{3}{2}} = 2^{\frac{3}{2}} \left[\frac{2}{5} - 1 \right]$$

$$= 2\sqrt{2} \left(-\frac{3}{5} \right) = -\frac{6}{5}\sqrt{2}$$

Question67

If $\int \frac{2x^2+5x+9}{\sqrt{x^2+x+1}} dx = x\sqrt{x^2+x+1} + \alpha\sqrt{x^2+x+1} + \beta \log_e \left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| + C$, where C is the constant of integration, then $\alpha + 2\beta$ is equal to _____.

JEE Main 2025 (Online) 24th January Evening Shift

Answer: 16

Solution:

$$2x^2 + 5x + 9 = A(x^2 + x + 1) + B(2x + 1) + C$$

$$A = 2 \quad B = \frac{3}{2} \quad C = \frac{11}{2}$$

$$2 \int \sqrt{x^2 + x + 1} dx + \frac{3}{2} \int \frac{2x + 1}{\sqrt{x^2 + x + 1}} dx + \frac{11}{2} \int \frac{dx}{\sqrt{x^2 + x + 1}}$$

$$2 \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx + 3\sqrt{x^2 + x + 1} + \frac{11}{2} \int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$2 \left(\frac{x + \frac{1}{2}}{2} \sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right) \right) + 3\sqrt{x^2 + x + 1}$$

$$+ \frac{11}{2} \ln \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right) + C$$

$$\alpha = \frac{7}{2}$$

$$\alpha + 2\beta = 16$$

Question 68

If $\int \frac{(\sqrt{1+x^2}+x)^{10}}{(\sqrt{1+x^2}-x)^9} dx = \frac{1}{m} \left(\left(\sqrt{1+x^2} + x \right)^n \left(n\sqrt{1+x^2} - x \right) \right) + C$ where C is the constant of integration and $m, n \in \mathbb{N}$, then $m + n$ is equal to _____.

JEE Main 2025 (Online) 4th April Evening Shift

Answer: 379

Solution:

$$\sqrt{1+x^2} + x = \sec \theta + \tan \theta = t$$

$$\sqrt{1+x^2} = \sec \theta = \frac{t^2 + 1}{2t}$$

$$x = \tan \theta = \frac{t^2 - 1}{2t}$$

The given expression becomes

$$\frac{1}{m} t^2 \left(n \cdot \frac{t^2 + 1}{2t} - \frac{t^2 - 1}{2t} \right) = \frac{t^{n-1}}{2m} ((n-1)t^2 + n + 1)$$

By compare

$$\begin{aligned} n &= 19 \\ m &= 360 \\ \therefore n + m &= 379 \end{aligned}$$

Question 69

If $\int \left(\frac{1}{x} + \frac{1}{x^3} \right) \left(\sqrt[23]{3x^{-24} + x^{-26}} \right) dx = -\frac{\alpha}{3(\alpha+1)} (3x^\beta + x^\gamma)^{\frac{\alpha+1}{\alpha}} + C, x > 0, (\alpha, \beta, \gamma \in \mathbf{Z})$, where C is the constant of integration, then $\alpha + \beta + \gamma$ is equal to _____.

JEE Main 2025 (Online) 7th April Evening Shift

Answer: 19

Solution:

$$\begin{aligned} &\int \left(\frac{1}{x^2} + \frac{1}{x^4} \right) \left(\frac{3}{x} + \frac{1}{x^3} \right)^{\frac{1}{23}} dx \\ \text{using } t &= \frac{3}{x} + \frac{1}{x^3} \Rightarrow dt = -3 \left(\frac{1}{x^2} + \frac{1}{x^4} \right) dx \\ &\int \frac{t^{1/23} dt}{-3} = -\frac{t^{24/23}}{\left(\frac{24}{23} \right)(-3)} + C \\ &\Rightarrow \alpha = 23, \beta = -1, \gamma = -3 \\ &\alpha + \beta + \gamma = 19 \end{aligned}$$

Question 70

If $\int_0^1 \frac{1}{\sqrt{3+x} + \sqrt{1+x}} dx = a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are rational numbers, then $2a + 3b - 4c$ is equal to :

[27-Jan-2024 Shift 1]

Options:

- A. 4
- B. 10
- C. 7
- D. 8

Answer: D

Solution:

$$\int_0^1 \frac{1}{\sqrt{3+x} + \sqrt{1+x}} dx = \int_0^1 \frac{\sqrt{3+x} - \sqrt{1+x}}{(3+x) - (1+x)} dx$$

$$\frac{1}{2} \left[\int_0^1 \sqrt{3+x} dx - \int_0^1 (\sqrt{1+x}) dx \right]$$

$$\frac{1}{2} \left[\frac{2}{3} \frac{(3+x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2(1+x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1$$

$$\frac{1}{2} \left[\frac{2}{3} (8 - 3\sqrt{3}) - \frac{2}{3} (2^{\frac{3}{2}} - 1) \right]$$

$$\frac{1}{3} [8 - 3\sqrt{3} - 2\sqrt{2} + 1]$$

$$= 3 - \sqrt{3} - \frac{2}{3}\sqrt{2} = a + b\sqrt{2} + c\sqrt{3}$$

$$a = 3, b = -\frac{2}{3}, c = -1$$

$$2a + 3b - 4c = 6 - 2 + 4 = 8$$

Question 71

If (a, b) be the orthocentre of the triangle whose vertices are (1, 2), (2, 3) and (3, 1), and

$$I_1 = \int_a^b x \sin(4x - x^2) dx, I_2 = \int_a^b \sin(4x - x^2) dx, \text{ then } 36 \frac{I_1}{I_2} \text{ is equal to :}$$

[27-Jan-2024 Shift 1]

Options:

A.

72

B.

88

C.

80

D.

66

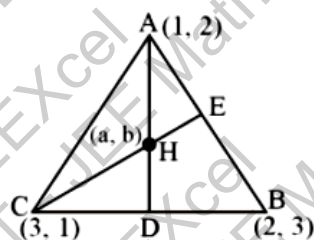
Answer: A

Solution:

Equation of CE

$$y - 1 = -(x - 3)$$

$$x + y = 4$$



orthocentre lies on the line $x + y = 4$

$$\text{so, } a + b = 4$$

$$I_1 = \int_a^b x \sin(x(4-x)) dx \dots\dots\dots (i)$$

Using king rule

$$I_1 = \int_a^h (4-x) \sin(x(4-x)) dx \dots\dots\dots (ii)$$

$$(i) + (ii)$$

$$2I_1 = \int_a^b 4 \sin(x(4-x)) dx$$

$$2I_1 = 4I_2$$

$$I_1 = 2I_2$$

$$\frac{I_1}{I_2} = 2$$

$$\frac{36I_1}{I_2} = 72$$

Question72

For $0 < a < 1$, the value of the integral $\int_0^{\pi} \frac{dx}{1 - 2a \cos x + a^2}$ is :

[27-Jan-2024 Shift 2]

Options:

A.

$$\frac{\pi^2}{\pi + a^2}$$

B.

$$\frac{\pi^2}{\pi - a^2}$$

C.

$$\frac{\pi}{1-a^2}$$

D.

$$\frac{\pi}{1+a^2}$$

Answer: C

Solution:

$$I = \int_0^{\pi} \frac{dx}{1-2a \cos x + a^2}; 0 < a < 1$$

$$I = \int_0^{\pi} \frac{dx}{1+2a \cos x + a^2}$$

$$2I = 2 \int_0^{\pi/2} \frac{2(1+a^2)}{(1+a^2)^2 - 4a^2 \cos^2 x} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2(1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \sec^2 x - 4a^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{2 \cdot (1+a^2) \cdot \sec^2 x}{(1+a^2)^2 \cdot \tan^2 x + (1-a^2)^2} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\frac{2 \cdot \sec^2 x}{1+a^2} \cdot dx}{\tan^2 x + \left(\frac{1-a^2}{1+a^2}\right)^2}$$

$$\Rightarrow I = \frac{2}{(1-a^2)} \left[\frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi}{1-a^2}$$

Question 73

The integral $\int \frac{(x^8 - x^2) dx}{(x^{12} + 3x^6 + 1) \tan^{-1} \left(x^3 + \frac{1}{x^3} \right)}$ is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

$$\log_e \left(\left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right| \right)^{1/3} + C$$

B.

$$\log_e \left(\left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right| \right)^{1/2} + C$$

C.

$$\log_e \left(\left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right| \right) + C$$

D.

$$\log_e \left(\left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right| \right)^3 + C$$

Answer: A

Solution:

$$I = \int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx$$

$$\text{Let } \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$$

$$\Rightarrow \frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \cdot \left(3x^2 - \frac{3}{x^4}\right) dx = dt$$

$$\Rightarrow \frac{x^6}{x^{12} + 3x^6 + 1} \cdot \frac{3x^6 - 3}{x^4} dx = dt$$

$$I = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln |t| + C$$

$$I = \frac{1}{3} \ln \left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right| + C$$

$$I = \ln \left| \tan^{-1} \left(x^3 + \frac{1}{x^3} \right) \right|^{1/3} + C$$

Hence option (1) is correct

Question 74

$$\text{For } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\text{if } y(x) = \int \frac{\operatorname{cosec} x + \sin x}{\operatorname{cosec} x \sec x + \tan x \sin^2 x} dx$$

$$\text{and } \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} y(x) = 0 \text{ then } y\left(\frac{\pi}{4}\right) \text{ is equal to}$$

[29-Jan-2024 Shift 1]

Options:

A.

$$\tan^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

B.

$$\frac{1}{2} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

C.

$$-\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

D.

$$\frac{1}{\sqrt{2}} \tan^{-1} \left(-\frac{1}{2} \right)$$

Answer: D

Solution:

$$y(x) = \int \frac{(1 + \sin^2 x) \cos x}{1 + \sin^4 x} dx$$

Put $\sin x = t$

$$= \int \frac{1+t^2}{t^4+1} dt = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t-\frac{1}{t}}{\sqrt{2}} \right) + C$$

$$x = \frac{\pi}{2}, t = 1 \quad \therefore C = 0$$

$$y \left(\frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \tan^{-1} \left(-\frac{1}{2} \right)$$

Question 75

$$\text{If } \int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^3 x \sin(x-\theta)}} dx = A \sqrt{\cos \theta \tan x - \sin \theta} + B \sqrt{\cos \theta - \sin \theta \cot x} + C,$$

where C is the integration constant, then AB is equal to

[29-Jan-2024 Shift 2]

Options:

A.

$$4 \operatorname{cosec}(2\theta)$$

B.

$$4 \sec \theta$$

C.

$$2 \sec \theta$$

D.

$$8 \operatorname{cosec}(2\theta)$$

Answer: D

Solution:

$$\int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^2 x \sin(x-\theta)}} dx$$

$$I = \int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^2 x (\sin x \cos \theta - \cos x \sin \theta)}} dx$$

$$= \int \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x \cos^2 x \sqrt{\tan x \cos \theta - \sin \theta}} dx + \int \frac{\cos^{\frac{3}{2}} x}{\sin^2 x \cos^{\frac{3}{2}} x \sqrt{\cos \theta - \cot x \sin \theta}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x \cos \theta - \sin \theta}} dx + \int \frac{\operatorname{cosec}^2 x}{2} \sqrt{\cos \theta - \cot^2 x \sin \theta} dx$$

$$I = I_1 + I_2 \dots \dots \{ \text{Let} \}$$

For I_1 , let $\tan x \cos \theta - \sin \theta = t^2$

$$\frac{\sec^2 x}{\cos \theta} dx = \frac{2t dt}{\cos \theta}$$

For I_2 , let $\cos \theta - \cot x \sin \theta = z^2$

$$\operatorname{cosec}^2 x dx = \frac{2z dz}{\sin \theta}$$

$$I = I_1 + I_2$$

$$= \int \frac{2t dt}{\cos \theta t} + \int \frac{2z dz}{\sin \theta z}$$

$$= \frac{2t}{\cos \theta} + \frac{2z}{\sin \theta}$$

$$= 2 \sec \theta \sqrt{\tan x \cos \theta - \sin \theta} + 2 \operatorname{cosec} \theta \sqrt{\cos \theta - \cot x \sin \theta}$$

Comparing

$$AB = 8 \operatorname{cosec} 2 \theta$$

Question 76

Let $f(x) = \int_0^x g(t) \log_e \left(\frac{1-t}{1+t} \right) dt$, where g is a continuous odd function.

If $\int_{-\pi/2}^{\pi/2} \left(f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx = \left(\frac{\pi}{\alpha} \right)^2 - \alpha$, then α is equal to.....

[27-Jan-2024 Shift 2]

Answer: 2

Solution:

$$f(x) = \int_0^x g(t) \ln\left(\frac{1-t}{1+t}\right) dt$$

$$f(-x) = \int_0^{-x} g(t) \ln\left(\frac{1-t}{1+t}\right) dt$$

$$f(-x) = -\int_0^x g(-y) \ln\left(\frac{1+y}{1-y}\right) dy$$

$$= -\int_0^x g(y) \ln\left(\frac{1-y}{1+y}\right) dy \quad (g \text{ is odd})$$

$$f(-x) = -f(x) \Rightarrow f \text{ is also odd}$$

Now,

$$I = \int_{-\pi/2}^{\pi/2} \left(f(x) + \frac{x^2 \cos x}{1+e^x} \right) dx \dots\dots(1)$$

$$I = \int_{-\pi/2}^{\pi/2} \left(f(-x) + \frac{x^2 e^x \cos x}{1+e^x} \right) dx \dots\dots(2)$$

$$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx = 2 \int_0^{\pi/2} x^2 \cos x dx$$

$$I = (x^2 \sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx$$

$$= \frac{\pi^2}{4} - 2(-x \cos x + \int \cos x dx)_0^{\pi/2}$$

$$= \frac{\pi^2}{4} - 2(0+1) = \frac{\pi^2}{4} - 2 \Rightarrow \left(\frac{\pi}{2}\right)^2 - 2$$

$$\therefore \alpha = 2$$

Question 77

If the value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{x^2 \cos x}{1+\pi^x} + \frac{1 + \sin^2 x}{1+e^{\sin x^{2023}}} \right) dx = \frac{\pi}{4}(\pi + a) - 2,$$

then the value of a is

[29-Jan-2024 Shift 1]

Options:

A.

3

B.

$-\frac{3}{2}$

C.

2

D.

3/2

Answer: A

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \left(\frac{x^2 \cos x}{1 + \pi^x} + \frac{1 + \sin^2 x}{1 + e^{\sin x}} \right) dx$$

$$I = \int_{-\pi/2}^{\pi/2} \left(\frac{x^2 \cos x}{1 + \pi^{-x}} + \frac{1 + \sin^2 x}{1 + e^{\sin(-x)}} \right) dx$$

On Adding, we get

$$2I = \int_{-\pi/2}^{\pi/2} (x^2 \cos x + 1 + \sin^2 x) dx$$

On solving

$$I = \frac{\pi^2}{4} + \frac{3\pi}{4} - 2$$

$$a = 3$$

Question 78

If $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin 2x} dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$, where α, β and γ are rational numbers, then $3\alpha + 4\beta - \gamma$ is equal to ____

[29-Jan-2024 Shift 2]

Answer: 6

Solution:

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1 - \sin 2x} dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} |\sin x - \cos x| dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin x - \cos x) dx$$

$$= -1 + 2\sqrt{2} - \sqrt{3}$$

$$= \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$$

$$\alpha = -1, \beta = 2, \gamma = -1$$

$$3\alpha + 4\beta - \gamma = 6$$

Question 79

The value of $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^3}{(n^2 + k^2)(n^2 + 3k^2)}$ is :

[30-Jan-2024 Shift 1]

Options:

A.

$$\frac{(2\sqrt{3} + 3)\pi}{24}$$

B.

$$\frac{13\pi}{8(4\sqrt{3} + 3)}$$

C.

$$\frac{13(2\sqrt{3} - 3)\pi}{8}$$

D.

$$\frac{\pi}{8(2\sqrt{3} + 3)}$$

Answer: B

Solution:

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n n^3}{n^4 \left(1 + \frac{k^2}{n^2}\right) \left(1 + \frac{3k^2}{n^2}\right)} \\
 &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sum_{k=1}^n n^3}{\left(1 + \frac{k^2}{n^2}\right) \left(1 + \frac{3k^2}{n^2}\right)} \\
 &= \int_0^1 \frac{dx}{3(1+x^2) \left(\frac{1}{3} + x^2\right)} \\
 &= \int_0^1 \frac{1}{3} \times \frac{3}{2} \frac{(x^2+1) - \left(x^2 + \frac{1}{3}\right)}{(1+x^2) \left(x^2 + \frac{1}{3}\right)} dx \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{x^2 + \left(\frac{1}{\sqrt{3}}\right)^2} - \frac{1}{1+x^2} \right] dx \\
 &= \frac{1}{2} [\sqrt{3} \tan^{-1}(\sqrt{3}x)]_0^1 - \frac{1}{2} (\tan^{-1}x)_0^1 \\
 &= \frac{\sqrt{3}}{2} \left(\frac{\pi}{3}\right) - \frac{1}{2} \left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{3}} - \frac{\pi}{8} \\
 &= \frac{13\pi}{8 \cdot (4\sqrt{3} + 3)}
 \end{aligned}$$

Question80

The value $9 \int_0^9 \left[\sqrt{\frac{10x}{x+1}} \right] dx$, where $[t]$ denotes the greatest integer less than or equal to t , is

[30-Jan-2024 Shift 1]

Answer: 155

Solution:

$$\frac{10x}{x+1} = 1 \Rightarrow x = \frac{1}{9}$$

$$\frac{10x}{x+1} = 4 \Rightarrow x = \frac{2}{3}$$

$$\frac{10x}{x+1} = 9 \Rightarrow x = 9$$

$$\begin{aligned}
 I &= 9 \left(\int_0^{1/9} 0 \, dx + \int_{1/9}^{2/3} 1 \, dx + \int_{2/3}^9 2 \, dx \right) \\
 &= 155
 \end{aligned}$$

Question 81

Let $y = f(x)$ be a thrice differentiable function in $(-5, 5)$. Let the tangents to the curve $y = f(x)$ at $(1, f(1))$ and $(3, f(3))$ make angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively with positive x -axis. If $27 \int_1^3 ((f'(t))^2 + 1) f''(t) dt = \alpha + \beta\sqrt{3}$ where α, β are integers, then the value of $\alpha + \beta$ equals

[30-Jan-2024 Shift 2]

Options:

A.

-14

B.

26

C.

-16

D.

36

Answer: B

Solution:

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$$

$$\left(\frac{dy}{dx}\right)_{(1, f(1))} = f'(1) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \Rightarrow f'(1) = \frac{1}{\sqrt{3}}$$

$$\left(\frac{dy}{dx}\right)_{(3, f(3))} = f'(3) = \tan \frac{\pi}{4} = 1 \Rightarrow f'(3) = 1$$

$$27 \int_1^3 ((f'(t))^2 + 1) f''(t) dt = \alpha + \beta\sqrt{3}$$

$$I = \int_1^3 ((f'(t))^2 + 1) f''(t) dt$$

$$f'(t) = z \Rightarrow f''(t) dt = dz$$

$$z = f'(3) = 1$$

$$z = f'(1) = \frac{1}{\sqrt{3}}$$

$$I = \int_{1/\sqrt{3}}^1 (z^2 + 1) dz = \left(\frac{z^3}{3} + z \right) \Big|_{1/\sqrt{3}}^1$$

$$= \left(\frac{1}{3} + 1 \right) - \left(\frac{1}{3} \cdot \frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \right)$$

$$= \frac{4}{3} - \frac{10}{9\sqrt{3}} = \frac{4}{3} - \frac{10}{27}\sqrt{3}$$

$$\alpha + \beta\sqrt{3} = 27 \left(\frac{4}{3} - \frac{10}{27}\sqrt{3} \right) = 36 - 10\sqrt{3}$$

$$\alpha = 36, \beta = -10$$

$$\alpha + \beta = 36 - 10 = 26$$

Question82

Let $f : R \rightarrow R$ be defined $f(x) = ae^{2x} + be^x + cx$. If $f(0) = -1$, $f'(\log_e 2) = 21$ and $\int_0^{\ln 4} \log_e 4(f(x) - cx) dx = \frac{39}{2}$, then the value of $|a + b + c|$ equals :

[30-Jan-2024 Shift 2]

Options:

A.

16

B.

10

C.

12

D.

8

Answer: D

Solution:

$$f(x) = ae^{2x} + be^x + cx$$

$$f(0) = -1$$

$$a + b = -1$$

$$f'(x) = 2ae^{2x} + be^x + c$$

$$f'(\ln 2) = 21$$

$$8a + 2b + c = 21$$

$$\int_0^{\ln 4} (ae^{2x} + be^x) dx = \frac{39}{2}$$

$$\left[\frac{ae^{2x}}{2} + be^x \right]_0^{\ln 4} = \frac{39}{2} \Rightarrow 8a + 4b - \frac{a}{2} - b = \frac{39}{2}$$

$$15a + 6b = 39$$

$$15a - 6a - 6 = 39$$

$$9a = 45 \Rightarrow a = 5$$

$$b = -6$$

$$c = 21 - 40 + 12 = -7$$

$$a + b + c = -8$$

$$|a + b + c| = 8$$

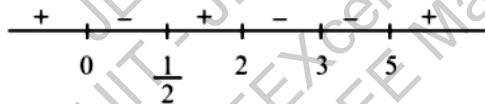
Question83

Let $S = (-1, \infty)$ and $f : S \rightarrow \mathbb{R}$ be defined as $f(x) = \int_{-1}^x (e^t - 1)^{11} (2t - 1)^5 (t - 2)^7 (t - 3)^{12} (2t - 10)^{61} dt$. Let $p =$ Sum of square of the values of x , where $f(x)$ attains local maxima on S . and $q =$ Sum of the values of x , where $f(x)$ attains local minima on S . Then, the value of $p^2 + 2q$ is _____ [31-Jan-2024 Shift 1]

Answer: 27

Solution:

$$f'(x) = (e^x - 1)^{11} (2x - 1)^5 (x - 2)^7 (x - 3)^{12} (2x - 10)^{61}$$



Local minima at $x = \frac{1}{2}, x = 5$

Local maxima at $x = 0, x = 2$

$$\therefore p = 0 + 4 = 4, q = \frac{1}{2} + 5 = \frac{11}{2}$$

$$\text{Then } p^2 + 2q = 16 + 11 = 27$$

Question84

If the integral $525 \int_0^{\frac{\pi}{2}} \sin 2x \cos^{\frac{11}{2}} x \left(1 + \cos^{\frac{5}{2}} x\right)^{\frac{1}{2}} dx$ is equal to $(n\sqrt{2} - 64)$, then n is equal to _____

[31-Jan-2024 Shift 1]

Answer: 176

Solution:

$$I = \int_0^{\frac{\pi}{2}} \sin 2x \cdot (\cos x)^{\frac{11}{2}} \left(1 + (\cos x)^{\frac{5}{2}}\right)^{\frac{1}{2}} dx$$

$$\text{Put } \cos x = t^2 \Rightarrow \sin x dx = -2t dt$$

$$\therefore I = 4 \int_0^1 t^2 \cdot t^{11} \sqrt{(1+t^5)}(t) dt$$

$$I = 4 \int_0^1 t^{14} \sqrt{1+t^5} dt$$

$$\text{Put } 1+t^5 = k^2$$

$$\Rightarrow 5t^4 dt = 2k dk$$

$$\therefore I = 4 \cdot \int_1^{\sqrt{2}} (k^2 - 1)^2 \cdot k \frac{2k}{5} dk$$

$$I = \frac{8}{5} \int_1^{\sqrt{2}} k^6 - 2k^4 + k^2 dk$$

$$I = \frac{8}{5} \left[\frac{k^7}{7} - \frac{2k^5}{5} + \frac{k^3}{3} \right]_1^{\sqrt{2}}$$

$$I = \frac{8}{5} \left[\frac{8\sqrt{2}}{7} - \frac{8\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3} \right]$$

$$I = \frac{8}{5} \left[\frac{22\sqrt{2}}{105} - \frac{8}{105} \right]$$

$$\therefore 525 \cdot I = 176\sqrt{2} - 64$$

Question85

Let $f, g : (0, \infty) \rightarrow \mathbb{R}$ be two functions defined by $f(x) = \int_{-x}^x (|t| - t^2)e^{-t^2} dt$ and $g(x) = \int_0^{2x} t^{1/2} e^{-t} dt$. Then the value of $(f(\sqrt{\log_e 9}) + g(\sqrt{\log_e 9}))$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

6

B.

9

C.

8

D.

10

Answer: C

Solution:

$$f(x) = \int_{-x}^x (t - t^2) e^{-t^2} dt$$

$$\Rightarrow f(x) = 2 \cdot (|x| - x^2) e^{-x^2} \dots \dots \dots (1)$$

$$g(x) = \int_0^{x^2} t^{\frac{1}{2}} e^{-t} dt$$

$$g'(x) = x e^{-x^2} (2x) = 0$$

$$f'(x) + g'(x) = 2x e^{-x^2} - 2x^2 e^{-x^2} + 2x^2 e^{-x^2}$$

Integrating both sides w.r.t. x

$$f(x) + g(x) = \int_0^a 2x e^{-x^2} dx$$

$$x^2 = t$$

$$\Rightarrow \int_0^{\sqrt{a}} e^{-t} dt = [-e^{-t}]_0^{\sqrt{a}}$$

$$= -e^{(\log_e(9)^{-1})} + 1$$

$$\Rightarrow 9(f(x) + g(x)) = \left(1 - \frac{1}{9}\right) 9 = 8$$

Question 86

$$\left| \frac{120}{\pi^3} \int_0^{\pi} \frac{x^2 \sin x \cos x}{\sin^4 x + \cos^4 x} dx \right| \text{ is equal to } \underline{\hspace{2cm}}$$

[31-Jan-2024 Shift 2]

Answer: 15

Solution:

$$\begin{aligned}
 & \int_0^{\pi} \frac{x^2 \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} (x^2 - (\pi - x)^2) dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x (2\pi x - \pi^2)}{\sin^4 x + \cos^4 x} dx \\
 &= 2\pi \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\
 &= 2\pi \cdot \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos^4 x}{\sin^4 x + \cos^4 x} dx \\
 &= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\
 &= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{1 - 2\sin^2 x \times \cos^2 x} \\
 &= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2 - \sin^2 2x} dx \\
 &= -\frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \cos^2 2x} dx
 \end{aligned}$$

Let $\cos 2x = t$

Question 87

The value of the integral $\int_0^{\frac{\pi}{4}} \frac{x dx}{\sin^4(2x) + \cos^4(2x)}$ equals :

[1-Feb-2024 Shift 1]

Options:

A.

$$\frac{\sqrt{2}\pi^2}{8}$$

B.

$$\frac{\sqrt{2}\pi^2}{16}$$

C.

$$\frac{\sqrt{2}\pi^2}{32}$$

D.

$$\frac{\sqrt{2}\pi^2}{64}$$

Answer: C

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{x \, dx}{\sin^4(2x) + \cos^4(2x)}$$

$$\text{Let } 2x = t \text{ then } dx = \frac{1}{2} dt$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{t \, dt}{\sin^4 t + \cos^4 t}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - t\right) dt}{\sin^4\left(\frac{\pi}{2} - t\right) + \cos^4\left(\frac{\pi}{2} - t\right)}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} dt}{\sin^4 t + \cos^4 t} = I$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t + \cos^4 t}$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{\sec^4 t \, dt}{\tan^4 t + 1}$$

$$\text{Let } \tan = y \text{ then } \sec^2 t \, dt = dy$$

$$2I = \frac{\pi}{8} \int_0^{\infty} \frac{(1+y^2) dy}{1+y^4}$$

$$= \frac{\pi}{16} \int_0^{\infty} \frac{1 + \frac{1}{y^2}}{y^2 + \frac{1}{y^2}} dy$$

$$\text{Put } y - \frac{1}{y} = p$$

$$I = \frac{\pi}{16} \int_{-\infty}^{\infty} \frac{dp}{p^2 + (\sqrt{2})^2}$$

$$= \frac{\pi}{16\sqrt{2}} \left[\tan^{-1}\left(\frac{p}{\sqrt{2}}\right) \right]_{-\infty}^{\infty}$$

$$I = \frac{\pi^2}{16\sqrt{2}}$$

Question 88

If $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x \, dx}{(1 + e^{\sin x})(1 + \sin^4 x)} = \alpha\pi + \beta \log_e(3 + 2\sqrt{2})$, where α, β are integers, then $\alpha^2 + \beta^2$ equals

[1-Feb-2024 Shift 1]

Answer: None

Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \dots\dots\dots(1)$$

Apply king

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x(e^{\sin x})}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \dots\dots\dots(2)$$

adding (1) & (2)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx,$$

$$\sin x = t$$

$$I = \int_0^1 \frac{8\sqrt{2}}{1 + t^4} dx$$

$$I = 4\sqrt{2} \int_0^1 \left(\frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} - \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \right) dt$$

$$I = 4\sqrt{2} \int_0^1 \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} - \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt$$

$$\text{Let } t - \frac{1}{t} = z \text{ and } t + \frac{1}{t} = k$$

$$= 4\sqrt{2} \left[\int_{-\infty}^0 \frac{dz}{z^2 + 2} - \int_{\infty}^2 \frac{dk}{k^2 - 2} \right]$$

$$= 4\sqrt{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} \right]_{-\infty}^0 - \left[\frac{1}{2\sqrt{2}} \ln \left(\frac{k - \sqrt{2}}{k + \sqrt{2}} \right) \right]_{\infty}^2$$

$$= 4\sqrt{2} \left[\frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \left[\ln \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right] \right]$$

$$= 2\pi + 2 \ln(3 + 2\sqrt{2})$$

$$\alpha = 2$$

$$\beta = 2$$

Question89

The value of $\int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}} dx$ is equal to:

[1-Feb-2024 Shift 2]

Options:

A.

0

B.

1

C.

2

D.

-1

Answer: A

Solution:

$$I = \int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}} dx$$

Using $\int_0^{2a} f(x) dx$ where $f(2a-x) = -f(x)$

Here $f(1-x) = f(x)$

$\therefore I = 0$

Question 90

If $\int_0^{\frac{\pi}{3}} \cos^4 x dx = a\pi + b\sqrt{3}$, where a and b are rational numbers, then $9a + 8b$ is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

2

B.

1

C.

3

D.

3/2

Answer: A

Solution:

$$\begin{aligned}
 & \int_0^{\pi/3} \cos^4 x \, dx \\
 &= \int_0^{\pi/3} \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\
 &= \frac{1}{4} \int_0^{\pi/3} (1 + 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \left[\int_0^{\pi/3} dx + 2 \int_0^{\pi/3} \cos 2x \, dx + \int_0^{\pi/3} \frac{1 + \cos 4x}{2} dx \right] \\
 &= \frac{1}{4} \left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left(\frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right] \\
 &= \frac{1}{4} \left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2} \left(\frac{\pi}{3} \right) + \frac{1}{8} (\sin 4x)_0^{\pi/3} \right] \\
 &= \frac{1}{4} \left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} + \frac{1}{8} \times \left(-\frac{\sqrt{3}}{2} \right) \right] \\
 &= \frac{\pi}{2} + \frac{7\sqrt{3}}{64} \\
 \therefore a &= \frac{1}{8}; b = \frac{7}{64} \\
 \therefore 9a + 8b &= \frac{9}{8} + \frac{7}{8} = 2
 \end{aligned}$$

Question 91

Let $f : (0, \infty) \rightarrow \mathbb{R}$ and $F(x) = \int_0^x tf(t) \, dt$. If $F(x^2) = x^4 + x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 219

Solution:

$$F(x) = \int_0^x t \cdot f(t) dt$$

$$F^1(x) = xf(x)$$

$$\text{Given } F(x^2) = x^4 + x^5, \quad \text{let } x^2 = t$$

$$F(t) = t^2 + t^{5/2}$$

$$F'(t) = 2t + 5/2 t^{3/2}$$

$$t \cdot f(t) = 2t + 5/2 t^{3/2}$$

$$f(t) = 2 + 5/2 t^{1/2}$$

$$\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} 2 + \frac{5}{2} r$$

$$= 24 + 5/2 \left[\frac{12(13)}{2} \right]$$

$$= 219$$

Question92

The value of $\frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$ is _____
[24-Jan-2023 Shift 1]

Answer: 2

Solution:

$$I = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx \dots (1)$$

$$\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx \dots (2)$$

Adding (1)&(2)

$$II = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} 1 dx$$

Question93

$$\frac{3\sqrt{3}}{4} \int_{\frac{3\sqrt{2}}{4}}^{\frac{48}{\sqrt{9-4x^2}}} dx \text{ is equal to}$$

[24-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{6}$

D. 2π

Answer: D

Solution:

$$\frac{3\sqrt{3}}{4} \int \frac{48}{\sqrt{9-4x^2}} dx$$

We have $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$

Hence $\frac{3\sqrt{3}}{4} \int \frac{48}{\sqrt{9-4x^2}} dx = \frac{48}{2} \times \left[\sin^{-1} \frac{2x}{3} \right] \frac{3\sqrt{3}}{4}$

$$= 24 \times \left[\sin^{-1} \left(\frac{2}{3} \times \frac{3\sqrt{3}}{4} \right) - \sin^{-1} \left(\frac{2}{3} \times \frac{3\sqrt{2}}{4} \right) \right]$$

$$= 24 \times \left[\sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} \frac{1}{\sqrt{2}} \right]$$

$$= 24 \times \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= 24 \times \frac{\pi}{12} = 2\pi$$

Question94

Let f be a differentiable function defined on $\left[0, \frac{\pi}{2}\right]$ such that $f(x) > 0$ and

$$f(x) + \int_0^x f(t) \sqrt{1 - (\log_e f(t))^2} dt = e, \forall x \in \left[0, \frac{\pi}{2}\right].$$

Then $\left(6 \log_e f\left(\frac{\pi}{6}\right)\right)^2$ is equal to _____

[24-Jan-2023 Shift 2]

Answer: 27

Solution:

$$f(x) + \int_0^x f(t) \sqrt{1 - (\log_e f(t))^2} dt = e$$

$$\Rightarrow f(0) = e$$

$$f'(x) + f(x) \sqrt{1 - (\ln f(x))^2} = 0$$

$$f(x) = y$$

$$\frac{dy}{dx} = -y \sqrt{1 - (\ln y)^2}$$

$$\int \frac{dy}{y \sqrt{1 - (\ln y)^2}} = -\int dx$$

Put $\ln y = t$

$$\int \frac{dt}{\sqrt{1-t^2}} = -x + C$$

$$\sin^{-1} t = -x + C \Rightarrow \sin^{-1}(\ln y) = -x + C$$

$$\sin^{-1}(\ln f(x)) = -x + C$$

$$f(0) = e$$

$$\Rightarrow \frac{\pi}{2} = C$$

$$\Rightarrow \sin^{-1}(\ln f(x)) = -x + \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = -\frac{\pi}{6} + \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\ln f\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{3}$$

$$\Rightarrow \ln f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \text{ we need } \left(6 \times \frac{\sqrt{3}}{2}\right)^2 = 27.$$

Question 95

The minimum value of the function $f(x) = \int_0^x e^{|x-t|} dt$ is

[25-Jan-2023 Shift 1]

Options:

A. $2(e-1)$

B. $2e-1$

C. 2

D. $e(e-1)$

Answer: A

Solution:

For $x \leq 0$

$$f(x) = \int_0^x e^{t-x} dt = e^{-x}(e^2 - 1)$$

For $0 < x < 2$

$$f(x) = \int_0^x e^{x-t} dt + \int_x^2 e^{t-x} dt = e^x + e^{2-x} - 2$$

For $x \geq 2$

$$f(x) = \int_0^2 e^{x-t} dt = e^{x-2}(e^2 - 1)$$

For $x \leq 0$, $f(x)$ is $\downarrow$ and $x \geq 2$, $f(x)$ is $\uparrow$

$\therefore$ Minimum value of $f(x)$ lies in $x \in (0, 2)$

Applying A.M $\geq$ G.M

minimum value of $f(x)$ is $2(e-1)$

Question 96

The value of

$$\lim_{n \rightarrow \infty} \frac{1+2-3+4+5-6+\dots+(3n-2)+(3n-1)-3n}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

is :

[25-Jan-2023 Shift 1]

Options:

- A. $\frac{\sqrt{2}+1}{2}$
 B. $3(\sqrt{2}+1)$
 C. $\frac{3}{2}(\sqrt{2}+1)$
 D. $\frac{3}{2\sqrt{2}}$

Answer: C

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{0+3+6+9+\dots+n \text{ terms}}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}} \\ \lim_{n \rightarrow \infty} \frac{3n(n-1)}{2(\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4})} \\ = \frac{3}{2(\sqrt{2}-1)} = \frac{3}{2}(\sqrt{2}+1) \end{aligned}$$

Question 97

The integral $16 \int_1^2 \frac{dx}{x^3(x^2+2)^2}$ is equal to
[25-Jan-2023 Shift 2]

Options:

- A. $\frac{11}{6} + \log_e 4$
 B. $\frac{11}{12} + \log_e 4$
 C. $\frac{11}{12} - \log_e 4$
 D. $\frac{11}{6} - \log_e 4$

Answer: D

Solution:

$$\begin{aligned} I &= 16 \int_1^2 \frac{dx}{x^3(x^2+2)^2} \\ &= 16 \int_1^2 \frac{dx}{x^3 x^4 \left(1 + \frac{2}{x^2}\right)^2} \\ \text{Let, } 1 + \frac{2}{x^2} &= t \Rightarrow -\frac{4}{x^3} dx = dt \\ I &= -4 \int_3^{\frac{3}{2}} \frac{dt}{\left(\frac{2}{t-1}\right)^2 t^2} \\ I &= -4 \int_3^{\frac{3}{2}} \left(\frac{t-1}{2}\right)^2 \frac{dt}{t^2} \\ I &= -\frac{4}{4} \int_3^{\frac{3}{2}} \left(1 - \frac{2}{t} + \frac{1}{t^2}\right) dt \end{aligned}$$

$$I = -1 \left[t - 2 \ln |t| - \frac{1}{t} \right]_3^{\frac{3}{2}}$$

$$I = -1 \left[\left(\frac{3}{2} - 2 \ln \frac{3}{2} - \frac{2}{3} \right) - \left(3 - 2 \ln 3 - \frac{1}{3} \right) \right]$$

$$I = -1 \left[2 \ln 2 - \frac{11}{6} \right]$$

$$I = \frac{11}{6} - \ln 4$$

Question98

If $\int_{\frac{1}{3}}^3 \log_e x \, dx = \frac{m}{n} \log_e \left(\frac{n^2}{e} \right)$, where m and n are coprime natural numbers, then $m^2 + n^2 - 5$ is equal to .
[25-Jan-2023 Shift 2]

Answer: 20

Solution:

$$\begin{aligned} \int_{\frac{1}{3}}^3 \log_e x \, dx &= \int_{\frac{1}{3}}^1 (-\log_e x) \, dx + \int_1^3 (\log_e x) \, dx \\ &= -[x \log_e x - x]_{1/3}^1 + [x \log_e x - x]_1^3 \\ &= -\left[-1 - \left(\frac{1}{3} \log_e \frac{1}{3} - \frac{1}{3} \right) \right] + [3 \log_e 3 - 3 - (-1)] \\ &= -\left[\frac{2}{3} - \frac{1}{3} \log_e \frac{1}{3} \right] + [3 \log_e 3 - 2] \\ &= -\frac{4}{3} + \frac{8}{3} \log_e 3 \\ &= \frac{4}{3} (2 \log_e 3 - 1) \\ &= \frac{4}{3} \left(\log_e \frac{9}{e} \right) \\ \therefore m &= 4, n = 3 \\ \text{Now, } m^2 + n^2 - 5 &= 16 + 9 - 5 = 20 \end{aligned}$$

Question99

Let $[x]$ denote the greatest integer $\leq x$. Consider the function $f(x) = \max\{x^2, 1 + [x]\}$. Then the value of the integral $\int_0^2 f(x) \, dx$ is:
[29-Jan-2023 Shift 1]

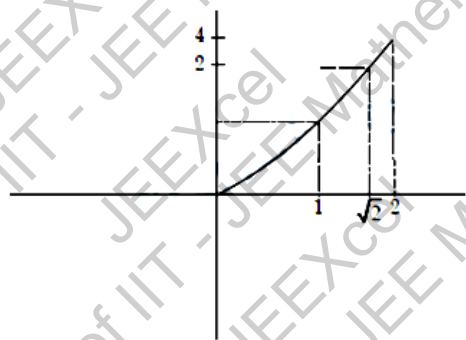
Options:

- A. $\frac{5+4\sqrt{2}}{3}$
- B. $\frac{8+4\sqrt{2}}{3}$
- C. $\frac{1+5\sqrt{2}}{3}$

D. $\frac{4+5\sqrt{2}}{3}$

Answer: A

Solution:



$$\begin{aligned} A &= \int_0^1 1 \cdot dx + \int_1^{\sqrt{2}} 2 \, dx + \int_{\sqrt{2}}^2 x^2 \, dx \\ &= 1 + 2\sqrt{2} - 2 + \frac{8}{3} - \frac{2\sqrt{2}}{3} \\ &= \frac{5}{3} + \frac{4\sqrt{2}}{3} \end{aligned}$$

Question 100

Let $f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x$, $x \in \mathbb{R}$ be a function which satisfies

$f(x) = x + \int_0^{\pi/2} \sin(x+y)f(y) \, dy$. Then $(a+b)$ is equal to
[29-Jan-2023 Shift 1]

Options:

A. $-\pi(\pi+2)$

B. $-2\pi(\pi+2)$

C. $-2\pi(\pi-2)$

D. $-\pi(\pi-2)$

Answer: B

Solution:

$$f(x) = x + \int_0^{\pi/2} (\sin x \cos y + \cos x \sin y) f(y) \, dy$$

$$f(x) = x + \int_0^{\pi/2} ((\cos y f(y) \, dy) \sin x + (\sin y f(y) \, dy) \cos x)$$

On comparing with

$$f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x, \, x \in \mathbb{R} \text{ then}$$

$$\Rightarrow \frac{a}{\pi^2 - 4} = \int_0^{\pi/2} \cos y f(y) \, dy \dots (2)$$

$$\Rightarrow \frac{b}{\pi^2 - 4} = \int_0^{\pi/2} \sin y f(y) \, dy \dots (3)$$

Add (2) and (3)

$$\frac{a+b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f(y) \, dy \dots (4)$$

$$\frac{a+b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f\left(\frac{\pi}{2} - y\right) \, dy \dots (5)$$

Add (4) and (5)

$$\begin{aligned} \frac{2(a+b)}{\pi^2-4} &= \int_0^{\pi/2} (\sin y + \cos y) \left(\frac{\pi}{2} + \frac{(a+b)}{\pi^2-4} (\sin y + \cos y) \right) dy \\ &= \pi + \frac{a+b}{\pi^2-4} \left(\frac{\pi}{2} + 1 \right) \\ (a+b) &= -2\pi(\pi+2) \end{aligned}$$

Question101

The value of the integral $\int_{1/2}^2 \frac{\tan^{-1}x}{x} dx$ is equal to
[29-Jan-2023 Shift 2]

Options:

A. $\pi \log_e 2$

B. $\frac{1}{2} \log_e 2$

C. $\frac{\pi}{4} \log_e 2$

D. $\frac{\pi}{2} \log_e 2$

Answer: D

Solution:

$$I = \int_{1/2}^2 \frac{\tan^{-1}x}{x} dx \dots (i)$$

$$\text{Put } x = \frac{1}{t} \quad dx = -\frac{1}{t^2} dt$$

$$I = - \int_2^{1/2} \frac{\tan^{-1} \frac{1}{t}}{\frac{1}{t}} \cdot \frac{1}{t^2} dt = - \int_2^{1/2} \frac{\tan^{-1} \frac{1}{t}}{t} dt$$

$$I = \int_{1/2}^2 \frac{\cot^{-1}t}{t} dt = \int_{1/2}^2 \frac{\cot^{-1}x}{x} dx \dots (ii)$$

Add both equation

$$2I = \int_{1/2}^2 \frac{\tan^{-1}x + \cot^{-1}x}{x} dx = \frac{\pi}{2} \int_{1/2}^2 \frac{dx}{x} = \frac{\pi}{2} (\ln 2)_{1/2}^2$$

$$= \frac{\pi}{2} \left(\ln 2 - \ell \ln \frac{1}{2} \right) = \pi \ell \ln 2$$

$$I = \frac{\pi}{2} \ell \ln 2$$

Question102

The value of the integral $\int_1^2 \left(\frac{t^4+1}{t^6+1} \right) dt$ is :
[29-Jan-2023 Shift 2]

Options:

A. $\tan^{-1} \frac{1}{2} + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$

B. $\tan^{-1} 2 - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$

C. $\tan^{-1}2 + \frac{1}{3}\tan^{-1}8 - \frac{\pi}{3}$

D. $\tan^{-1}\frac{1}{2} - \frac{1}{3}\tan^{-1}8 + \frac{\pi}{3}$

Answer: C

Solution:

$$\begin{aligned} I &= \int_1^2 \left(\frac{t^4+1}{t^6+1} \right) dt \\ &= \int_1^2 \frac{(t^4+1-t^2)+t^2}{(t^2+1)(t^4-t^2+1)} dt \\ &= \int_1^2 \left(\frac{1}{t^2+1} + \frac{t^2}{t^6+1} \right) dt \\ &= \int_1^2 \left(\frac{1}{t^2+1} + \frac{1}{3} \frac{3t^2}{(t^3)^2+1} \right) dt \\ &= \tan^{-1}(t) + \frac{1}{3} \tan^{-1}(t^3) \Big|_1^2 \\ &= (\tan^{-1}(2) - \tan^{-1}(1)) + \frac{1}{3} (\tan^{-1}(2^3) - \tan^{-1}(1^3)) \\ &= \tan^{-1}(2) + \frac{1}{3} \tan^{-1}(8) - \frac{\pi}{3} \end{aligned}$$

Question 103

If $[t]$ denotes the greatest integer $\leq t$, then the value of $\int_1^{\frac{3(e-1)^2}{e}} x^2 e^{[x] + [x^3]} dx$ is :
[30-Jan-2023 Shift 1]

Options:

A. $e^9 - e$

B. $e^8 - e$

C. $e^7 - 1$

D. $e^8 - 1$

Answer: B

Solution:

$$\begin{aligned} &\int_1^{\frac{3(e-1)^2}{e}} x^2 e^{[x] + [x^3]} dx \\ &\quad x^3 = t \\ &\quad 3x^2 dx = dt \\ &= \frac{e}{3} \int_1^8 e^{[t]} dt \\ &= \frac{e}{3} \left\{ \int_1^2 e dt + \int_2^3 e^2 dt + \dots + \int_7^8 e^7 dt \right\} \\ &= \frac{e}{3} (e + e^2 + \dots + e^7) \\ &= \frac{e^2}{3} (1 + e + \dots + e^6) = \frac{e^2}{3} \frac{(e^7 - 1)}{(e - 1)} \\ &\frac{3(e-1)^2}{e} \int_1^{\frac{3(e-1)^2}{e}} x^2 \times e^{[x] + [x^3]} dx = \frac{3}{e} (e-1) \times \frac{e^2}{3} \frac{(e^7 - 1)}{(e - 1)} \\ &= e(e^7 - 1) \\ &= e^8 - e \end{aligned}$$

Question104

$\lim_{x \rightarrow 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6 + 1} dt$ is equal to _____.

[30-Jan-2023 Shift 1]

Answer: 12

Solution:

$$48 \lim_{x \rightarrow 0} \frac{\int_0^x \frac{t^3}{t^6 + 1} dt}{x^4} \left(\frac{0}{0} \right)$$

Applying L'Hospitals Rule

$$\begin{aligned} & 48 \lim_{x \rightarrow 0} \frac{x^3}{x^6 + 1} \times \frac{1}{4x^3} \\ &= 12 \end{aligned}$$

Question105

$\lim_{n \rightarrow \infty} \frac{3}{n} \left\{ 4 + \left(2 + \frac{1}{n} \right)^2 + \left(2 + \frac{2}{n} \right)^2 + \dots + \left(3 - \frac{1}{n} \right)^2 \right\}$

is equal to

[30-Jan-2023 Shift 2]

Options:

A. 12

B. $\frac{19}{3}$

C. 0

D. 19

Answer: D

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{r=0}^{n-1} \left(2 + \frac{r}{n} \right)^2 \\ &= 3 \int_0^1 (2+x)^2 dx = 27 - 8 = 19 \end{aligned}$$

Question106

The value of $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)} dx$ is equal to

[31-Jan-2023 Shift 1]

Options:

A. $\frac{7}{2} - \sqrt{3} - \log_e \sqrt{3}$

B. $-2 + 3\sqrt{3} + \log_e \sqrt{3}$

C. $\frac{10}{3} - \sqrt{3} + \log_e \sqrt{3}$

D. $\frac{10}{3} - \sqrt{3} - \log_e \sqrt{3}$

Answer: C

Solution:

$$\begin{aligned} \int_{\pi/3}^{\pi/2} \left(\frac{2+3\sin x}{\sin x(1+\cos x)} \right) dx &= 2 \int_{\pi/3}^{\pi/2} \frac{dx}{\sin x + \sin x \cos x} + 3 \\ &= 3 \int_{\pi/3}^{\pi/2} \frac{dx}{1+\cos x} \\ \int_{\pi/3}^{\pi/2} \frac{dx}{1+\cos x} &= \int_{\pi/3}^{\pi/2} \frac{1-\cos x}{\sin^2 x} dx \\ &= \int_{\pi/3}^{\pi/2} (\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x) dx \\ &= (\operatorname{cosec} x - \cot x) \Big|_{\pi/3}^{\pi/2} = (1) - \left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) = 1 - \frac{1}{\sqrt{3}} \\ \int_{\pi/3}^{\pi/2} \frac{dx}{\sin x(1+\cos x)} &= \\ \int \frac{dx}{(2 \tan x/2)(1+1-\tan^2 x/2)} &= \\ = \int \frac{(1+\tan^2 x/2) \sec^2 x/2 dx}{2 \tan x/2} &= \\ \tan x/2 = t &= \\ \frac{1}{2} \int \left(\frac{1+t^2}{t} \right) dt &= \frac{1}{2} \left[\ell n t + \frac{t^2}{2} \right] \Big|_{\frac{1}{\sqrt{3}}}^1 \\ &= \frac{1}{2} \left[\left(0 + \frac{1}{2} \right) - \left(\ell n \frac{1}{\sqrt{3}} + \frac{1}{6} \right) \right] = \left(\frac{1}{3} + \ell n \sqrt{3} \right) \frac{1}{2} \\ &= \left(\frac{1}{6} + \frac{1}{2} \ell n \sqrt{3} \right) dt \\ 2 \left(\frac{1}{6} + \frac{1}{2} \ell n \sqrt{3} \right) + 3 \left(1 - \frac{1}{\sqrt{3}} \right) &= \\ = \frac{1}{3} + \ell n \sqrt{3} + 3 - \sqrt{3} &= \frac{10}{3} + \ell n \sqrt{3} - \sqrt{3} \end{aligned}$$

Question 107

Let a differentiable function f satisfy $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}$, $x \geq 3$. Then $12f(8)$ is equal to:

[31-Jan-2023 Shift 1]

Options:

A. 34

B. 19

C. 17

D. 1

Answer: C

Solution:

Differentiate w.r.t. x

$$f'(x) + \frac{f(x)}{x} = \frac{1}{2\sqrt{x+1}}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xf(x) = \int \frac{x}{2\sqrt{x+1}} dx$$

$$x+1 = t^2$$

$$= \int \frac{t^2-1}{2t} 2t dt$$

$$xf(x) = \frac{t^3}{3} - t + c$$

$$xf(x) = \frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + c$$

Also putting $x=3$ in given equation $f(3)+0=\sqrt{4} f(3)=2$

$$\Rightarrow C = 8 - \frac{8}{3} = \frac{16}{3}$$

$$f(x) = \frac{\frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + \frac{16}{3}}{x}$$

$$f(8) = \frac{9 - 3 + \frac{16}{3}}{8} = \frac{34}{24}$$

$$\Rightarrow 12f(8) = 17$$

Question 108

Let $\alpha \in (0, 1)$ and $\beta = \log_e(1 - \alpha)$. Let $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n}$, $x \in (0, 1)$.

Then the integral $\int_0^\alpha \frac{t^{50}}{1-t} dt$ is equal to
[31-Jan-2023 Shift 1]

Options:

A. $\beta - P_{50}(\alpha)$

B. $-(\beta + P_{50}(\alpha))$

C. $P_{50}(\alpha) - \beta$

D. $\beta + P_{50}(\alpha)$

Answer: B

Solution:

$$\begin{aligned} \int_0^\alpha \frac{t^{50} - 1 + 1}{1-t} dt &= \int_0^\alpha (1+t+\dots+t^{49}) dt + \int_0^\alpha \frac{1}{1-t} dt \\ &= \left(\frac{\alpha^{50}}{50} + \frac{\alpha^{49}}{49} + \dots + \frac{\alpha^1}{1} \right) + \left(\frac{\ln(1-t)}{-1} \right)_0^\alpha \\ &= -P_{50}(\alpha) - \ln(1-\alpha) \\ &= -P_{50}(\alpha) - \beta \end{aligned}$$

Question 109

Let $\alpha > 0$. If $\int_0^\alpha \frac{x}{\sqrt{x+\alpha} - \sqrt{x}} dx = \frac{16+20\sqrt{2}}{15}$, then α is equal to :

[31-Jan-2023 Shift 2]

Options:

- A. 2
B. 4
C. $\sqrt{2}$
D. $2\sqrt{2}$

Answer: A

Solution:

After rationalising

$$\begin{aligned} & \int_0^{\alpha} \frac{x}{\alpha} (\sqrt{x+\alpha} + \sqrt{x}) \\ &= \frac{1}{\alpha} \left[\frac{2}{5} (x+\alpha)^{5/2} - \alpha \frac{2}{3} (x+\alpha)^{3/2} + \frac{2}{5} x^{5/2} \right]_0^{\alpha} \\ &= \frac{1}{\alpha} \left(\frac{5}{2} (2\alpha)^{5/2} - \frac{2\alpha}{3} (2\alpha)^{3/2} + \frac{2}{5} \alpha^{5/2} - \frac{2}{5} \alpha^{5/2} + \frac{2}{3} \alpha^{5/2} \right) \\ &= \frac{1}{\alpha} \left(\frac{2^{7/2} \alpha^{5/2}}{5} - \frac{2^{5/2} \alpha^{5/2}}{3} + \frac{2}{3} \alpha^{5/2} \right) \\ &= \alpha^{3/2} \left(\frac{2^{7/2}}{5} - \frac{2^{5/2}}{3} + \frac{2}{3} \right) \\ &= \frac{\alpha^{3/2}}{15} (24\sqrt{2} - 20\sqrt{2} + 10) = \frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10) \end{aligned}$$

Now,

$$\frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10) = \frac{16 + 20\sqrt{2}}{15}$$

$$\Rightarrow \alpha = 2$$

Question 110

If $\phi(x) = \frac{1}{\sqrt{x}} \int_{\frac{\pi}{4}}^x (4\sqrt{2} \sin t - 3\phi'(t)) dt$, $x > 0$ then $\phi'\left(\frac{\pi}{4}\right)$ is equal to :

[31-Jan-2023 Shift 2]

Options:

- A. $\frac{8}{\sqrt{\pi}}$
B. $\frac{4}{6 + \sqrt{\pi}}$
C. $\frac{8}{6 + \sqrt{\pi}}$
D. $\frac{4}{6 - \sqrt{\pi}}$

Answer: C

Solution:

$$\begin{aligned} \phi'(x) &= \frac{1}{\sqrt{x}} [(4\sqrt{2} \sin x - 3\phi'(x)) \cdot 1 - 0] - \frac{1}{2} x^{-3/2} \\ &= \frac{1}{\sqrt{x}} \int_{\frac{\pi}{4}}^x (4\sqrt{2} \sin t - 3\phi'(t)) dt \\ \phi'\left(\frac{\pi}{4}\right) &= \frac{2}{\sqrt{\pi}} \left[4 - 3\phi'\left(\frac{\pi}{4}\right) \right] + 0 \end{aligned}$$

$$\left(1 + \frac{6}{\sqrt{\pi}}\right) \varphi'\left(\frac{\pi}{4}\right) = \frac{8}{\sqrt{\pi}}$$

$$\varphi'\left(\frac{\pi}{4}\right) = \frac{8}{\sqrt{\pi} + 6}$$

Question 111

Let $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$.

If $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$, then $f(4)$ is equal to
[25-Jan-2023 Shift 1]

Options:

A. $\frac{1}{2}(\log_e 17 - \log_e 19)$

B. $\log_e 17 - \log_e 18$

C. $\frac{1}{2}(\log_e 19 - \log_e 17)$

D. $\log_e 19 - \log_e 20$

Answer: A

Solution:

Put $x^2 = t$

$$\int \frac{dt}{(t+1)(t+3)} = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t+3} \right) dt$$

$$f(x) = \frac{1}{2} \ln \left(\frac{x^2+1}{x^2+3} \right) + C$$

$$f(3) = \frac{1}{2}(\ln 10 - \ln 12) + C$$

$$\Rightarrow C = 0$$

$$f(4) = \frac{1}{2} \ln \left(\frac{17}{19} \right)$$

Question 112

If $\int \sqrt{\sec 2x - 1} dx = \alpha \log_e \left| \cos 2x + \beta + \sqrt{\cos 2x \left(1 + \cos \frac{1}{\beta} x \right)} \right|$

+ constant, then $\beta - \alpha$ is equal to _____.

[30-Jan-2023 Shift 2]

Answer: 1

Solution:

$$\int \sqrt{\sec 2x - 1} dx = \int \sqrt{\frac{1 - \cos 2x}{\cos 2x}} dx$$

$$= \sqrt{2} \int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx$$

$$\begin{aligned}
 \text{put } \cos x = t &\Rightarrow -\sin x \, dx = dt \\
 &= -\sqrt{2} \int \frac{dt}{\sqrt{2t^2 - 1}} \\
 &= -\ln \left| \sqrt{2} \cos x + \sqrt{\cos 2x} \right| + c \\
 &= -\frac{1}{2} \ln \left| 2\cos^2 x + \cos 2x + 2\sqrt{\cos 2x} \cdot \sqrt{2} \cos x \right| + c \\
 &= -\frac{1}{2} \ln \left| \cos^2 x + \frac{1}{2} + \sqrt{\cos 2x} \cdot \sqrt{1 + \cos 2x} \right| + c \\
 \therefore \beta &= \frac{1}{2}, \alpha = -\frac{1}{2} \Rightarrow \beta - \alpha = 1
 \end{aligned}$$

Question 113

$\lim_{n \rightarrow \infty} \left(\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n} \right)$ is equal to :-

[1-Feb-2023 Shift 1]

Options:

- A. 0
- B. $\log_e 2$
- C. $\log_e \left(\frac{3}{2} \right)$
- D. $\log_e \left(\frac{2}{3} \right)$

Answer: B

Solution:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left(\frac{1}{1+n} + \dots + \frac{1}{n+n} \right) &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n+r} \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left(\frac{1}{1 + \frac{r}{n}} \right) \\
 &= \int_0^1 \frac{1}{1+x} dx = [\ln(1+x)]_0^1 = \ln 2.
 \end{aligned}$$

Question 114

$$\int_0^1 (x^{21} + x^{14} + x^7)(2x^{14} + 3x^7 + 6)^{1/7} dx = \frac{1}{l} (11)^{m/n}$$

where $l, m, n \in \mathbb{N}$, m and n are coprime then $l + m + n$ is equal to _____.

[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$$\begin{aligned}
 \int (x^{20} + x^{13} + x^6)(2x^{21} + 3x^{14} + 6x^7)^{1/7} dx \\
 2x^{21} + 3x^{14} + 6x^7 = t \\
 42(x^{20} + x^{13} + x^6) dx = dt
 \end{aligned}$$

$$\frac{1}{42} \int_0^{11} t^{\frac{1}{7}} dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}} \times \frac{1}{42} \right)_0^{11}$$

$$= \frac{1}{48} \left(t^{\frac{8}{7}} \right)_0^{11} = \frac{1}{48} (11)^{8/7}$$

$$l = 48, m = 8, n = 7$$

$$l + m + n = 63$$

Question 115

The value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$ is:

[1-Feb-2023 Shift 2]

Options:

A. $\frac{\pi^2}{6}$

B. $\frac{\pi^2}{12\sqrt{3}}$

C. $\frac{\pi^2}{3\sqrt{3}}$

D. $\frac{\pi^2}{6\sqrt{3}}$

Answer: D

Solution:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx \dots (1)$$

$$x \rightarrow -x$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{-x + \frac{\pi}{4}}{2 - \cos 2x} dx \dots (2)$$

$$(1) + (2)$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{\pi}{2}}{2 - \cos 2x} dx$$

$$I = \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{dx}{2 - \cos 2x}$$

$$I = \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{(1 + \tan^2 x) dx}{2(1 + \tan^2 x) - (1 - \tan^2 x)}$$

$$I = \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{dt}{3t^2 + 1}$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{3}} \tan^{-1} \sqrt{3}$$

$$I = \frac{\pi^2}{6\sqrt{3}}$$

Question 116

If $\int_0^{\pi} \frac{5^{\cos x} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx}{1 + 5^{\cos x}} = \frac{k\pi}{16}$, then k is equal to _____.

[1-Feb-2023 Shift 2]

Answer: 13

Solution:

$$I = \int_0^{\pi} \frac{5^{\cos x} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx}{1 + 5^{\cos x}}$$

$$I = \int_0^{\pi} \frac{5^{-\cos x} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx}{1 + 5^{\cos x}}$$

$$2I = \int_0^{\pi} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx$$

$$\text{not } I = \text{not } 2 \int_0^{\frac{\pi}{2}} (1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x) dx$$

$$I = \int_0^{\frac{\pi}{2}} (1 + \sin x(-\sin 3x) + \sin^2 x - \sin^3 x \sin 3x) dx$$

$$2I = \int_0^{\frac{\pi}{2}} (3 + \cos 4x + \cos^3 x \cos 3x - \sin^3 x \sin 3x) dx$$

$$2I = \int_0^{\frac{\pi}{2}} 3 + \cos 4x + \left(\frac{\cos 3x + 3 \cos x}{4} \right) \cos 3x - \sin 3x \left(\frac{3 \sin x - \sin 3x}{4} \right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(3 + \cos 4x + \frac{1}{4} + \frac{3}{4} \cos 4x \right) dx$$

$$2I = \frac{13}{4} \times \frac{\pi}{2} + \frac{7}{4} \left(\frac{\sin 4x}{4} \right)_0^{\frac{\pi}{2}} \Rightarrow I = \frac{13\pi}{16}$$

Question 117

Let $f(x)$ be a function satisfying $f(x) + f(\pi - x) = \pi^2$, $\forall x \in \mathbb{R}$. Then $\int_0^{\pi} f(x) \sin x dx$ is equal to :

[6-Apr-2023 shift 2]

Options:

A. $\frac{\pi^2}{2}$

B. π^2

C. $2\pi^2$

D. $\frac{\pi^2}{4}$

Answer: B

Solution:

$$I = \int_0^{\pi} f(x) \sin x \, dx \dots (1)$$

Apply king property

$$I = \int_0^{\pi} f(\pi - x) \sin(\pi - x) \, dx \dots (2)$$

Add

$$2I = \int_0^{\pi} f(x) + f(\pi - x) \sin x \, dx$$

$$2I = \int_0^{\pi} \pi^2 \sin x \, dx$$

$$2I = \pi^2 (\text{not})$$

$$I = \pi^2$$

Ans. Option 2

Question 118

Let $f(x) = \frac{x}{(1+x^n)^{\frac{1}{n}}}$, $x \in \mathbb{R} - \{-1\}$, $n \in \mathbb{N}$, $n > 2$. If $f^n(x) = n$ (f of of upto n times) (x), then

$\lim_{n \rightarrow \infty} \int_0^1 x^{n-2} (f^n(x)) \, dx$ is equal to _____.

[6-Apr-2023 shift 2]

Answer: 0

Solution:

$$\text{Let } f(x) = \frac{x}{(1+x^n)^{\frac{1}{n}}}, x \in \mathbb{R} - \{-1\}, n \in \mathbb{N}, n > 2.$$

If $f^n(x) = n$ (f of of upto n times) (x)

then $\lim_{n \rightarrow \infty} \int_0^1 x^{n-2} (f^n(x)) \, dx$

$$f(f(x)) = \frac{x}{(1+2x^n)^{\frac{1}{n}}}$$

$$f(f(f(x))) = \frac{x}{(1+3x^n)^{\frac{1}{n}}}$$

$$\text{Similarly } f^n(x) = \frac{x}{(1+n \cdot x^n)^{\frac{1}{n}}}$$

$$\text{Now } \lim_{n \rightarrow \infty} \int \frac{x^{n-2} \cdot x \, dx}{(1+n \cdot x^n)^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \int \frac{x^{n-1} \cdot dx}{(1+n \cdot x^n)^{\frac{1}{n}}}$$

$$\text{Now } 1 + nx^n = t$$

$$n^2 \cdot x^{n-1} \, dx = dt$$

$$x^{n-1} \, dx = \frac{dt}{n^2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_1^{1+n} \frac{dt}{t^{\frac{1}{n}}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} \left[\frac{t^{\frac{1}{n}-1}}{\frac{1}{n}-1} \right]_1^{1+n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} \left((1+n)^{\frac{n-1}{n}} - 1 \right) \text{ Now let } n = \frac{1}{h}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{\left(1 + \frac{1}{h}\right)^{1-h} - 1}{\frac{1}{h} \left(\frac{1-h}{h}\right)}$$

Using series expansion

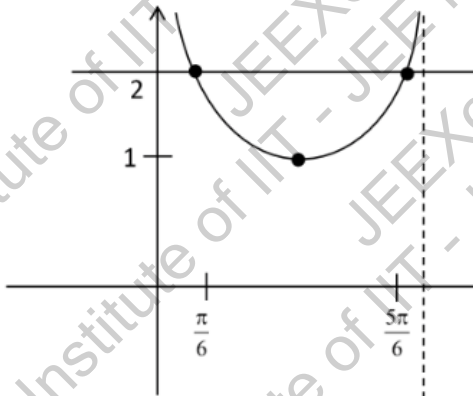
$\Rightarrow 0$

Question 119

Let $[t]$ denote the greatest integer $\leq t$. The $\int_{\pi/6}^{5\pi/6} (8[\operatorname{cosec} x] - 5[\cot x]) dx$ is equal to
[8-Apr-2023 shift 1]

Answer: 14

Solution:



$$\begin{aligned} & 8 \int_{\pi/6}^{5\pi/6} [\operatorname{cosec} x] dx \\ & 8 \int_{\pi/6}^{5\pi/6} dx = \frac{16\pi}{3} \\ & I = \int_{\pi}^{5\pi/6} [\cot x] dx \\ & x \rightarrow \pi - x \\ & I = \int_{\pi/6}^{5\pi/6} [-\cot x] dx \\ & 2I = \int_{\pi/6}^{5\pi/6} ([\cot x] + [-\cot x]) dx \\ & I = -\frac{1}{2} \int_{\pi/6}^{5\pi/6} dx \Rightarrow -\frac{1}{2} \left(\frac{4\pi}{6} \right) \\ & = -\pi/3 \\ & \therefore \frac{2}{\pi} \left[\frac{16\pi}{3} + \frac{5\pi}{3} \right] = \frac{2}{\pi} \left(\frac{21\pi}{3} \right) \\ & = 14 \end{aligned}$$

Question 120

Let $[t]$ denote the greatest integer function. If $\int_0^{24} [x^2] dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5}$, then $\alpha + \beta + \gamma + \delta$ is equal to _____.
[8-Apr-2023 shift 2]

Answer: 6

Solution:

$$\int_0^1 0 \, dx + \int_1^{\sqrt{2}} 1 \, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 \, dx + \int_{\sqrt{3}}^2 3 \, dx + \int_2^{\sqrt{5}} 4 \, dx + \int_{\sqrt{5}}^{2.4} 5 \, dx \\ = \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) + 4(\sqrt{5} - 2) + 5((2.4) - \sqrt{5}) \\ = 9 - \sqrt{2} - \sqrt{3} - \sqrt{5} \\ \alpha + \beta + \gamma + \delta = 9 - 1 - 1 - 1 = 6$$

Question 121

Let f be a continuous function satisfying $\int_0^t (f(x) + x^2) \, dx = \frac{4}{3}t^3, \forall t > 0$. Then $f\left(\frac{\pi^2}{4}\right)$ is equal to
[10-Apr-2023 shift 2]

Options:

A. $-\pi^2 \left(1 + \frac{\pi^2}{16}\right)$

B. $\pi \left(1 - \frac{\pi^3}{16}\right)$

C. $-\pi \left(1 + \frac{\pi^3}{16}\right)$

D. $\pi^2 \left(1 - \frac{\pi^3}{16}\right)$

Answer: B

Solution:

$$\int_0^t (f(x) + x^2) \, dx = \frac{4}{3}t^3, \forall t > 0 \\ (f(t^2) + t^4) = 2t \\ f(t^2) = 2t - t^4 \\ t = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi^2}{4}\right) = \frac{2\pi}{2} - \frac{\pi^4}{16} \\ = \pi - \frac{\pi^4}{16} = \pi \left(1 - \frac{\pi^3}{16}\right)$$

Question 122

The value of the integral $\int_{-\log_e 2}^{\log_e 2} e^x \left(\log_e \left(e^x + \sqrt{1 + e^{2x}} \right) \right) \, dx$ is equal to :
[11-Apr-2023 shift 1]

Options:

A. $\log_e \left(\frac{(2 + \sqrt{5})^2}{\sqrt{1 + \sqrt{5}}} \right) + \frac{\sqrt{5}}{2}$

$$\text{B. } \log_e \left(\frac{2(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}} \right) - \frac{\sqrt{5}}{2}$$

$$\text{C. } \log_e \left(\frac{\sqrt{2}(3-\sqrt{5})^2}{\sqrt{1+\sqrt{5}}} \right) + \frac{\sqrt{5}}{2}$$

$$\text{D. } \log_e \left(\frac{\sqrt{2}(2+\sqrt{5})^2}{\sqrt{1+\sqrt{5}}} \right) - \frac{\sqrt{5}}{2}$$

Answer: D

Solution:

$$I = \int_{-\ln 2}^{\ln 2} e^x (\ln(e^x + \sqrt{1+e^{2x}})) dx$$

$$\text{Put } e^x = t \Rightarrow e^x dx = dt$$

$$I = \int_{1/2}^2 \ln(t + \sqrt{1+t^2}) dt$$

Applying integration by parts.

$$= [t \ln(t + \sqrt{1+t^2})]_{1/2}^2 - \int_{1/2}^2 \frac{t}{t + \sqrt{1+t^2}} \left(1 + \frac{2t}{2\sqrt{1+t^2}}\right) dt$$

$$= 2 \ln(2 + \sqrt{5}) - \frac{1}{2} \ln\left(\frac{1 + \sqrt{5}}{2}\right) - \int_{1/2}^2 \frac{t}{\sqrt{1+t^2}} dt$$

$$= 2 \ln(2 + \sqrt{5}) - \frac{1}{2} \ln\left(\frac{1 + \sqrt{5}}{2}\right) - \frac{\sqrt{5}}{2}$$

$$= \ln \left(\frac{(2 + \sqrt{5})^2}{\left(\left(\frac{\sqrt{5} + 1}{2}\right)^{\frac{1}{2}}\right)^2} \right) - \frac{\sqrt{5}}{2}$$

Question 123

For $m, n > 0$, let $\alpha(m, n) = \int_0^2 t^m (1 + 3t)^n dt$. If $11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6$, then p is equal to

 .
[11-Apr-2023 shift 1]

Answer: 32

Solution:

$$\alpha(m, n) = \int_0^2 t^m (1 + 3t)^n dt$$

$$\text{If } 11\alpha(10, 6) + 18\alpha(11, 5) = p(14)^6 \text{ then } P$$

$$= 11 \int_0^2 \frac{t^{10} (1 + 3t)^6}{1} + 10 \int_0^2 t^{11} (1 + 3t)^5 dt$$

$$= 11 \left[(1 + 3t)^6 \cdot \frac{t^{11}}{11} - \int 6(1 + 3t)^5 \cdot 3 \frac{t^{11}}{11} \right]_0^2 + 18 \int_0^2 t^{11} (1 + 3t)^5 dt$$

$$= (t^{11} (1 + 3t)^6)_0^2$$

$$= 2^{11} (7)^6$$

$$= 2^5 (14)^6$$

$$= 32(14)^6$$

Question 124

Let the function $f : [0, 2] \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{\min\{x^2, x - [x]\}} & x \in [0, 1) \\ e^{[x - \log_e x]} & x \in [1, 2) \end{cases}$$

where $[t]$ denotes the greatest integer less than or equal to t . Then the value of the integral $\int_0^2 xf(x) dx$ is

[11-Apr-2023 shift 2]

Options:

A. $(e - 1)\left(e^2 + \frac{1}{2}\right)$

B. $1 + \frac{3e}{2}$

C. $2e - \frac{1}{2}$

D. $2e - 1$

Answer: A

Solution:

$$f : [0, 2] \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} \min\{x^2, \{x\}\} & ; x \in [0, 1) \\ [x - \log_e x] & ; x \in [1, 2) \end{cases}$$

$$f(x) = \begin{cases} e^{x^2} & ; x \in [0, 1) \\ e^{x - \log_e x} & ; x \in [1, 2) \end{cases}$$

$$\begin{aligned} \int_0^2 xf(x) dx &= \int_0^1 x \cdot e^{x^2} dx + \int_1^2 x \cdot e^{x - \log_e x} dx \\ &= \frac{1}{2}(e - 1) + \frac{1}{2}(4 - 1)e \\ &\Rightarrow 2e - \frac{1}{2} \end{aligned}$$

Question 125

If $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $\int_0^{\frac{\pi}{2}} f(\sin 2x) \sin x dx + \alpha \int_0^{\frac{\pi}{4}} f(\cos 2x) \cos x dx = 0$, then the value of α is
[11-Apr-2023 shift 2]

Options:

A. $-\sqrt{3}$

B. $\sqrt{3}$

C. $-\sqrt{2}$

D. $\sqrt{2}$

Answer: C

Solution:

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} F(\sin 2x) \sin x \, dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cdot \cos x \, dx = 0$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} F(\sin 2x) \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\sin 2x) \cdot \sin x \, dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cdot \cos x \, dx = 0$$

$$\int_0^a F(x) \, dx = \int_0^a F(a-x) \, dx$$

$$\text{Let } x = t + \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \sin \left(\frac{\pi}{4} - x \right) \, dx + \int_0^{\frac{\pi}{4}} F(\cos 2t) \sin \left(t + \frac{\pi}{4} \right) + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \left\{ \sin \left(\frac{\pi}{4} - x \right) + \sin \left(x + \frac{\pi}{4} \right) + \alpha \cos x \right\} \, dx = 0.$$

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \{ (\sqrt{2} + \alpha) \cos x \} \, dx = 0$$

$$(\sqrt{2} + \alpha) \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

$$\therefore \text{in interval } \left(0, \frac{\pi}{4} \right) \Rightarrow F(\cos 2x) \& \cos x \text{ is NOT Zero.}$$

$$\therefore \sqrt{2} + \alpha = 0$$

$$\alpha = -\sqrt{2}$$

Question 126

$$\text{If } \int_{-0.15}^{0.15} |100x^2 - 1| \, dx = \frac{k}{3000}, \text{ then } k \text{ is equal to } \underline{\hspace{2cm}}.$$

[12-Apr-2023 shift 1]

Answer: 575

Solution:

$$\int_{-0.15}^{0.15} |100x^2 - 1| \, dx = 2 \int_0^{0.15} |100x^2 - 1| \, dx$$

$$\text{Now } 100x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{100} \Rightarrow x = 0.1$$

$$I = 2 \left[\int_0^{0.1} (1 - 100x^2) \, dx + \int_{0.1}^{0.15} (100x^2 - 1) \, dx \right]$$

$$I = 2 \left[x - \frac{100}{3} x^3 \right]_0^{0.1} + 2 \left[\frac{100x^3}{3} - x \right]_{0.1}^{0.15}$$

$$= 2 \left[0.1 - \frac{0.1}{3} \right] + 2 \left[\frac{0.3375}{3} - 0.15 - \frac{0.1}{3} + 0.1 \right]$$

$$= 2 \left[0.2 - \frac{0.2}{3} + 0.1125 - 0.15 \right]$$

$$= 2 \left[\frac{5}{100} - \frac{2}{30} + \frac{1125}{10000} \right] = 2 \left(\frac{1500 - 2000 + 3375}{30000} \right)$$

$$= \frac{575}{3000} \Rightarrow k = 575$$

Question 127

The value of $\frac{\int_0^{\frac{\pi}{4}} e^{-x} \tan^{50} x \, dx}{\int_0^{\frac{\pi}{4}} e^{-x} (\tan^{49} x + \tan^{51} x) \, dx}$ is

[13-Apr-2023 shift 2]

Options:

- A. 25
- B. 51
- C. 50
- D. 49

Answer: C

Solution:

$$\begin{aligned} \text{let } I_1 &= \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx \\ I_2 &= \int_0^{\pi/4} e^{-x} (\tan^{49} x + \tan^{51} x) \, dx \\ &= \int_0^{\pi/4} e^{-x} \tan^{49} x (\sec^2 x) \, dx \\ &= \left[e^{-x} \frac{\tan^{50} x}{50} \right]_0^{\pi/4} + \frac{1}{50} \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx \\ &= \frac{e^{-\pi/4}}{50} + \frac{1}{50} \int_0^{\pi/4} e^{-x} \tan^{50} x \, dx = \frac{I_1}{50} \\ \text{then } \frac{I_1}{I_2} &= 50 \end{aligned}$$

Question 128

Let $f_n = \int_0^{\pi/2} \left(\sum_{k=1}^n \sin^{k-1} x \right) \left(\sum_{k=1}^n (2k-1) \sin^{k-1} x \right) \cos x \, dx$, $n \in \mathbb{N}$. Then $f_{21} - f_{20}$ is equal to

[13-Apr-2023 shift 2]

Answer: 41

Solution:

$$\begin{aligned} f_n &= \int_0^{\pi/2} \left(\sum_{k=1}^n \sin^{k-1} x \right) \left(\sum_{k=1}^n (2k-1) \sin^{k-1} x \right) \cos x \, dx \\ \sin x &= t \\ \cos x \, dx &= dt \\ f_n &= \int_0^1 \left(\sum_{k=1}^n t^{k-1} \right) \left(\sum_{k=1}^n (2k-1) t^{k-1} \right) dt \\ &= \int_0^1 (1+t+t^2+\dots+t^n)(1+3t+5t^2+\dots+(2n+1)t^n) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 (1+t+t^2+\dots+t^{n-1})(1+3t+5t^2+\dots+(2n-1)t^{n-1}) dt \\
 &+ \int_0^1 (1+3t+5t^2+\dots+(2n+1))t^n dt \\
 &+ \int_0^1 (1+t+t^2+\dots+t^{n-1})(2n+1)t^n dt \\
 f_{n+1} - f_n &= \int_0^1 (1+3t+5t^2+\dots+(2n+1)t^n)t^n dt \\
 &+ \int_0^1 (1+t+t^2+\dots+t^{n+1})((2n+1)t^n) dt \\
 \text{put } n &= 20 \\
 f_{21} - f_{20} &= \int_0^1 (1+3t+5t^2+\dots+41 \cdot t^{20})t^{20} dt + \int_0^1 (1+t+t^2+\dots+t^{19})(41 \cdot t^{20}) dt \\
 &= \left(\frac{1}{21} + \frac{3}{22} + \frac{5}{23} + \dots + \frac{39}{40} + \frac{41}{41} \right) + \left(\frac{41}{21} + \frac{41}{22} + \frac{41}{40} \right) \\
 &= \frac{1+41}{21} + \frac{3+41}{22} + \dots + \frac{39+41}{40} + 1 = 40 + 1 = 41
 \end{aligned}$$

Question 129

If $\int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{(2-4x)})} dx = \frac{1}{\alpha} \log_e \left(\frac{\alpha+1}{\beta} \right)$, $\alpha, \beta > 0$, then $\alpha^4 - \beta^4$ is equal to

[15-Apr-2023 shift 1]

Options:

- A. 19
- B. -21
- C. 21
- D. 0

Answer: C

Solution:

$$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{(2-4x)})} \dots (i)$$

$$x \rightarrow 1-x$$

$$I = \int_0^1 \frac{e^{2-4x} dx}{(5+2x-2x^2)(1+e^{(2-4x)})} \dots (ii)$$

Add (i) and (ii)

$$2I \int_0^1 \frac{dx}{5+2x-2x^2} = \int_0^1 \frac{dx}{2 \left(\frac{11}{4} - \left(x - \frac{1}{2} \right)^2 \right)}$$

$$I = \frac{1}{\sqrt{11}} \ln \left(\frac{\sqrt{11}+1}{\sqrt{10}} \right)$$

$$\alpha = \sqrt{11}$$

$$\beta = \sqrt{10}$$

$$\alpha^4 - \beta^4 = 121 - 100 = 21$$

Question 130

Let $I(x) = \int \frac{x^2(x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx$. If $I(0) = 0$, then $I\left(\frac{\pi}{4}\right)$ is equal to :

[6-Apr-2023 shift 1]

Options:

A. $\log_e \frac{(\pi+4)^2}{16} + \frac{\pi^2}{4(\pi+4)}$

B. $\log_e \frac{(\pi+4)^2}{32} - \frac{\pi^2}{4(\pi+4)}$

C. $\log_e \frac{(\pi+4)^2}{16} - \frac{\pi^2}{4(\pi+4)}$

D. $\log_e \frac{(\pi+4)^2}{32} + \frac{\pi^2}{4(\pi+4)}$

Answer: B

Solution:

$$I(x) = \int \frac{x^2 (x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx$$

Let $x \tan x + 1 = t$

$$I = x^2 \left(\frac{-1}{x \tan x + 1} \right) + \int \frac{2x}{x \tan x + 1} dx$$

$$I = x^2 \left(\frac{-1}{x \tan x + 1} \right) + 2 \int \frac{2x}{x \tan x + 1} dx$$

$$I = x^2 \left(\frac{-1}{x \tan x + 1} \right) + 2 \ln |x \sin x + \cos x| + C$$

As $I(0) = 0 \Rightarrow C = 0$

$$I\left(\frac{\pi}{4}\right) = \ln\left(\frac{(\pi+4)^2}{32}\right) - \frac{\pi^2}{4(\pi+4)}$$

Question 131

Let $I(x) = \int \frac{(x+1)}{x(1+xe^x)^2} dx$, $x > 0$. If $\lim_{x \rightarrow \infty} I(x) = 0$, then $I(1)$ is equal to

[8-Apr-2023 shift 1]

Options:

A. $\frac{e+1}{e+2} - \log_e(e+1)$

B. $\frac{e+2}{e+1} + \log_e(e+1)$

C. $\frac{e+2}{e+1} - \log_e(e+1)$

D. $\frac{e+1}{e+2} + \log_e(e+1)$

Answer: D

Solution:

Question 132

The integral $\int \left(\left(\frac{x}{2} \right)^x + \left(\frac{2}{x} \right)^x \right) \log_2 x \, dx$ is equal to

[8-Apr-2023 shift 2]

Options:

A. $\left(\frac{x}{2}\right)^x \log_2\left(\frac{2}{x}\right) + C$

B. $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

C. $\left(\frac{x}{2}\right)^x \log_2\left(\frac{x}{2}\right) + C$

D. $\left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x + C$

Answer: B

Solution:

$$\begin{aligned} & \int (x^x 2^{-x} + 2^x x^{-x}) \log_2^x dx \\ & \int (e^{x \ln x} \cdot e^{-x \ln 2} + e^{x \ln 2} \cdot e^{-x \ln x}) dx \\ & \int (e^{x \ln x - x \ln 2} + e^{x \ln 2 - x \ln x}) \frac{\ln x}{\ln 2} dx \\ & \text{let } x \ln x - x \ln 2 = t \\ & (\ln x + 1 - \ln 2) dx = dt \end{aligned}$$

Question 133

If $I(x) = \int e^{\sin^2 x} (\cos x \sin 2x - \sin x) dx$ and $I(0) = 1$, then $I\left(\frac{\pi}{3}\right)$ is equal to :

[10-Apr-2023 shift 1]

Options:

A. $e^{\frac{3}{4}}$

B. $-e^{\frac{3}{4}}$

C. $\frac{1}{2}e^{\frac{3}{4}}$

D. $-\frac{1}{2}e^{\frac{3}{4}}$

Answer: C

Solution:

$$\begin{aligned} I &= \int e^{\sin^2 x} \sin 2x \cos x dx - \int e^{\sin^2 x} \sin x dx \\ &= \cos x \int e^{\sin^2 x} \sin 2x dx - \int ((-\sin x) \int e^{\sin^2 x} \sin 2x dx) dx - \int e^{\sin^2 x} \sin x dx \\ \sin^2 x &= t \\ \sin 2x dx &= dt \\ &= \cos x \int e^t dt + \int (\sin x \int e^t dt) dx - \int e^{\sin^2 x} \sin x dx \\ &= e^{\sin^2 x} \cos x + \int e^{\sin^2 x} \sin x dx - \int e^{\sin^2 x} \sin x dx \\ I &= e^{\sin^2 x} \cos x + C \\ I(0) &= 1 \\ \Rightarrow 1 &= 1 + C \\ \Rightarrow C &= 0 \\ \therefore I &= e^{\sin^2 x} \cos x \end{aligned}$$

$$I\left(\frac{\pi}{3}\right) = e^{\sin^2 \frac{\pi}{3}} \cos \frac{\pi}{3} \\ = \frac{e^{\frac{3}{4}}}{2}$$

Question 134

For $\alpha, \beta, \gamma, \delta \in \mathbb{N}$, if $\int \left(\left(\frac{x}{e} \right)^{2x} + \left(\frac{e}{x} \right)^{2x} \right) \log_e x dx = \frac{1}{\alpha} \left(\frac{x}{e} \right)^{\beta x} - \frac{1}{\gamma} \left(\frac{e}{x} \right)^{\delta x} + C$, where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ and C is constant of integration, then $\alpha + 2\beta + 3\gamma - 4\delta$ is equal to
[10-Apr-2023 shift 2]

Options:

- A. 4
- B. -4
- C. -8
- D. 1

Answer: A

Solution:

$$x = e^{\ln x} \\ \int \left(\left(\frac{x}{e} \right)^{2x} + \left(\frac{e}{x} \right)^{2x} \right) \log_e x dx = \int [e^{2(x \ln x - x)} + e^{-2(x \ln x - x)}] \ln x dx \\ x \ln x - x = t \\ \ln x \cdot dx = dt \\ \int (e^{2t} + e^{-2t}) dt \\ \frac{e^{2t}}{2} - \frac{e^{-2t}}{2} + C \\ = \frac{1}{2} \left(\frac{x}{e} \right)^{2x} - \frac{1}{2} \left(\frac{e}{x} \right)^{2x} + C \\ \alpha = B = \gamma = \delta = 2 \\ \alpha + 2B + 3\gamma - 4\delta = 4$$

Question 135

$$\int_0^{\infty} \frac{6}{e^{3x} + 6e^{2x} + 11e^x + 6} dx =$$

[13-Apr-2023 shift 1]

Options:

- A. $\log_e \left(\frac{32}{27} \right)$
- B. $\log_e \left(\frac{256}{81} \right)$
- C. $\log_e \left(\frac{512}{81} \right)$
- D. $\log_e \left(\frac{64}{27} \right)$

Answer: A

Solution:

$$\begin{aligned} I &= \int_0^{\infty} \frac{6}{(e^x+1)(e^x+2)(e^x+3)} dx \\ &= 6 \int_0^{\infty} \left(\frac{\frac{1}{2}}{e^x+1} + \frac{-1}{e^x+2} + \frac{\frac{1}{2}}{e^x+3} \right) dx \\ &= 3 \int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx - 6 \int_0^{\infty} \frac{e^{-x}}{1+2e^{-x}} dx + 3 \int_0^{\infty} \frac{e^{-x}}{1+3e^{-x}} dx \\ &= 3[-\ln(1+e^{-x})]_0^{\infty} + 6 \left[\frac{1}{2} \ln(1+2e^{-x}) \right]_0^{\infty} \\ &\quad - \frac{3}{3} [\ln(1+3e^{-x})]_0^{\infty} \\ &= 3 \ln 2 - 3 \ln 3 + \ln 4 \\ &= 3 \ln \frac{2}{3} + \ln 4 \\ &= \ln \frac{32}{27} \end{aligned}$$

Question 136

Let $f(x) = \int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$, $\left| x \right| < \frac{2}{\sqrt{3}}$. If $f(0) = 0$ and $f(1) = \frac{1}{\alpha\beta} \tan^{-1} \left(\frac{\alpha}{\beta} \right)$, $\alpha, \beta > 0$, then $\alpha^2 + \beta^2$ is equal to
[15-Apr-2023 shift 1]

Answer: 28

Solution:

$$\begin{aligned} f(x) &= \int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}} \\ x &= \frac{1}{t} \\ &= \int \frac{\frac{-1}{t^2} dt}{\frac{(3t^2+4)}{t^2} \frac{\sqrt{4t^2-3}}{t}} \\ &= \int \frac{-dt \cdot t}{(3t^2+4)\sqrt{4t^2-3}} : \text{ Put } 4t^2-3 = z^2 \\ &= \frac{1}{4} \int \frac{z dx}{\left(3 \left(\frac{z^2+3}{4} \right) + 4 \right) z} \\ &= \int \frac{-dz}{3z^2+25} = -\frac{1}{3} \int \frac{dz}{z^2 + \left(\frac{5}{\sqrt{3}} \right)^2} \\ &= -\frac{1}{3} \cdot \frac{\sqrt{3}}{5} \tan^{-1} \left(\frac{\sqrt{3}z}{5} \right) + C \\ &= -\frac{1}{5\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}}{5} \sqrt{4t^2-3} \right) + C \\ f(x) &= -\frac{1}{5\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}}{5} \sqrt{\frac{4-3x^2}{x^2}} \right) + C \\ \because f(0) = 0 \therefore C &= \frac{\pi}{10\sqrt{3}} \end{aligned}$$

$$f(1) = -\frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + \frac{\pi}{10\sqrt{3}}$$

$$f(1) = \frac{1}{5\sqrt{3}} \cot^{-1}\left(\frac{\sqrt{3}}{5}\right) = \frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$

$$\alpha = 5 : \beta = \sqrt{3} \therefore \alpha^2 + \beta^2 = 28$$

Question 137

The value of $12 \int_0^3 |x^2 - 3x + 2| dx$ is
[24-Jan-2023 Shift 1]

Answer: 22

Solution:

$$12 \int_0^3 |x^2 - 3x + 2| dx$$

$$= 12 \int_0^3 \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} dx$$

$$\text{If } x - \frac{3}{2} = t$$

$$dx = dt$$

$$= 24 \int_0^{3/2} \left(t^2 - \frac{1}{4}\right) dt$$

$$= 24 \left[-\int_0^{1/2} \left(t^2 - \frac{1}{4}\right) dt + \int_{1/2}^{3/2} \left(t^2 - \frac{1}{4}\right) dt \right] = 22$$

Question 138

If $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx = g(x) + c$, $g(1) = 0$, then $g\left(\frac{1}{2}\right)$ is equal to:

[26-Jun-2022-Shift-2]

Options:

A. $\log_e \left(\frac{\sqrt{3}-1}{\sqrt{3}+1} \right) + \frac{\pi}{3}$

B. $\log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) + \frac{\pi}{3}$

C. $\log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) - \frac{\pi}{3}$

D. $\frac{1}{2} \log_e \left(\frac{\sqrt{3}-1}{\sqrt{3}+1} \right) - \frac{\pi}{6}$

Answer: A

Solution:

$$\therefore \int_x^1 \sqrt{\frac{1-x}{1+x}} dx = g(x) + c$$

$$\int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx = g\left(\frac{1}{2}\right) - g(1)$$

$$\therefore g\left(\frac{1}{2}\right) = \int_1^{\frac{1}{2}} \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$$

$$\cos x = \cos 2\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos 2\theta} \cdot \frac{\sin \theta}{\cos \theta} (-2 \sin 2\theta) d\theta$$

$$= - \int_0^{\frac{\pi}{6}} \frac{4 \sin^2 \theta}{\cos 2\theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{6}} \frac{(1 - 2 \sin^2 \theta) - 1}{\cos 2\theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{6}} (1 - \sec 2\theta) d\theta$$

$$= \frac{\pi}{3} - 2 \cdot \frac{1}{2} [\ln |\sec 2\theta + \tan 2\theta|]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} - [\ln |2 + \sqrt{3}| - \ln 1]$$

$$= \frac{\pi}{3} - \ln \left(\frac{1}{2 - \sqrt{3}} \right)$$

$$= \frac{\pi}{3} - \ln \left| \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \right|$$

Question 139

If $\int \frac{(x^2+1)e^x}{(x+1)^2} dx = f(x)e^x + C$, where C is a constant, then $\frac{d^3f}{dx^3}$ at $x = 1$ is equal to :

[27-Jun-2022-Shift-1]

Options:

A. $-\frac{3}{4}$

B. $\frac{3}{4}$

C. $-\frac{3}{2}$

D. $\frac{3}{2}$

Answer: B

Solution:

$$I = \int \frac{e^x(x^2+1)}{(x+1)^2} dx = f(x)e^x + c$$

$$= \int \frac{e^x(x^2 - 1 + 1 + 1)}{(x+1)^2} dx$$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$$

$$= e^x \left(\frac{x-1}{x+1} \right) + c$$

$$\therefore f(x) = \frac{x-1}{x+1}$$

$$f(x) = 1 - \frac{2}{x+1}$$

$$f'(x) = 2 \left(\frac{1}{x+1} \right)^2$$

$$f''(x) = -4 \left(\frac{1}{x+1} \right)^3$$

$$f'''(x) = \frac{12}{(x+1)^4}$$

for $x = 1$

$$f'''(1) = \frac{12}{2^4} = \frac{12}{16} = \frac{3}{4}$$

Question 140

The value of the integral

$$\int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)}$$

is equal to

[24-Jun-2022-Shift-2]

Options:

- A. 2π
- B. 0
- C. π
- D. $\frac{\pi}{2}$

Answer: C

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)} \dots (i)$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^{-x})(\sin^6 x + \cos^6 x)} \dots (ii)$$

(i) and (ii)

From equation (i) & (ii)

$$2I = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sin^6 x + \cos^6 x}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{dx}{\sin^6 x + \cos^6 x} = \int_0^{\pi/2} \frac{dx}{1 - \frac{3}{4} \sin^2 2x}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{4 \sec^2 2x dx}{4 + \tan^2 2x} = 2 \int_0^{\pi/4} \frac{4 \sec^2 2x}{4 + \tan^2 2x} dx$$

when $x = 0, t = 0$ Now, $\tan 2x = t$ when $x = \frac{\pi}{4}, t \rightarrow \infty$

$$2 \sec^2 2x dx = dt$$

$$\therefore I = 2 \int_0^{\infty} \frac{2 dt}{4 + t^2} = 2 \left(\tan^{-1} \frac{t}{2} \right)_0^{\infty}$$

$$= 2 \cdot \frac{\pi}{2} = \pi$$

Question141

$$\lim_{n \rightarrow \infty} \left(\frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \frac{n^2}{(n^2+9)(n+3)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

is equal to :

[24-Jun-2022-Shift-2]

Options:

A. $\frac{\pi}{8} + \frac{1}{4} \log_e 2$

B. $\frac{\pi}{4} + \frac{1}{8} \log_e 2$

C. $\frac{\pi}{4} - \frac{1}{8} \log_e 2$

D. $\frac{\pi}{8} + \log_e \sqrt{2}$

Answer: A

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n^2}{(n^2+r^2)(n+r)} \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \frac{1}{\left[1 + \left(\frac{r}{n}\right)^2\right] \left[1 + \left(\frac{r}{n}\right)\right]} \\ &= \int_0^1 \frac{1}{(1+x^2)(1+x)} dx \\ &= \frac{1}{2} \int_0^1 \left[\frac{1}{1+x} - \frac{(x-1)}{(1+x^2)} \right] dx \\ &= \frac{1}{2} \left[\ln(1+x) - \frac{1}{2} \ln(1+x^2) + \tan^{-1} x \right]_0^1 \\ &= \frac{1}{2} \left[\frac{\pi}{4} + \frac{1}{2} \ln 2 \right] = \frac{\pi}{8} + \frac{1}{4} \ln 2 \end{aligned}$$

Question142

The value of $\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 + \cos^2 x)(e^{\cos x} + e^{-\cos x})} dx$ is equal to:

[25-Jun-2022-Shift-1]

Options:

A. $\frac{\pi^2}{4}$

B. $\frac{\pi^2}{2}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

Answer: C

Solution:

$$\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1 - \cos^2 x)(e^{\cos x} - e^{-\cos x})} dx$$

$$\text{Let } \cos x = t$$

$$\sin x dx = dt$$

$$= \int_{-1}^1 \frac{-e^t dt}{(1+t^2)(e^t - e^{-t})}$$

$$I = \int_{-1}^1 \frac{e^t}{(1+t^2)(e^t - e^{-t})} dt \dots (i)$$

$$I = \int_{-1}^1 \frac{e^{-t}}{(1+t^2)(e^{-t} + e^t)} dt \dots (ii)$$

Adding (i) and (ii)

Adding (i) and (ii)

$$2I = \int_{-1}^1 \frac{dt}{1-t^2}$$

$$2I = \tan^{-1} \Big|_{-1}^1$$

$$2I = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

Question 143

Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that

$$\int \left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x + 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

, for all $x > 0$, where c is an arbitrary constant. Then :
[25-Jun-2022-Shift-1]

Options:

A. g is decreasing in $\left(0, \frac{\pi}{4}\right)$

B. g' is increasing in $\left(0, \frac{\pi}{4}\right)$

C. $g + g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

D. $g - g'$ is increasing in $\left(0, \frac{\pi}{2}\right)$

Answer: D

Solution:

$$\int \left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x - 1)^2} \right) dx = \frac{xg(x)}{e^x + 1} + c$$

On differentiating both sides w.r.t. x , we get

$$\left(\frac{x(\cos x - \sin x)}{e^x + 1} + \frac{g(x)(e^x + 1 - xe^x)}{(e^x - 1)^2} \right)$$

$$= \frac{(e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)}{(e^x + 1)^2}$$

$$(e^x + 1)x(\cos x - \sin x) - g(x)(e^x + 1 - xe^x)$$

$$= (e^x + 1)(g(x) + xg'(x)) - e^x \cdot x \cdot g(x)$$

$$\Rightarrow g'(x) = \cos x - \sin x$$

$$\Rightarrow g(x) = \sin x - \cos x + C$$

$g(x)$ is increasing in $(0, \pi/4)$

$$g''(x) = -\sin x - \cos x < 0$$

$$\Rightarrow g'(x) \text{ is decreasing function let } h(x) = g(x) + g'(x) = 2 \cos x + C \Rightarrow h(x) = g'(x) + g''(x) = -2 \sin x < 0$$

$$\Rightarrow h \text{ is decreasing let } \phi(x) = g(x) - g'(x) = 2 \sin x + C \Rightarrow \phi'(x) = g(x) - g''(x) = 2 \cos x > 0 \Rightarrow \phi \text{ is increasing}$$

Question 144

If $b_n = \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx}{\sin x} dx$, $n \in \mathbb{N}$, then

[25-Jun-2022-Shift-2]

Options:

A. $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in A.P. with common difference -2

B. $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in an A.P. with common difference 2

C. $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in a G.P.

D. $\frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in an A.P. with common difference -2

Answer: D

Solution:

$$\begin{aligned} b_n - b_{n-1} &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 nx - \cos^2 (n-1)x}{\sin x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{-\sin(2n-1)x \cdot \sin x}{\sin x} dx \\ &= \left[-\frac{\cos(2n-1)x}{2n-1} \right]_0^{\pi/2} = -\frac{1}{2n-1} \end{aligned}$$

So, $b_3 - b_2, b_4 - b_3, b_5 - b_4$ are in H.P.

$\Rightarrow \frac{1}{b_3 - b_2}, \frac{1}{b_4 - b_3}, \frac{1}{b_5 - b_4}$ are in A.P. with common difference -2 .

Question 145

The value of $b > 3$ for which

$$12 \int_3^b \frac{1}{(x^2-1)(x^2-4)} dx = \log_e \left(\frac{49}{40} \right)$$

, is equal to

[25-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I = \int \frac{1}{(x^2-1)(x^2-4)} dx = \frac{1}{3} \int \left(\frac{1}{x^2-4} - \frac{1}{x^2-1} \right) dx$$

$$= \frac{1}{3} \left(\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \right) + C$$

$$12I = \ln \left| \frac{x-2}{x+2} \right| + 2 \ln \left| \frac{x-1}{x+1} \right| + C$$

$$12 \int_3^b \frac{dx}{(x^2-4)(x^2-1)}$$

$$= \ln \left(\frac{b-2}{b+2} \right) - 2 \ln \left(\frac{b-1}{b+1} \right) - \left(\ln \left(\frac{1}{5} \right) - 2 \ln \left(\frac{1}{2} \right) \right)$$

$$= \ln \left(\left(\frac{b-2}{b+2} \right) \cdot \frac{(b+1)^2}{(b-1)^2} \right) - \left(\ln \frac{4}{5} \right)$$

$$\text{So, } \frac{49}{40} = \frac{(b-2)}{(b+2)} \cdot \frac{(b+1)^2}{(b-1)^2} \cdot \frac{5}{4}$$

$$\Rightarrow b = 6$$

Question 146

Let $f(x) = \max\{|x+1|, |x+2|, \dots, |x+5|\}$. Then $\int_{-6}^0 f(x) dx$ is equal to

[26-Jun-2022-Shift-1]

Answer: 21

Solution:

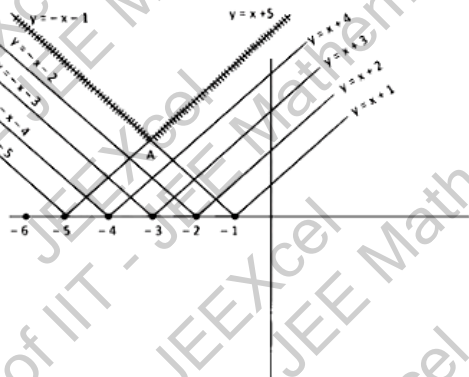
For $|x+1|$ critical point, $x+1=0 \Rightarrow x=-1$

For $|x+2|$ critical point, $x+2=0 \Rightarrow x=-2$

For $|x+3|$ critical point, $x+3=0 \Rightarrow x=-3$

For $|x+4|$ critical point, $x+4=0 \Rightarrow x=-4$

For $|x+5|$ critical point, $x+5=0 \Rightarrow x=-5$



Here maximum function is represent by the dotted line.

$\therefore$ Point of intersection A of line $y = -x-1$ and $y = x+5$:

$$-x-1 = x+5$$

$$\Rightarrow 2x = -6$$

$$\Rightarrow x = -3$$

$$\therefore y = -3+5 = 2$$

$$\therefore \text{Point } A = (-3, 2)$$

$$\therefore \int_{-6}^0 f(x) dx$$

$$= \int_{-6}^{-3} (-x-1) dx + \int_{-3}^0 (x+5) dx$$

$$= \left(-\frac{x^2}{2} - x \right)_{-6}^{-3} + \left(\frac{x^2}{2} + 5x \right)_{-3}^0$$

$$= \left[\left(-\frac{9}{2} + 3 \right) - \left(-\frac{36}{2} + 6 \right) \right] + \left[0 - \left(\frac{9}{2} - 15 \right) \right]$$

$$= \left(-\frac{3}{2} + 12 \right) + \frac{21}{2}$$

$$= \frac{21}{2} = 10.5$$

Question 147

The value of the integral

$$\frac{48}{\pi^4} \int_0^{\pi} \left(\frac{3\pi x^2}{2} - x^3 \right) \frac{\sin x}{1 + \cos^2 x} dx \text{ is equal to } \underline{\hspace{2cm}}$$

[26-Jun-2022-Shift-1]

Answer: 6

Solution:

$$I = \frac{48}{\pi^4} \int_0^\pi \left[\left(\frac{\pi-x}{2} \right)^3 - \frac{3\pi^2}{4} \left(\frac{\pi-x}{2} \right) + \frac{\pi^3}{4} \right] \frac{\sin x \, dx}{1 + \cos^2 x}$$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$I = \frac{48}{\pi^4} \int_0^\pi \left[-\left(\frac{\pi-x}{2} \right)^3 + \frac{3\pi^2}{4} \left(\frac{\pi-x}{2} \right) + \frac{\pi^3}{4} \right] \frac{\sin x \, dx}{1 + \cos^2 x}$$

Adding these two equations, we get

$$2I = \frac{48}{\pi^4} \int_0^\pi \frac{\pi^3}{2} \cdot \frac{\sin x \, dx}{1 + \cos^2 x}$$

$$\Rightarrow I = \frac{12}{\pi} [-\tan^{-1}(\cos x)]_0^\pi = \frac{12}{\pi} \cdot \frac{\pi}{2} = 6$$

Question 148

The integral $\frac{24}{\pi} \int_0^{\sqrt{2}} \frac{(2-x^2)dx}{(2+x^2)\sqrt{4+x^4}}$ is equal to
[26-Jun-2022-Shift-2]

Answer: 6

Solution:

$$I = \frac{24}{\pi} \int_0^{\sqrt{2}} \frac{2-x^2}{(2+x^2)\sqrt{4+x^4}} dx$$

$$\text{Let } x = \sqrt{2}t \Rightarrow dx = \sqrt{2}dt$$

$$I = \frac{24}{\pi} \int_0^1 \frac{(2-2t^2) \cdot \sqrt{2}dt}{(2+2t^2)\sqrt{4+4t^4}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_0^1 \frac{\left(\frac{1}{t^2}-1\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{\left(t+\frac{1}{t}\right)^2-2}}$$

$$\text{Let } t - \frac{1}{t} = u$$

$$\Rightarrow \left(1 - \frac{1}{t^2}\right)dt = du$$

$$= \frac{12\sqrt{2}}{\pi} \int_{-\infty}^2 \frac{-du}{u\sqrt{4^2-2}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_2^{\infty} \frac{du}{u^2 \sqrt{-\left(\frac{\sqrt{2}}{u}\right)^2}}$$

$$= \frac{12\sqrt{2}}{\pi} \int_{\frac{1}{\sqrt{2}}}^0 \frac{-\frac{1}{\sqrt{2}}dp}{\sqrt{1-p^2}}$$

$$= \frac{12}{\pi} [\sin^{-1}p]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{12}{\pi} \cdot \frac{\pi}{4} = 3$$

Question 149

The value of the integral

$\int_{-2}^2 \frac{x^3 + x}{(e^{|x|} + 1)} dx$ is equal to:

[27-Jun-2022-Shift-1]

Options:

A. $5e^2$

B. $3e^{-2}$

C. 4

D. 6

Answer: D

Solution:

$$I = \int_{-2}^2 \frac{|x^3 + x|}{e^{|x|} + 1} dx \dots (i)$$

$$I = \int_{-2}^2 \frac{|x^3 + x|}{e^{-|x|} + 1} dx \dots (ii)$$

$$2I = \int_{-2}^2 |x^3 + x| dx$$

$$2I = 2 \int_0^2 (x^3 + x) dx$$

$$I = \int_0^2 (x^3 + x) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^2$$

$$= \left(\frac{16}{4} + \frac{4}{2} \right) - 0$$

$$= 4 + 2 = 6$$

Question 150

Let f be a differentiable function in $\left(0, \frac{\pi}{2}\right)$. If $\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x$, then $\frac{1}{\sqrt{3}} f' \left(\frac{1}{\sqrt{3}} \right)$ is equal

to

[27-Jun-2022-Shift-2]

Options:

A. $6 - 9\sqrt{2}$

B. $6 - \frac{9}{\sqrt{2}}$

C. $\frac{9}{2} - 6\sqrt{2}$

D. $\frac{9}{\sqrt{2}} - 6$

Answer: B

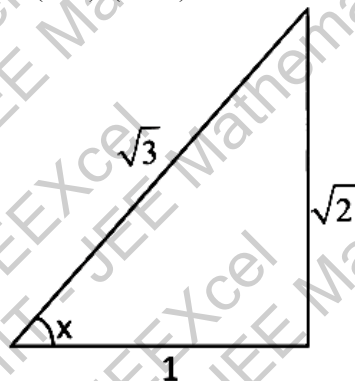
Solution:

$$\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x$$

$$\Rightarrow \sin x \cos^2 x f(\cos x) = 3 \sin^2 x \cos x - \sin x$$

$$\Rightarrow f(\cos x) = 3 \tan x - \sec^2 x$$

$$\Rightarrow f'(\cos x) \cdot (-\sin x) = 3 \sec^2 x - 2 \sec^2 x \tan x$$



$$\text{Put } \cos x = \frac{1}{\sqrt{3}}$$

$$\therefore f'\left(\frac{1}{\sqrt{3}}\right) \left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = 9 - 6\sqrt{2}$$

$$\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right) = 6 - \frac{9}{\sqrt{2}}$$

Question 151

The integral $\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx$, where $[.]$ denotes the greatest integer function, is equal to

[27-Jun-2022-Shift-2]

Options:

A. $1 + 6 \log_e \left(\frac{6}{7} \right)$

B. $1 - 6 \log_e \left(\frac{6}{7} \right)$

C. $\log_e \left(\frac{7}{6} \right)$

D. $1 - 7 \log_e \left(\frac{6}{7} \right)$

Answer: A

Solution:

$$\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx, \text{ let } \frac{1}{x} = t$$

$$\frac{-1}{x^2} dx = dt$$

$$= \int_{\infty}^1 \frac{1}{7^{\left[\frac{1}{t}\right]}} dt = \int_1^{\infty} \frac{1}{7^{\left[\frac{1}{t}\right]}} dt$$

$$= \int_1^2 \frac{1}{7^1} dt + \int_2^3 \frac{1}{7^2} dt + \dots$$

$$= \frac{1}{7} \left[-\frac{1}{t} \right]_1^2 + \frac{1}{7^2} \left[-\frac{1}{t} \right]_2^3 + \frac{1}{7^3} \left[-\frac{1}{t} \right]_3^4 + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{7^n} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^n}{n} - 7 \sum_{n=1}^{\infty} \frac{\left(\frac{1}{7}\right)^{n+1}}{n+1}$$

$$= -\log\left(1 - \frac{1}{7}\right) + 7\log\left(1 - \frac{1}{7}\right) + 1$$

$$= 1 + 6\log\frac{6}{7}$$

Question 152

Let $[t]$ denote the greatest integer less than or equal to t . Then, the value of the integral

$$\int_0^1 [-8x^2 + 6x - 1] dx \text{ is equal to}$$

[28-Jun-2022-Shift-1]

Options:

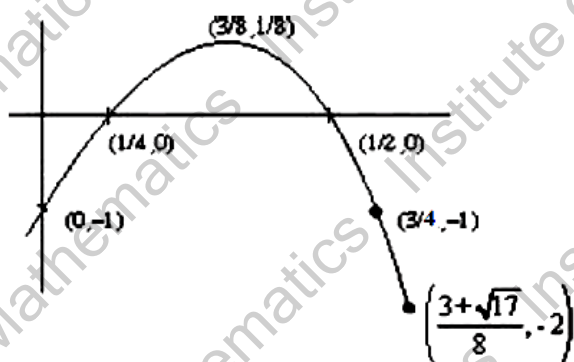
- A. -1
- B. $-\frac{5}{4}$
- C. $\frac{\sqrt{17}-13}{8}$
- D. $\frac{\sqrt{17}-16}{8}$

Answer: C

Solution:

$$\int_0^1 [-8x^2 + 6x - 1] dx$$

$$= \int_0^{1/4} -1 dx + \int_{1/4}^{1/2} 0 dx + \int_{1/2}^{3/4} -1 dx$$



$$+ \int_{3/4}^{\frac{3+\sqrt{17}}{8}} -2 dx + \int_{\frac{3+\sqrt{17}}{8}}^1 \frac{3+\sqrt{17}}{8} dx$$

$$= -[x]_0^{1/4} + 0 - [x]_{1/2}^{3/4} + -2[x]_{3/4}^{\frac{3+\sqrt{17}}{8}} - 3[x]_{\frac{3+\sqrt{17}}{8}}^1$$

$$= -\left(\frac{1}{4} - 0\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - 2\left(\frac{3+\sqrt{17}}{8} - \frac{3}{4}\right) - 3\left(1 - \frac{3+\sqrt{17}}{8}\right)$$

$$= -\frac{1}{4} - \frac{1}{4} + \frac{-6 - 2\sqrt{17}}{8} + \frac{3}{2} - 3 + \frac{9 + 3\sqrt{17}}{8}$$

$$= \frac{\sqrt{17}-13}{8}$$

Question 153

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$ and

let $g(x) = \int_x^{\pi/4} (f'(t) \sec t + \tan t \sec t f(t)) dt$ for $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} g(x)$ is equal to :

[28-Jun-2022-Shift-2]

Options:

- A. 2
- B. 3
- C. 4
- D. -3

Answer: B

Solution:

Given: $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$

$$g(x) = \int_{x \frac{\pi}{4}}^{\pi/4} (f'(t) \sec t + \tan t \sec t f(t)) dt$$

$$= [\sec t + f(t)]_x^{\pi/4} = 2 - \sec x f(x)$$

$$\text{Now, } \lim_{x \rightarrow \frac{\pi}{2}} g(x) = \lim_{h \rightarrow 0} g\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow 0} 2 - (\sec h) f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \rightarrow 0} \left[2 - \frac{f\left(\frac{\pi}{2} - h\right)}{\sin h} \right]$$

$$= \lim_{h \rightarrow 0} \left[2 + \frac{f'\left(\frac{\pi}{2} - h\right)}{\cos h} \right] = 3$$

Question 154

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $f(x) + f(x+k) = n$, for all $x \in \mathbb{R}$ where $k > 0$ and n is a positive integer. If $I_1 = \int_0^{4nk} f(x) dx$ and $I_2 = \int_{-k}^{3k} f(x) dx$, then :

[28-Jun-2022-Shift-2]

Options:

- A. $I_1 + 2I_2 = 4nk$
- B. $I_1 + 2I_2 = 2nk$
- C. $I_1 + nI_2 = 4n^2k$
- D. $I_1 + nI_2 = 6n^2k$

Answer: C

Solution:

$$f: \mathbb{R} \rightarrow \mathbb{R} \text{ and } f(x) + f(x+k) = n \quad \forall x \in \mathbb{R}$$

$$x \rightarrow x+k$$

$$f(x+k) + f(x+2k) = n$$

$$\therefore f(x+2k) = f(x)$$

So, period of $f(x)$ is $2k$

$$\text{Now, } I_1 = \int_0^{4nk} f(x) dx = 2n \int_0^{2k} f(x) dx$$

$$= 2n \left[\int_0^k f(x) dx + \int_k^{2k} f(x) dx \right]$$

$$x = t+k \Rightarrow dx = dt \text{ (in second integral)}$$

$$= 2n \left[\int_0^k f(x) dx + \int_0^k f(t+k) dt \right]$$

$$= 2n^2k$$

$$\text{Now, } I_2 = \int_{-k}^{3k} f(x) dx = 2 \int_0^{2k} f(x) dx$$

$$I_2 = 2(nk)$$

$$\therefore I_1 + nI_2 = 4n^2k$$

Question 155

$$\int_0^5 \cos \left(\pi \left(x - \left[\frac{x}{2} \right] \right) \right) dx$$

where $[t]$ denotes greatest integer less than or equal to t , is equal to:

[29-Jun-2022-Shift-1]

Options:

A. -3

B. -2

C. 2

D. 0

Answer: D

Solution:

We know,

$\left[\frac{x}{2} \right]$ is discontinuous at 1, 2, 3, 4,.....

$\therefore [x]$ is discontinuous at 2, 4, 6, 8,.....

In between 0 to 5 it is discontinuous at 2 and 4.

Break the integration into 3 parts

(1) 0 to 2

(2) 2 to 4

(3) 4 to 5

$$\therefore \int_0^5 \cos \left(\pi \left(x - \left[\frac{x}{2} \right] \right) \right) dx$$

$$= \int_0^2 \cos(\pi(x-0)) dx + \int_2^4 \cos(\pi(x-1)) dx + \int_4^5 \cos(\pi(x-2)) dx$$

$$= \int_0^2 \cos \pi x dx + \int_2^4 \cos(\pi x - \pi) dx + \int_4^5 \cos(\pi x - 2\pi) dx$$

$$= \int_0^2 \cos \pi x dx - \int_2^4 \cos \pi x dx + \int_4^5 \cos \pi x dx$$

$$= \left[\frac{\sin \pi x}{\pi} \right]_0^2 - \left[\frac{\sin \pi x}{\pi} \right]_2^4 + \left[\frac{\sin \pi x}{\pi} \right]_4^5$$

$$= 0 - 0 + 0$$

$$= 0$$

Question 156

Let f be a real valued continuous function on $[0, 1]$ and $f(x) = x + \int_0^1 (x-t)f(t)dt$. Then, which of the following points (x, y) lies on the curve $y = f(x)$?
[29-Jun-2022-Shift-2]

Options:

- A. (2, 4)
- B. (1, 2)
- C. (4, 17)
- D. (6, 8)

Answer: D

Solution:

$$f(x) = \left(1 + \int_0^1 f(t)dt\right)x - \int_0^1 tf(t)dt$$

$$f(x) = Ax - B$$

$$A = 1 + \int_0^1 f(t)dt = 1 + \int_0^1 (At - B)dt$$

$$\Rightarrow A = 2(1 - B)$$

$$\text{Also } B = \int_0^1 tf(t)dt = \int_0^1 (At^2 - Bt)dt$$

$$A = \frac{9}{2}B$$

From (2), (3)

$$A = \frac{18}{13}, B = \frac{4}{13}$$

$$\text{so } f(6) = 8$$

Question 157

If $\int_0^2 (\sqrt{2x} - \sqrt{2x-x^2})dx = \int_0^1 \left(1 - \sqrt{1-y^2} - \frac{y^2}{2}\right)dy + \int_1^2 \left(2 - \frac{y^2}{2}\right)dy + I$, then I equals
[29-Jun-2022-Shift-2]

Options:

A. $\int_0^1 (1 + \sqrt{1-y^2})dy$

B. $\int_0^1 \left(\frac{y^2}{2} - \sqrt{1-y^2} + 1\right)dy$

C. $\int_0^1 (1 - \sqrt{1-y^2})dy$

D. $\int_0^1 \left(\frac{y^2}{2} + \sqrt{1-y^2} + 1\right)dy$

Answer: C

Solution:

$$\text{LHS} = \int_0^2 (\sqrt{2x} - \sqrt{2x-x^2}) dx = \frac{8}{3} - \frac{\pi}{2}$$

$$\text{RHS} = \int_0^1 \left(1 - \sqrt{1-y^2} - \frac{y^2}{2}\right) dy + \int_1^2 \left(2 - \frac{y^2}{2}\right) dy + I$$

$$I + \frac{5}{3} - \frac{\pi}{4}$$

$$\text{So, } I = 1 - \frac{\pi}{4} = \int_0^1 (1 - \sqrt{1-y^2}) dy$$

Question 158

Let $f(\theta) = \sin \theta + \int_{-\pi/2}^{\pi/2} (\sin \theta + t \cos \theta) f(t) dt$. Then the value of $\left| \int_0^{\pi/2} f(\theta) d\theta \right|$ is

[24-Jun-2022-Shift-1]

Answer: 1

Solution:

$$f(\theta) = \sin \theta \left(1 + \int_{-\pi/2}^{\pi/2} f(t) dt \right) + \cos \theta \left(\int_{-\pi/2}^{\pi/2} t f(t) dt \right)$$

$$\text{Clearly } f(\theta) = a \sin \theta + b \cos \theta$$

$$\text{Where } a = 1 + \int_{-\pi/2}^{\pi/2} (a \sin t + b \cos t) dt \Rightarrow a = 1 + 2b \dots (i) \text{ and } b = \int_{-\pi/2}^{\pi/2} (at \sin t + bt \cos t) dt \Rightarrow b = 2a \dots (ii) \text{ from (i) and (ii) we get}$$

$$a = -\frac{1}{3} \text{ and } b = -\frac{2}{3}$$

$$\text{So } f(\theta) = -\frac{1}{3}(\sin \theta + 2 \cos \theta)$$

$$\Rightarrow \left| \int_0^{\pi/2} f(\theta) d\theta \right| = \frac{1}{3}(1 + 2 \times 1) = 1$$

Question 159

$$\text{Let } \max_{0 \leq x \leq 2} \left\{ \frac{9-x^2}{5-x} \right\} = \alpha \text{ and } \min_{0 \leq x \leq 2} \left\{ \frac{9-x^2}{5-x} \right\} = \beta.$$

$$\text{If } \int_{\beta}^{2\alpha-1} \max \left\{ \frac{9-x^2}{5-x}, x \right\} dx = \alpha_1 + \alpha_2 \log_e \left(\frac{8}{15} \right) \text{ then } \alpha_1 + \alpha_2 \text{ is equal to}$$

[24-Jun-2022-Shift-1]

Answer: 34

Solution:

$$\text{Let } f(x) = \frac{x^2-9}{x-5} \Rightarrow f'(x) = \frac{(x-1)(x-9)}{(x-5)^2}$$

$$\text{So, } \alpha = f(1) = 2 \text{ and } \beta = \min(f'(0), f'(2)) = \frac{5}{3}$$

$$\begin{aligned} \text{Now, } \int_{-1}^3 \max \left\{ \frac{x^2-9}{x-5}, x \right\} dx &= \int_{-1}^{9/5} \frac{x^2-9}{x-5} dx + \int_{9/5}^3 x dx \\ &= \int_{-1}^{9/5} \left(x+5 + \frac{16}{x-5} \right) dx + \frac{x^2}{2} \Big|_{9/5}^3 \\ &= \frac{28}{25} + 14 + 16 \ln \left(\frac{8}{15} \right) + \frac{72}{25} = 18 + 16 \ln \left(\frac{8}{15} \right) \end{aligned}$$

$$\text{Clearly } \alpha_1 = 18 \text{ and } \alpha_2 = 16, \text{ so } \alpha_1 + \alpha_2 = 34.$$

Question 160

The integral $\int \frac{\left(1 - \frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1 + \frac{2}{\sqrt{3}} \sin 2x\right)} dx$ is equal to

[26-Jul-2022-Shift-2]

Options:

A. $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)} \right| + C$

B. $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{3}\right)} \right| + C$

C. $\log_e \left| \frac{\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)}{\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)} \right| + C$

D. $\frac{1}{2} \log_e \left| \frac{\tan\left(\frac{x}{2} - \frac{\pi}{12}\right)}{\tan\left(\frac{x}{2} - \frac{\pi}{6}\right)} \right| + C$

Answer: A

Solution:

$$\begin{aligned} &= \int \frac{\left(1 - \frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1 + \frac{2}{\sqrt{3}} \sin 2x\right)} dx \\ &= \int \frac{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \sqrt{2} \sin\left(\frac{\pi}{4} - x\right)}{\left(\frac{2}{\sqrt{3}}\right) \left(\sin \frac{\pi}{3} + \sin 2x\right)} dx \\ &= \int \frac{\frac{(\sqrt{3}-1)}{\sqrt{2}} \sin\left(\frac{\pi}{4} - x\right)}{\left(\sin \frac{\pi}{3} + \sin 2x\right)} dx \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{\frac{\sqrt{3}-1}{2\sqrt{2}} \sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
 &= \frac{1}{2} \int \frac{2 \sin \frac{\pi}{12} \sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
 &= \frac{1}{2} \int \frac{\cos\left(\frac{\pi}{6}-x\right) - \cos\left(\frac{\pi}{3}-x\right)}{\sin\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}-x\right)} dx \\
 &= \frac{1}{2} \left[\int \operatorname{cosec}\left(\frac{\pi}{6}+x\right) dx - \int \sec\left(\frac{\pi}{3}-x\right) dx \right] \\
 &= \frac{1}{2} \left[\ln \left| \tan\left(\frac{\pi}{12} + \frac{x}{2}\right) \right| - \int \operatorname{cosec}\left(\frac{\pi}{3}-x\right) dx \right] \\
 &= \frac{1}{2} \left[\ln \left| \tan\left(\frac{\pi}{12} + \frac{x}{2}\right) \right| - \ln \left| \frac{\pi}{6} + \frac{x}{2} \right| \right] + C \\
 &= \frac{1}{2} \ln \left| \frac{\tan\left(\frac{\pi}{12} + \frac{x}{2}\right)}{\tan\left(\frac{\pi}{6} + \frac{x}{2}\right)} \right| + C
 \end{aligned}$$

Question 161

For $I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dx$, if $I\left(\frac{\pi}{4}\right) = 2^{1011}$, then

[29-Jul-2022-Shift-2]

Options:

A. $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

B. $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

C. $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

D. $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Answer: A

Solution:

Given,

$$\begin{aligned}
 I(x) &= \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dx \\
 &= \int \frac{\sec^2 x}{\sin^{2022} x} dx - \int \frac{2022}{\sin^{2022} x} dx \\
 &= \int \frac{1}{\sin^{2022} x} \cdot \sec^2 x dx - \int \frac{2022}{\sin^{2022} x} dx \\
 &= \frac{1}{\sin^{2022} x} \cdot \tan x - \int \left(\frac{-2022}{\sin^{2023} x} \cdot \cos x \cdot \tan x \right) dx - \int \frac{2022}{\sin^{2022} x} dx + C \\
 &= \frac{\tan x}{\sin^{2022} x} + \int \left(\frac{2022}{\sin^{2023} x} \cdot \cos x \cdot \frac{\sin x}{\cos x} \right) dx - \int \frac{2022}{\sin^{2022} x} dx + C \\
 &= \frac{\tan x}{\sin^{2022} x} + \int \frac{2022}{\sin^{2022} x} dx - \int \frac{2022}{\sin^{2022} x} dx + C \\
 &= \frac{\tan x}{\sin^{2022} x} + C
 \end{aligned}$$

$$\text{Given, } I\left(\frac{\pi}{4}\right) = 2^{1011} \therefore I\left(\frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)^{2022}} + C$$

$$\Rightarrow 2^{1011} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^{2022}} + C$$

$$\Rightarrow C = 2^{1011} - 2^{1011} = 0$$

$$\therefore I(x) = \frac{\tan x}{\sin^{2022} x}$$

$$\therefore I\left(\frac{\pi}{3}\right) = \frac{\tan\frac{\pi}{3}}{\left(\sin\frac{\pi}{3}\right)^{2022}} = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}}$$

$$I\left(\frac{\pi}{6}\right) = \frac{\frac{1}{\sqrt{3}}}{\left(\frac{1}{2}\right)^{2022}} = \frac{1}{\sqrt{3}} \times (2)^{2022}$$

From option (A),

$$\begin{aligned} & 3^{1010} \cdot I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) \\ &= 3^{1010} \cdot \sqrt{3} \cdot \left(\frac{2}{\sqrt{3}}\right)^{2022} - \frac{(2)^{2022}}{\sqrt{3}} \\ &= 3^{1010} \cdot \sqrt{3} \times \frac{2^{2022}}{3^{1011}} - \frac{2^{2022}}{\sqrt{3}} \\ &= \frac{2^{2022}}{\sqrt{3}} - \frac{2^{2022}}{\sqrt{3}} = 0 \end{aligned}$$

Question 162

For any real number x , let $[x]$ denote the largest integer less than equal to x . Let f be a real

valued function defined on the interval $[-10, 10]$ by $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$.

Then the value of $\int_{-10}^{10} f(x) \cos \pi x \, dx$ is :
[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. 2
- C. 1
- D. 0

Answer: A

Solution:

Case 1:

Let $0 \leq x < 1$

then $[x] = 0$, which is even

$$\therefore f(x) = 1 + [x] - x$$

$$= 1 + 0 - x$$

$$= 1 - x$$

Case 2 :

Let $1 \leq x < 2$

then $[x] = 1$, which is odd

$$\therefore f(x) = x - [x]$$

$$= x - 1$$

Case 3 :

Let $2 \leq x < 3$ then $[x] = 2$, which is even

$$\begin{aligned}\therefore f(x) &= 1 + [x] - x \\ &= 1 + 2 - x \\ &= 3 - x\end{aligned}$$

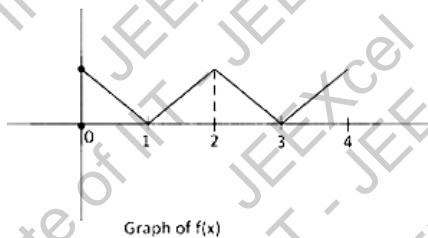
Case 4 :

Let $3 \leq x < 4$

then $[x] = 3$, which is odd

$$\begin{aligned}\therefore f(x) &= x - [x] \\ &= x - 3\end{aligned}$$

$$\therefore f(x) = \begin{cases} 1-x & 0 \leq x < 1 \\ x-1 & 1 \leq x < 2 \\ 3-x & 2 \leq x < 3 \\ x-3 & 3 \leq x < 4. \end{cases}$$



$\therefore f(x)$ is periodic and period of $f(x) = 2$

And period of $\cos \pi x = \frac{2\pi}{\pi} = 2$

$\therefore$ Period of $f(x) \cos \pi x = 2$

Now,

$$I = \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \int_{-10}^{10 \times 2} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \int_0^{10 \times 2} f(x) \cos \pi x \, dx$$

$$= \frac{\pi^2}{10} \times 10 \int_0^2 f(x) \cos \pi x \, dx$$

$$= \pi^2 \int_0^2 f(x) \cos \pi x \, dx$$

$$\therefore I = \pi^2 \left[\int_0^1 f(x) \cos \pi x \, dx + \int_1^2 f(x) \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\int_0^1 (1-x) \cos \pi x \, dx + \int_1^2 (x-1) \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\int_0^1 \cos \pi x \, dx - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \int_1^2 \cos \pi x \, dx \right]$$

$$= \pi^2 \left[\frac{1}{\pi} [\sin \pi x]_0^1 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - \frac{1}{\pi} [\sin \pi x]_1^2 \right]$$

$$= \pi^2 \left[0 - \int_0^1 x \cos \pi x \, dx + \int_1^2 x \cos \pi x \, dx - 0 \right]$$

$$= \pi^2 \left[- \left[x \cdot \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_0^1 + \left[x \cdot \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x \right]_1^2 \right] \text{ [As } \int x \cos \pi x \, dx = x \cdot \int \cos \pi x - \int \left(1 \cdot \frac{\sin \pi x}{\pi} \right) dx = x \cdot \frac{\sin \pi x}{\pi} + \frac{1}{\pi^2} \cos \pi x + c]$$

$$= \pi^2 \left[- \left[\left(1 \cdot \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cos \pi \right) - \left(0 + \frac{1}{\pi^2} \cos 0 \right) \right] + \left[\left(2 \cdot \frac{\sin 2\pi}{\pi} + \frac{1}{\pi^2} \cos 2\pi \right) - \left(1 \cdot \frac{\sin \pi}{\pi} + \frac{1}{\pi^2} \cos \pi \right) \right] \right]$$

$$= \pi^2 \left[- \left\{ \left(-\frac{1}{\pi^2} \right) - \left(\frac{1}{\pi^2} \right) \right\} + \left\{ \left(\frac{1}{\pi^2} \right) - \left(-\frac{1}{\pi^2} \right) \right\} \right]$$

$$= \pi^2 \left[- \left(-\frac{2}{\pi^2} \right) + \frac{2}{\pi^2} \right]$$

$$= \pi^2 \left[\frac{2}{\pi^2} + \frac{2}{\pi^2} \right]$$

$$= \pi^2 \times \frac{4}{\pi^2}$$

$$= 4$$

Question 163

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \left(\frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \dots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

is equal to
[25-Jul-2022-Shift-2]

Options:

- A. $\frac{1}{2}$
- B. 1
- C. 2
- D. -2

Answer: C

Solution:

$$I = \lim_{n \rightarrow \infty} \frac{1}{2^n} \left(\frac{1}{\sqrt{1 - \frac{1}{2^n}}} + \frac{1}{\sqrt{1 - \frac{2}{2^n}}} + \frac{1}{\sqrt{1 - \frac{3}{2^n}}} + \dots + \frac{1}{\sqrt{1 - \frac{2^n - 1}{2^n}}} \right)$$

Let $2^n = t$ and if $n \rightarrow \infty$ then $t \rightarrow \infty$

$$I = \lim_{n \rightarrow \infty} \frac{1}{t} \left(\sum_{r=1}^{t-1} \frac{1}{\sqrt{1 - \frac{r}{t}}} \right)$$

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x}} \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$= \left[2x^{\frac{1}{2}} \right]_0^1 = 2$$

Question 164

Let $|t|$ denote the greatest integer less than or equal to t . Then the value of the integral

$$\int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$$

is equal to
[25-Jul-2022-Shift-2]

Options:

- A. $\frac{52(1-e)}{e}$
- B. $\frac{52}{e}$
- C. $\frac{52(2+e)}{e}$
- D. $\frac{104}{e}$

Answer: B

Solution:

$$I = \int_{-3}^{101} ([\sin(\pi x)] + e^{[\cos(2\pi x)]}) dx$$

$[\sin \pi x]$ is periodic with period 2 and $e^{[\cos(2\pi x)]}$ is periodic with period 1.

So,

$$I = 52 \int_0^2 ([\sin \pi x] + e^{\cos 2\pi x}) dx$$
$$= 52 \left\{ \int_1^2 -1 dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1} dx + \int_{\frac{5}{4}}^{\frac{7}{4}} e^{-1} dx + \int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{3}{4}}^{\frac{5}{4}} e^0 dx + \int_{\frac{7}{4}}^2 e^0 dx \right\}$$
$$= \frac{52}{e}$$

Question 165

Let f be a twice differentiable function on C . If $f'(0) = 4$ and $f(x) + \int_0^x (x-t)f'(t)dt = (e^{2x} + e^{-2x}) \cos 2x + \frac{2}{a}x$, then $(2a+1)^5 a^2$ is equal to ____

[25-Jul-2022-Shift-2]

Answer: 8

Question 166

Let $a_n = \int_{-1}^n \left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} \right) dx$ for every $n \in \mathbb{N}$. Then the sum of all the elements of the set $\{n \in \mathbb{N} : a_n \in (2, 30)\}$ is ____

[25-Jul-2022-Shift-2]

Answer: 5

Solution:

$$\therefore a_n = \int_{-1}^n \left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} \right) dx$$
$$= \left[x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots + \frac{x^n}{n^2} \right]_{-1}^n$$
$$a_n = \frac{n+1}{1^2} + \frac{n^2-1}{2^2} + \frac{n^3-1}{3^2} + \frac{n^4-1}{4^2} + \dots + \frac{n^n+(-1)^{n+1}}{n^2}$$

$$\text{Here, } a_1 = 2, a_2 = \frac{2+1}{1} + \frac{2^2-1}{2} = 3 + \frac{3}{2} = \frac{9}{2}$$

$$a_3 = 4 + 2 + \frac{28}{9} = \frac{100}{9}$$

$$a_4 = 5 + \frac{15}{4} + \frac{65}{9} + \frac{255}{16} > 31$$

$\therefore$ The required set is $\{2, 3\}$. $\therefore a_n \in (2, 30)$

$\therefore$ Sum of elements = 5.

Question 167

If $a = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2n}{n^2 + k^2}$ and $f(x) = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$, $x \in (0, 1)$, then :

[26-Jul-2022-Shift-1]

Options:

A. $2\sqrt{2}f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

B. $f\left(\frac{a}{2}\right)f'\left(\frac{a}{2}\right) = \sqrt{2}$

C. $\sqrt{2}f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$

D. $f\left(\frac{a}{2}\right) = \sqrt{2}f'\left(\frac{a}{2}\right)$

Answer: C

Solution:

$$a = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2n}{n^2 + k^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{2}{1 + \left(\frac{k}{n}\right)^2}$$

$$a = \int_0^1 \frac{2}{1+x^2} dx = 2 \tan^{-1} x \Big|_0^1 = \frac{\pi}{2}$$

$$f(x) = \sqrt{\frac{1 - \cos x}{1 + \cos x}}, x \in (0, 1)$$

$$f(x) = \frac{1 - \cos x}{\sin x} = \operatorname{cosec} x - \cot x$$

$$f'(x) = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$$

$$\left. \begin{aligned} f\left(\frac{a}{2}\right) &= f\left(\frac{\pi}{4}\right) = \sqrt{2} - 1 \\ f'\left(\frac{a}{2}\right) &= f'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \end{aligned} \right\} f'\left(\frac{a}{2}\right) = \sqrt{2} \cdot f\left(\frac{a}{2}\right)$$

Question 168

If $n(2n+1) \int_0^1 (1-x^n)^{2n} dx = 1177 \int_0^1 (1-x^n)^{2n+1} dx$, then $n \in \mathbb{N}$ is equal to _____.

[26-Jul-2022-Shift-1]

Answer: 24

Solution:

$$\begin{aligned} \int_0^1 (1-x^n)^{2n+1} dx &= \int_0^1 1 \cdot (1-x^n)^{2n+1} dx \\ &= \left[(1-x^n)^{2n+1} \cdot x \right]_0^1 - \int_0^1 x \cdot (2n+1)(1-x^n)^{2n} \cdot nx^{n-1} dx \end{aligned}$$

$$\begin{aligned}
 &= n(2n+1) \int_0^1 (1 - (1-x^n))(1-x^n)^{2n} dx \\
 &= n(2n+1) \int_0^1 (1-x^n)^{2n} dx - n(2n+1) \int_0^1 (1-x^n)^{2n+1} dx \\
 &= (1+n(2n+1)) \int_0^1 (1-x^n)^{2n+1} dx = n(2n+1) \int_0^1 (1-x^n)^{2n} dx \\
 &= (2n^2+n+1) \int_0^1 (1-x^n)^{2n+1} dx = 1177 \int_0^1 (1-x^n)^{2n+1} dx \\
 &\therefore 2n^2+n+1 = 1177 \\
 &2n^2+n-1176 = 0 \\
 &\therefore n = 24 \text{ or } -\frac{49}{2} \\
 &\therefore n = 24
 \end{aligned}$$

Question 169

$\int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$ is equal to
[26-Jul-2022-Shift-2]

Options:

- A. $10(\pi+4)$
- B. $10(\pi+2)$
- C. $20(\pi-2)$
- D. $20(\pi+2)$

Answer: D

Solution:

$$\begin{aligned}
 I &= \int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx \\
 &= 20 \int_0^{\pi} (1 + |\sin 2x|) dx \\
 &= 40 \int_0^{\frac{\pi}{2}} (1 + \sin 2x) dx \\
 &= 40 \left(x - \frac{\cos 2x}{2} \right) \Big|_0^{\frac{\pi}{2}} \\
 &= 40 \left(\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2} \right) = 20(\pi+2)
 \end{aligned}$$

Question 170

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2-x]$, $a \in \mathbb{R}$ where $[t]$ is the greatest integer less than or equal to t . If

$\lim_{x \rightarrow -1} f(x)$ exists, then the value of $\int_0^4 f(x) dx$ is equal to

[27-Jul-2022-Shift-1]

Options:

- A. -1
- B. -2
- C. 1

D. 2

Answer: B

Solution:

$$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2-x]a \in \mathbb{R}$$

Now,

$$\therefore \lim_{x \rightarrow -1} f(x) \text{ exist}$$

$$\therefore \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$\Rightarrow a \sin\left(\frac{-2\pi}{2}\right) + 3 = a \sin\left(\frac{-\pi}{2}\right) + 2$$

$$\Rightarrow -a = 1 \Rightarrow a = -1$$

$$\text{Now, } \int_0^4 f(x) dx = \int_0^4 \left(-\sin\left(\frac{\pi[x]}{2}\right) + [2-x]\right) dx$$

$$= \int_0^1 1 dx + \int_1^2 -1 dx + \int_2^3 -1 dx + \int_3^4 (1-2) dx$$

$$= 1 - 1 - 1 - 1 = -2$$

Question 171

Let $f(x) = 2 + |x| - |x-1| + |x+1|$, $x \in \mathbb{R}$.

Consider

$$(S1): f'\left(-\frac{3}{2}\right) + f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) = 2$$

$$(S2): \int_{-2}^2 f(x) dx = 12$$

Then,

[27-Jul-2022-Shift-2]

Options:

A. both (S1) and (S2) are correct

B. both (S1) and (S2) are wrong

C. only (S1) is correct

D. only (S2) is correct

Answer: D

Solution:

$$f(x) = 2 + |x| - |x-1| + |x+1|, x \in \mathbb{R}$$

$$\therefore f(x) = \begin{cases} -x, & x < -1 \\ x+2, & -1 \leq x < 0 \\ 3x+2, & 0 \leq x < 1 \\ x+4, & x \geq 1. \end{cases}$$

$$\therefore f'\left(-\frac{3}{2}\right) + f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) = -1 + 1 + 3 + 1 = 4$$

$$\text{and } \int_{-2}^2 f(x) dx = \int_{-2}^{-1} f(x) dx + \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

$$= \left[-\frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{(x+2)^2}{2}\right]_{-1}^0 + \left[\frac{(3x+2)^2}{6}\right]_0^1 + \left[\frac{(x+4)^2}{2}\right]_1^2$$

$$= \frac{3}{2} + \frac{3}{2} + \frac{7}{2} + \frac{11}{2} = \frac{24}{2} = 12$$

$\therefore$ Only (S2) is correct

Question 172

$\int_0^2 \left(|2x^2 - 3x| + \left\lfloor x - \frac{1}{2} \right\rfloor \right) dx$, where $[t]$ is the greatest integer function, is equal to:
[27-Jul-2022-Shift-2]

Options:

A. $\frac{7}{6}$

B. $\frac{19}{12}$

C. $\frac{31}{12}$

D. $\frac{3}{2}$

Answer: B

Solution:

$$\begin{aligned} & \int_0^2 \left| 2x^2 - 3x \right| dx + \int_0^2 \left\lfloor x - \frac{1}{2} \right\rfloor dx \\ &= \int_0^{3/2} (3x - 2x^2) dx + \int_{3/2}^2 (2x^2 - 3x) dx + \int_0^{1/2} -1 dx + \int_{1/2}^{3/2} 0 dx + \int_{3/2}^2 1 dx \\ &= \left(\frac{3x^2}{2} - \frac{2x^3}{3} \right) \Big|_0^{3/2} + \left(\frac{2x^3}{3} - \frac{3x^2}{2} \right) \Big|_{3/2}^2 - \frac{1}{2} + \frac{1}{2} \\ &= \left(\frac{27}{8} - \frac{27}{12} \right) + \left(\frac{16}{3} - 6 - \frac{27}{12} + \frac{27}{8} \right) \\ &= \frac{19}{12} \end{aligned}$$

Question 173

Let $f(x) = \min\{[x-1], [x-2], \dots, [x-10]\}$ where $[t]$ denotes the greatest integer $\leq t$. Then

$\int_0^{10} f(x) dx + \int_0^{10} (f(x))^2 dx + \int_0^{10} |f(x)| dx$ is equal to _____.

[27-Jul-2022-Shift-2]

Answer: 385

Solution:

$$\because f(x) = \min\{[x-1], [x-2], \dots, [x-10]\} = [x-10]$$

$$\text{Also } |f(x)| = \begin{cases} -f(x), & \text{if } x \leq 10 \\ f(x), & \text{if } x \geq 10. \end{cases}$$

$$\begin{aligned} \therefore & \int_0^{10} f(x) dx + \int_0^{10} (f(x))^2 dx + \int_0^{10} (-f(x)) dx \\ &= \int_0^{10} (f(x))^2 dx \\ &= 10^2 + 9^2 + 8^2 + \dots + 1^2 \\ &= \frac{10 \times 11 \times 21}{6} = 385 \end{aligned}$$

Question 174

Let f be a differentiable function satisfying $f(x) = \frac{2}{\sqrt{3}} \int_0^{\sqrt{3}} f\left(\frac{\lambda^2 x}{3}\right) d\lambda$, $x > 0$ and $f(1) = \sqrt{3}$. If $y = f(x)$ passes through the point $(\alpha, 6)$, then α is equal to _____.

[27-Jul-2022-Shift-2]

Answer: 12

Solution:

$$\therefore f(x) = \frac{2}{\sqrt{3}} \int_0^{\sqrt{3}} f\left(\frac{\lambda^2 x}{3}\right) d\lambda, x > 0$$

On differentiating both sides w.r.t., x , we get

$$f'(x) = \frac{2}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{\lambda^2}{3} f'\left(\frac{\lambda^2 x}{3}\right) d\lambda$$

$$f'(x) = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \lambda \cdot \frac{2\lambda}{3} f'\left(\frac{\lambda^2 x}{3}\right) d\lambda$$

$$\therefore \sqrt{3} f'(x) = \left[\frac{\lambda}{x} \cdot f\left(\frac{\lambda^2 x}{3}\right) \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{1}{x} f\left(\frac{\lambda^2 x}{3}\right) dx$$

$$\sqrt{3} x f'(x) = \sqrt{3} f(x) - \frac{\sqrt{3}}{2} f(x)$$

$$x f'(x) = \frac{f(x)}{2}$$

On integrating we get : $\ln y = \frac{1}{2} \ln x + \ln c$ $\therefore f(1) = \sqrt{3}$ then $c = \sqrt{3}$

$\therefore (\alpha, 6)$ lies on

$$\therefore y = \sqrt{3x}$$

$$\therefore 6 = \sqrt{3\alpha} \Rightarrow \alpha = 12$$

Question 175

If $\int_0^{\sqrt{3}} \frac{15x^3}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} dx = \alpha\sqrt{2} + \beta\sqrt{3}$, where α, β are integers, then $\alpha + \beta$ is equal to _____.

[28-Jul-2022-Shift-1]

Answer: 10

Solution:

$$\text{Put } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15 \tan^3 \theta \cdot \sec^2 \theta d\theta}{\sqrt{1 + \tan^2 \theta} + \sqrt{\sec^6 \theta}}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15 \tan^2 \theta \sec^2 \theta d\theta}{\sec \theta \sqrt{1 + \sec \theta}}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{15(\sec^2 \theta - 1) \sec \theta \tan \theta d\theta}{(\sqrt{1 + \sec \theta})}$$

Now put $1 + \sec \theta = t^2$

$$\Rightarrow \sec \theta \tan \theta d\theta = 2t dt$$

$$\Rightarrow I = \int_{\frac{\sqrt{2}}{2}}^{\sqrt{3}} \frac{15((t^2 - 1)^2 - 1)2t dt}{t}$$

$$\Rightarrow I = 30 \int_{\frac{\sqrt{2}}{2}}^{\sqrt{3}} (t^4 - 2t^2 + 1 - 1) dt$$

$$\Rightarrow I = 30 \int_{\frac{\sqrt{2}}{2}}^{\sqrt{3}} (t^4 - 2t^2) dt$$

$$\Rightarrow I = 30 \int_{\frac{\sqrt{2}}{2}}^{\sqrt{3}} (t^4 - 2t^2) dt$$

$$\Rightarrow I = 30 \left(\frac{t^5}{5} - \frac{2t^3}{3} \right) \Big|_{\frac{\sqrt{2}}{2}}^{\sqrt{3}}$$

$$= 30 \left[\left(\frac{9\sqrt{3}}{5} - 2\sqrt{3} \right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} \right) \right]$$

$$= (54\sqrt{3} - 60\sqrt{3}) - (24\sqrt{2} - 40\sqrt{2})$$

$$= 16\sqrt{2} - 6\sqrt{3}$$

$$\therefore \alpha = 16 \text{ and } \beta = -6$$

$$\alpha + \beta = 10$$

Question 176

Let $I_n(x) = \int_0^x \frac{1}{(t^2 + 5)^n} dt$, $n = 1, 2, 3, \dots$. Then :

[28-Jul-2022-Shift-2]

Options:

A. $50I_6 - 9I_5 - xI_5'$

B. $50I_6 - 11I_5 - xI_5'$

C. $50I_6 - 9I_5 - I_5'$

D. $50I_6 - 11I_5 - I_5'$

Answer: A

Solution:

$$I_n(x) = \int_0^x \frac{1}{(t^2 + 5)^n} dt$$

$$= \int_0^x \frac{1}{(t^2 + 5)^n} \times \int_0^1 dt$$

$$= \frac{t}{(t^2 + 5)^n} \Big|_0^x - \int_0^x \frac{-2nt}{(t^2 + 5)^{n+1}} \times t dt$$

$$= \frac{x}{(x^2 + 5)^n} + \int_0^x 2n \left(\frac{t^2 + 5 - 5}{(t^2 + 5)^{n+1}} \right) dt$$

$$I_n(x) = \frac{x}{(x^2 + 5)^n} + 2nI_n(x) - 10nI_{n+1}(x)$$

$$10nI_{n+1}(x) - (2n - 1)I_n(x) = xI_n'(x)$$

For $n = 5$

$$50I_6(x) - 9I_5(x) = xI_5'(x)$$

Question 177

The value of the integral $\int_0^{\frac{\pi}{2}} 60 \frac{\sin(6x)}{\sin x} dx$ is equal to _____.

[28-Jul-2022-Shift-2]

Answer: 104

Solution:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} 60 \cdot \frac{\sin 6x}{\sin x} dx \\ &= 60 \cdot 2 \int_0^{\frac{\pi}{2}} (3 - 4 \sin^2 x)(4 \cos^2 x - 3) \cos x dx \\ &= 120 \int_0^{\frac{\pi}{2}} (3 - 4 \sin^2 x)(1 - 4 \sin^2 x) \cos x dx \\ \text{Let } \sin x &= t \Rightarrow \cos x dx = dt \\ &= 120 \int_0^1 (3 - 4t^2)(1 - 4t^2) dt \\ &= 120 \int_0^1 (3 - 16t^2 + 16t^4) dt \\ &= 120 \left[3t - \frac{16t^3}{3} + \frac{16t^5}{5} \right]_0^1 \\ &= 104 \end{aligned}$$

Question 178

The integral $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 2 \sin x + \cos x} dx$ is equal to:

[29-Jul-2022-Shift-1]

Options:

- A. $\tan^{-1}$
- B. $\tan^{-1}(2) - \frac{\pi}{4}$
- C. $\frac{1}{2} \tan^{-1}(2) - \frac{\pi}{8}$
- D. $\frac{1}{2}$

Answer: B

Solution:

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{1}{3 + 2 \sin x + \cos x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{(1 + \tan^2 x/2) dx}{3(1 + \tan^2 x/2) + 2(2 \tan x/2) + (1 - \tan^2 x/2)} \\ \text{Let } \tan x/2 &= t \Rightarrow \sec^2 x/2 dx = 2dt \\ I &= \int_0^1 \frac{2dt}{4 + 2t^2 + 4t} \end{aligned}$$

$$= \int_0^1 \frac{dt}{t^2 + 2t + 2} = \int_0^1 \frac{dt}{(t+1)^2 + 1}$$

$$= \tan^{-1}(t+1) \Big|_0^1 = \tan^{-1}2 - \frac{\pi}{4}$$

Question 179

If $f(\alpha) = \int_1^\alpha \frac{\log_{10} t}{1+t} dt$, $\alpha > 0$, then $f(e^3) + f(e^{-3})$ is equal to :

[29-Jul-2022-Shift-1]

Options:

- A. 9
- B. $\frac{9}{2}$
- C. $\frac{9}{\log_e(10)}$
- D. $\frac{9}{2\log_a(10)}$

Answer: D

Solution:

$$f(\alpha) = \int_1^\alpha \frac{\log_{10} t}{1+t} dt \dots \dots (i)$$

$$f\left(\frac{1}{\alpha}\right) = \int_1^{\frac{1}{\alpha}} \frac{\log_{10} t}{1+t} dt$$

Substituting $t \rightarrow \frac{1}{p}$

$$f\left(\frac{1}{\alpha}\right) = \int_1^\alpha \frac{\log_{10}\left(\frac{1}{p}\right)}{1+\frac{1}{p}} \left(-\frac{1}{p^2}\right) dp$$

$$= \int_1^\alpha \frac{\log_{10} p}{p(p+1)} dp = \int_1^\alpha \left(\frac{\log_{10} t}{t} - \frac{\log_{10} t}{t+1} \right) dt \dots \dots (ii)$$

By (i) + (ii)

$$f(\alpha) + f\left(\frac{1}{\alpha}\right) = \int_1^\alpha \frac{\log_{10} t}{t} dt = \int_1^\alpha \frac{\ln t}{t} \cdot \log_{10} e \, dt$$

$$= \frac{(\ln \alpha)^2}{2 \log_e 10}$$

$$\alpha = e^3 \Rightarrow f(e^3) + f(e^{-3}) = \frac{9}{2 \log_e 10}$$

Question 180

If $[t]$ denotes the greatest integer $\leq t$, then the value of $\int_0^1 [2x - |3x^2 - 5x + 2| + 1] dx$ is:

[29-Jul-2022-Shift-2]

Options:

- A. $\frac{\sqrt{37} + \sqrt{13} - 4}{6}$
- B. $\frac{\sqrt{37} - \sqrt{13} - 4}{6}$

C. $\frac{-\sqrt{37}-\sqrt{13}+4}{6}$

D. $\frac{-\sqrt{37}+\sqrt{13}+4}{6}$

Answer: A

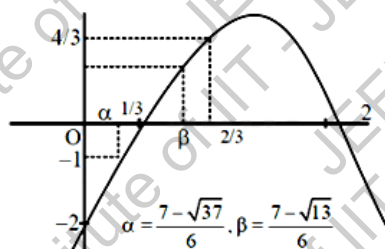
Solution:

$$I = \int_0^1 [2x - 13x^2 - 3x - 2x + 2] + 1] dt$$

$$I = \int_0^1 [2x - (3x - 2)(x - 1)] dt + \int_0^1 1 dt$$

$$I = \int_0^{2/3} [(2x - (3x^2 - 5x + 2))] dt + \int_{2/3}^1 (2x + (3x^2 - 5x + 2)) dt + 1$$

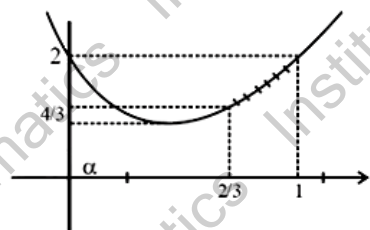
$$I = \int_0^{2/3} [-3x^2 + 7x - 2] dt + \int_{2/3}^1 (3x^2 - 3x + 2) dt + 1$$



$$\int_0^{\alpha} (-2) dt + \int_{\alpha}^{1/3} (-1) dt + \int_{1/3}^{\beta} 0 dt + \int_{\beta}^{2/3} 1 dt + \int_{2/3}^1 1 dt$$

$$= -2\alpha - \left(\frac{1}{3} - \alpha \right) + \frac{2}{3} - \beta = -\alpha - \beta + \frac{1}{3}$$

$$y = 3x^2 - 3x + 2$$



When $x \in \left(\frac{2}{3}, 1 \right)$

$$3x^2 - 3x + 2 \in \left(\frac{4}{3}, 2 \right)$$

$$[3x^2 - 3x + 2] = 1$$

$$\therefore \int_{2/3}^1 [3x^2 - 3x + 2] dt = 1 \left(1 - \frac{2}{3} \right) = \frac{1}{3}$$

$$\text{Hence } I = \left(\frac{1}{3} - (\alpha + \beta) \right) + \left(\frac{1}{3} \right) + 1$$

$$= \frac{5}{3} - \left(\frac{7 - \sqrt{37}}{6} + \frac{7 - \sqrt{13}}{6} \right)$$

$$= \frac{-2}{3} + \frac{\sqrt{37} + \sqrt{13}}{6}$$

$$= \frac{\sqrt{37} + \sqrt{13} - 4}{6}$$

Question 181

The integral $\int \frac{e^{3\log 2x} + 5e^{2\log 2x}}{e^{4\log x} + 5e^{3\log x} - 7e^{2\log x}} dx$, $x > 0$, is equal to (where, c is a constant of integration)

[25 Feb 2021 Shift 2]

Options:

A. $\log_e x^2 + 5x - 7 | + c$

B. $4\log_e x^2 + 5x - 7 | + c$

C. $\frac{1}{4}\log_e x^2 + 5x - 7 | + c$

D. $\log_e \sqrt{x^2 + 5x - 7} + c$

Answer: B

Solution:

$$I = \int \frac{e^{3\log_e(2x)} + 5e^{2\log_e(2x)}}{e^{4\log_e(x)} + 5e^{3\log_e(x)} - 7e^{2\log_e(x)}} dx$$

$$= \int \frac{e^{\log_e(2x)^3} + 5e^{\log_e(2x)^2}}{e^{\log_e x^4} + 5e^{\log_e x^3} - 7e^{\log_e x^2}} dx$$

[using property $a \log x = \log x^a$]

$$= \int \frac{8x^3 + 5(2x)^2}{x^4 + 5(x)^3 - 7x^2} dx \quad [\text{using property } a^{\log x} = x^a]$$

$$= \int \frac{8x^3 + 20x^2}{x^4 + 5x^3 - 7x^2} dx = \int \frac{4x^2(2x + 5)}{x^2(x^2 + 5x - 7)} dx$$

$$= \int \frac{4(2x + 5)}{x^2 + 5x - 7} dx$$

Let $x^2 + 5x - 7 = t$, then $(2x + 5)dx = dt$

$$I = \int \frac{4dt}{t} = 4\log_e t + c$$

Put $t = x^2 + 5x - 7$

$$I = 4\log_e |x^2 + 5x - 7| + c$$

Question 182

The value of the integral

$$\int \frac{\sin \theta \sin 2\theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta) \sqrt{2\sin^4 \theta + 3\sin^2 \theta + 6}}{1 - \cos 2\theta} d\theta$$

is (where, c is a constant of integration)

[25 Feb 2021 Shift 1]

Options:

A. $\frac{1}{18}[11 - 18\sin^2 \theta + 9\sin^4 \theta - 2\sin^6 \theta]^{\frac{3}{2}} + c$

B. $\frac{1}{18}[9 - 2\cos^6 \theta - 6\cos^4 \theta - 6\cos^2 \theta]^{\frac{3}{2}} + c$

C. $\frac{1}{18}[9 - 2\sin^6 \theta - 3\sin^4 \theta - 6\sin^2 \theta]^{\frac{3}{2}} + c$

D. $\frac{1}{18}[11 - 18\cos^2 \theta + 9\cos^4 \theta - 2\cos^6 \theta]^{\frac{3}{2}} + c$

Answer: D

Solution:

Let

$$\int \left[\frac{\sin \theta \cdot \sin 2\theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta) \sqrt{2\sin^4 \theta + 3\sin^2 \theta + 6}}{1 - \cos 2\theta} \right] d\theta$$

$$\therefore \sin 2A = 2 \sin A \cos A$$

$$\text{and } 1 - \cos 2A = 2\sin^2 A \sin \theta \cdot 2 \sin \theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta)$$

$$1 = \int \frac{\sqrt{2\sin^4 \theta + 3\sin^2 \theta + 6}}{2\sin^2 \theta} d\theta$$

$$I = \int \cos \theta (\sin^6 \theta + \sin^4 \theta + \sin^2 \theta)$$

$$\sqrt{2\sin^4 \theta + 3\sin^2 \theta + 6} d\theta$$

$$= \int (t^6 + t^4 + t^2) \sqrt{2t^4 + 3t^2 + 6} dt$$

$$= \int (t^5 + t^3 + t) \sqrt{2t^6 + 3t^4 + 6t^2} dt$$

$$\text{Let } 2t^6 + 3t^4 + 6t^2 = z$$

$$\therefore dz = (12t^5 + 12t^3 + 12t) dt$$

$$\therefore dz = 12(t^5 + t^3 + t) dt$$

$$\text{Now, } \frac{1}{12} \int \sqrt{z} dz = \frac{1}{12} \times \frac{z^{3/2}}{3/2} + c$$

$$= \frac{1}{18} z^{3/2} + c$$

$$= \frac{1}{18} [2t^6 + 3t^4 + 6t^2]^{3/2} + c$$

$$= \frac{1}{18} [2\sin^6 \theta + 3\sin^4 \theta + 6\sin^2 \theta]^{3/2} + c$$

$$= \frac{1}{18} [(1 - \cos^2 \theta) \{2(1 - \cos^2 \theta)^3 + 3 - 3\cos^2 \theta + 6\}]^{3/2} + c$$

$$= \frac{1}{18} [(1 - \cos^2 \theta)(2\cos^4 \theta - 7\cos^2 \theta + 11)]^{3/2} + c$$

$$= \frac{1}{18} [-2\cos^6 \theta + 9\cos^4 \theta - 18\cos^2 \theta + 11]^{3/2} + c$$

$$= \frac{1}{18} [11 - 18\cos^2 \theta + 9\cos^4 \theta - 2\cos^6 \theta]^{3/2}$$

Question 183

For $x > 0$, if $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is equal to

[26 Feb 2021 Shift 2]

Options:

A. 1

B. -1

C. $\frac{1}{2}$

D. 0

Answer: C

Solution:

$$f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$$

$$\text{Then, } f(e) = \int_1^e \frac{\log_e t}{1+t} dt \dots (i)$$

$$\text{and } f\left(\frac{1}{e}\right) = \int_1^{\frac{1}{e}} \frac{\log_e t}{1+t} dt \dots (ii)$$

$$\text{Let } t = \frac{1}{u}, dt = \frac{-1}{u^2} du \text{ and put in Eq. (ii), we get}$$

$$f\left(\frac{1}{e}\right) = \int_1^e \frac{\log\left(\frac{1}{u}\right)}{1+\frac{1}{u}} \cdot \frac{-1}{u^2} du = \int_1^e \frac{\log u}{u(u+1)} du$$

Using change of variable

$$f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{t(t+1)} dt \dots (iii)$$

From Eqs. (i) and (iii), we get

$$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{\log t}{t(1+t)} dt = \int_1^e \frac{e^{\log t}}{t} \frac{gt}{t}$$

Take $\log t = v$, then $\frac{1}{t} dt = dv$

$$f(e) + f\left(\frac{1}{e}\right) = \int_0^1 v dv = \left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\therefore f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$$

Question 184

If $I_{m \cdot n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$, for $m, n \geq 1$ and $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \alpha I_{m \cdot n}$, $\alpha \in \mathbb{R}$, then α equals

[26 Feb 2021 Shift 2]

Answer: 1

Solution:

$$\text{Given, } I_{mn} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Using substitution put } x = \frac{1}{t+1}$$

$$\text{Then, } dx = \frac{-1}{(t+1)^2} dt$$

$$I_{mn} = \int_{\infty}^0 (-1) \frac{1}{(t+1)^{m-1}} \cdot \frac{t^{n-1}}{(t+1)^{n-1}} \cdot \frac{1}{(t+1)^2} dt$$

$$= - \int_{\infty}^0 \frac{t^{n-1}}{(t+1)^{m+n}} dt \dots (i)$$

Similarly,

$$I_{mn} = \int_0^1 x^{n-1} (1-x)^{m-1} dx \dots (ii)$$

$$\Rightarrow I_{mn} = \int_0^{\infty} \frac{t^{m-1}}{(t+1)^{m+n}} dt$$

From Eqs. (i) and (ii), we get

$$2I_{mn} = \int_0^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$2I_{mn} = \int_0^1 \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt + \int_1^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt \text{ Let } I_1 = \int_1^{\infty} \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$\text{Let } t = \frac{1}{z}, \text{ then } dt = \frac{-1}{z^2} dz$$

$$I_1 = \int_1^0 (-1) \frac{\left(\frac{1}{z}\right)^{n-1} + \left(\frac{1}{z}\right)^{m-1}}{\left(\frac{1}{z} + 1\right)^{m+n}} \cdot \frac{1}{z^2} dz = - \int_1^0 \frac{z^{n-1} + z^{m-1}}{(z+1)^{m+n}} dz$$

$$2I_{mn} = \int_0^1 \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$- \int_1^0 \frac{z^{n-1} + z^{m-1}}{(z+1)^{m+n}} dz$$

$$= 2 \int_0^1 \frac{t^{n-1} + t^{m-1}}{(t+1)^{m+n}} dt$$

$$\Rightarrow \alpha = 1$$

Question 185

The value of $\sum_{n=1}^{100} \int_n^{n+1} e^{x-[x]} dx$, where $[x]$ is the greatest integer $\leq x$, is
[26 Feb 2021 Shift 1]

Options:

- A. $100(e-1)$
- B. $100(1-e)$
- C. $100e$
- D. $100(1+e)$

Answer: A

Solution:

Let 'x' be any real number, then $x = [x] + \{x\}$, where $[x]$ is integer part of x and $\{x\}$ is fractional part of x. Then, $x - [x] = \{x\}$. Also period of $\{x\} = 1$

Now, $\sum_{n=1}^{100} \int_n^{n+1} e^{x-[x]} dx = \sum_{n=1}^{100} \int_n^{n+1} e^{\{x\}} dx$

[Difference between upper and lower limit is 1 unit]

$$= \int_0^1 e^{\{x\}} dx + \int_1^2 e^{\{x\}} dx + \dots + \int_{99}^{100} e^{\{x\}} dx$$

$$= e^x \Big|_0^1 + e^{(x-1)} \Big|_1^2 + \dots + e^{(x-99)} \Big|_{99}^{100}$$

$$= (e-1) + (e-1) + \dots + (e-1) = 100(e-1)$$

Question 186

The value of $\int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx$ is
[26 Feb 2021 Shift 1]

Options:

- A. $\frac{\pi}{4}$
- B. 4π
- C. $\frac{\pi}{2}$
- D. 2π

Answer: A

Solution:

$$\text{Let } I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx \dots (i)$$

Using the property, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2(\pi/2 - \pi/2 - x)}{1+3^{\pi/2 - \pi/2 - x}} dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^{-x}} dx \quad [\because \cos(-x) = \cos x]$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^x} dx + \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx$$

$$\begin{aligned}
 &= \int_{-\pi/2}^{\pi/2} \frac{(1+3^x)\cos^2 x}{1+3^x} dx = \int_{-\pi/2}^{\pi/2} \cos^2 x dx \\
 &= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2x}{2} dx \\
 &= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2} [\pi] \\
 \Rightarrow 2I &= \pi/2 \Rightarrow I = \frac{\pi}{4}
 \end{aligned}$$

Question 187

The value of the integral $\int_0^{\pi} |\sin 2x| dx$ is

[26 Feb 2021 Shift 1]

Answer: 2

Solution:

$$\begin{aligned}
 \text{Let } I &= \int_0^{\pi} |\sin 2x| dx \\
 &= 2 \int_0^{\pi/2} |\sin 2x| dx \quad [\because \sin 2x \text{ is periodic function}] \\
 &= 2 \int_0^{\pi/2} \sin 2x dx \quad [\sin 2x \text{ is positive in range } (0, \pi/2)] \\
 &= 2 \left[-\frac{\cos 2x}{2} \right]_0^{\pi/2} \\
 &= -[\cos \pi - \cos 0] = -(-1 - 1) = 2 \\
 I &= 2
 \end{aligned}$$

Question 188

The value of $\int_{-2}^2 |3x^2 - 3x - 6| dx$ is

[25 Feb 2021 Shift 2]

Answer: 19

Solution:

$$\begin{aligned}
 \int_{-2}^2 |3x^2 - 3x - 6| dx &= I \quad (\text{say}) \\
 I &= 3 \int_{-2}^2 |x^2 - x - 2| dx \\
 &= 3 \left[\int_{-2}^{-1} (x^2 - x - 2) dx + \int_{-1}^2 (-x^2 + x + 2) dx \right] \\
 &= 3 \left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^{-1}_{-2} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right)^2_1 \right] \\
 &= 19
 \end{aligned}$$

Question 189

If $I_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx$, then

[25 Feb 2021 Shift 2]

Options:

A. $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in AP

B. $I_2+I_4, I_3+I_5, I_4+I_6$ are in AP

C. $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in GP

D. $I_2+I_4, (I_3+I_5)^2, I_4+I_6$ are in GP

Answer: A

Solution:

$$I_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx = \int_{\pi/4}^{\pi/2} \cot^{n-2} x (\cot^2 x) \, dx$$

$$I_n = \int_{\pi/4}^{\pi/2} \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int_{\pi/4}^{\pi/2} \cot^{n-2} x \, dx$$

$$I_n + I_{n-2} = \int_{\pi/4}^{\pi/2} \cot^{n-2} x \cdot \operatorname{cosec}^2 x \, dx$$

Now, let $\cot x = t$, then $\operatorname{cosec}^2 x \, dx = -dt$, limit will be

$$\begin{aligned} I_n + I_{n-2} &= \int_1^0 -t^{n-2} \, dt \\ &= \left[\frac{-(t)^{n-1}}{n-1} \right]_1^0 = - \left\{ \frac{0}{n-1} - \frac{(1)^{n-1}}{n-1} \right\} \end{aligned}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

Now, put $n=4$

$$\Rightarrow I_2 + I_4 = \frac{1}{3}, \text{ then } \frac{1}{I_2 + I_4} = 3 \dots (i)$$

Put $n=5$

$$\Rightarrow I_3 + I_5 = \frac{1}{4}, \text{ then } \frac{1}{I_3 + I_5} = 4 \dots (ii)$$

Put $n=6$

$$\Rightarrow I_4 + I_6 = \frac{1}{5}, \text{ then } \frac{1}{I_4 + I_6} = 5 \dots (iii)$$

Here, from Eqs. (i), (ii) and (iii), we conclude

$$\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5} \text{ and } \frac{1}{I_4 + I_6} \text{ are in AP with common difference 1.}$$

Question 190

The value of $\int_{-1}^1 x^2 e^{[x^3]} \, dx$, where $[t]$ denotes the greatest integer $\leq t$, is

[25 Feb 2021 Shift 1]

Options:

A. $\frac{e-1}{3e}$

B. $\frac{e+1}{3}$

C. $\frac{e+1}{3e}$

D. $\frac{1}{3e}$

Answer: C

Solution:

Given, $\int_{-1}^1 x^2 e^{[x^3]} dx$, where $[t]$ is greatest integer function.

$\therefore [x^3] = 0 \forall x \in (0, 1)$

and $[x^3] = -1 \forall x \in (-1, 0)$

So, $\int_{-1}^1 x^2 e^{[x^3]} dx = \int_{-1}^0 x^2 e^{-1} dx + \int_0^1 x^2 e^0 dx$

$= \frac{1}{e} \int_{-1}^0 x^2 dx + \int_0^1 x^2 dx = \frac{1}{e} \times \left[\frac{x^3}{3} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^1$

$= \frac{1}{e} \times \left[0 + \frac{1}{3} \right] + \left[\frac{1}{3} \right] = \frac{1}{3e} + \frac{1}{3} = \left(\frac{1+e}{3e} \right)$

Question 191

Let $f(x)$ be a differentiable function defined on $[0, 2]$, such that $f'(x) = f'(2-x)$, for all $x \in (0, 2)$, $f(0) = 1$ and $f(2) = e^2$. Then, the value of $\int_0^2 f(x) dx$ is
[24 Feb 2021 Shift 2]

Options:

A. $1 - e^2$

B. $1 + e^2$

C. $2(1 - e^2)$

D. $2(1 + e^2)$

Answer: B

Solution:

Given, $f(0) = 1 \dots (i)$

$f(2) = e^2 \dots (ii)$

$f'(x) = f'(2-x)$

Integrating w.r.t. x ,

$f(x) = -f(2-x) + C$

Put $x = 0$

$f(0) = -f(2) + C$

$\Rightarrow 1 = -e^2 + C$ [from Eqs. (i) and (ii)]

$\Rightarrow C = 1 + e^2$

$\therefore f(x) = -f(2-x) + 1 + e^2$

or $f(x) + f(2-x) = 1 + e^2 \dots (iii)$

Let $I = \int_0^2 f(x) dx \dots (iv)$

Also, $I = \int_0^2 f(2-x) dx \dots (v)$

Now, adding Eqs. (iv) and (v),

$2I = \int_0^2 [f(x) + f(2-x)] dx$

$$2I = \int_0^2 (1 + e^x) dx \text{ [from Eq. (iii)]}$$

$$2I = 2(1 + e^2)$$

$$\therefore I = (1 + e^2)$$

Question 192

The value of the integral $\int_1^3 [x^2 - 2x - 2] dx$, where $[x]$ denotes the greatest integer less than or equal to x , is
[24 Feb 2021 Shift 2]

Options:

A. $-\sqrt{2} - \sqrt{3} + 1$

B. $-\sqrt{2} - \sqrt{3} - 1$

C. -5

D. -4

Answer: B

Solution:

$$\text{Let } I = \int_1^3 [x^2 - 2x - 2] dx$$

$$= \int_1^3 [x^2 - 2x + 1 - 3] dx = \int_1^3 (x - 1)^2 - 3 \, dx$$

$$= \int_1^3 [(x - 1)^2] dx + \int_1^3 -3 dx$$

$$\text{Put } x - 1 = t; dx = dt, \text{ when } x = 1, t = 0 \text{ and } x = 3, t = 2$$

$$\therefore I = -3[x]_1^3 + \int_0^2 [t^2] dt$$

$$= -6 + \int_0^1 0 dt + \int_1^{\sqrt{2}} 1 dt + \int_{\sqrt{2}}^{\sqrt{3}} 2 dt + \int_{\sqrt{3}}^2 3 dt$$

$$= -6 + (0) + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= -6 + \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$$

$$I = -1 - \sqrt{2} + \sqrt{3}$$

Question 193

$$\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3} \text{ is equal to :}$$

24 Feb 2021 Shift 1

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. 0

D. $\frac{1}{15}$

Answer: A

Solution:

$$\lim_{x \rightarrow 0^+} \frac{\int_0^{x^2} \sin \sqrt{t} \, dt}{x^3} = \lim_{x \rightarrow 0^+} \frac{(\sin x)2x}{3x^2}$$
$$= \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right) \times \frac{2}{3} = \frac{2}{3}$$

Question 194

If $\int_{-a}^a (|x| + |x - 2|) dx = 22$, ($a > 2$) and $[x]$ denotes the greatest integer $\leq x$, then $\int_a^{-a} (x + [x]) dx$ is equal to

24 Feb 2021 Shift 1

Answer: 3

Solution:

$$\int_{-a}^0 (-2x + 2) dx + \int_0^2 (x + 2 - x) dx + \int_2^a (2x - 2) dx = 22$$
$$\Rightarrow x^2 - 2x \Big|_0^{-a} + 2x \Big|_0^2 + x^2 - 2x \Big|_2^a = 22$$
$$\Rightarrow a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$$
$$\Rightarrow 2a^2 = 18 \Rightarrow a = 3$$
$$\therefore \int_3^{-3} (x + [x]) dx = -(-3 - 2 - 1 + 1 + 2) = 3$$

Question 195

The integral $\int \frac{(2x-1) \cos \sqrt{(2x-1)^2 + 5}}{\sqrt{4x^2 - 4x + 6}} dx$ is equal to (where, c is a constant of integration)

[18 Mar 2021 Shift 1]

Options:

- A. $\frac{1}{2} \sin \sqrt{(2x-1)^2 + 5} + c$
- B. $\frac{1}{2} \cos \sqrt{(2x+1)^2 + 5} + c$
- C. $\frac{1}{2} \cos \sqrt{(2x-1)^2 + 5} + c$
- D. $\frac{1}{2} \sin \sqrt{(2x+1)^2 + 5} + c$

Answer: A

Solution:

$$\text{Let } I = \int \frac{(2x-1) \cos \sqrt{(2x-1)^2+5}}{\sqrt{4x^2-4x+6}} dx$$

$$= \int \frac{(2x-1) \cos \sqrt{(2x-1)^2+5}}{\sqrt{(2x-1)^2+5}} dx$$

$$\text{Putting } (2x-1)^2+5 = z^2$$

$$\Rightarrow 2(2x-1) \times 2 \cdot dx = 2z dz$$

$$\Rightarrow (2x-1)dx = \frac{1}{2}z dz$$

$$\therefore I = \int \frac{\cos z}{z} \cdot \frac{1}{2}z \cdot dz = \frac{1}{2} \int \cos z dz = \frac{1}{2} \sin z + C$$

$$= \frac{1}{2} \sin \sqrt{(2x-1)^2+5} + C$$

[Note You can also substitute $\sqrt{(2x-1)^2+5} = z$ and then proceed.]

Question 196

If $f(x) = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2} dx$, ($x \geq 0$), $f(0) = 0$ and $f(1) = \frac{1}{K}$, then the value of K is
[18 Mar 2021 Shift 1]

Answer: 4

Solution:

$$\text{Let } I = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2} dx$$

$$= \int \frac{5x^8+7x^6}{x^{14}(x^{-5}+x^{-7}+2)^2} dx$$

$$= \int \frac{\frac{5x^8+7x^6}{x^{14}}}{(x^{-5}+x^{-7}+2)^2} dx$$

$$\Rightarrow I = \int \frac{5x^{-6}+7x^{-8}}{(x^{-5}+x^{-7}+2)^2} dx$$

$$\text{Putting } x^{-5}+x^{-7}+2 = z$$

$$\Rightarrow -(5x^{-6}+7x^{-8})dx = dz$$

$$\therefore I = - \int \frac{dz}{z^2} = - \left(\frac{1}{-z} \right) + C$$

$$\Rightarrow \Rightarrow I = \frac{1}{x^{-5}+x^{-7}+2} + C$$

$$\Rightarrow f(x) = \frac{x^7}{x^2+1+2x^7} + C$$

$$\text{Given, } f(0) = 0$$

$$\Rightarrow C = 0$$

$$\therefore f(x) = \frac{x^7}{x^2+1+2x^7}$$

$$\therefore f(1) = \frac{1}{1+1+2}$$

$$= \frac{1}{4} = \frac{1}{K}$$

$$\text{Hence, } K = 4.$$

Question 197

For real numbers α, β, γ and δ , if $\int \frac{(x^2-1) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx$

$$= \alpha \log_e \left[\tan^{-1}\left(\frac{x^2+1}{x}\right) \right] + \beta \tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where C is an arbitrary constant, then the value of $10(\alpha + \beta\gamma + \delta)$ is equal to.....
[16 Mar 2021 Shift 2]

Answer: 6

Solution:

$$\text{Let } I = \int \frac{(x^2-1) + \tan^{-1}\left(\frac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx$$

$$\Rightarrow I = \int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx + \int \frac{1}{x^4+3x^2+1} dx$$

$$\text{Again let } I_1 = \int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx$$

$$\text{and } I_2 = \int \frac{dx}{x^4+3x^2+1}$$

$$\therefore I = I_1 + I_2 \dots (i)$$

$$\text{Now, } I_1 = \int \frac{(x^2-1)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx$$

$$\text{Let } \tan^{-1}\left(\frac{x^2+1}{x}\right) = t$$

$$\Rightarrow \frac{x^2-1}{(x^4+3x^2+1)} dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t} = \log|t| + C_1 = \log\left|\tan^{-1}\left(\frac{x^2+1}{x}\right)\right| + C_1$$

$$I_2 = \int \frac{1}{x^4+3x^2+1} dx = \frac{1}{2} \int \frac{(x^2+1) - (x^2-1)}{x^4+3x^2+1} dx$$

$$= \frac{1}{2} \int \frac{x^2+1}{x^4+3x^2+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+3x^2+1} dx$$

$$= \frac{1}{2} \int \frac{1+1/x^2}{x^2+3+\frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1-1/x^2}{x^2+3+\frac{1}{x^2}} dx$$

$$I_2 = \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+5} dx - \frac{1}{2} \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2+1} dx$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2+1}{x}\right) + C_2$$

$$I = \log\left|\tan^{-1}\left(\frac{x^2+1}{x}\right)\right| + \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$$= \alpha \log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right) + \beta \tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2+1}{x}\right) + C \text{ (given)}$$

$$\therefore \alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \gamma = \frac{1}{\sqrt{5}} \text{ and } \delta = -\frac{1}{2}$$

$$\therefore \text{Required value of } 10(\alpha + \beta\gamma + \delta)$$

$$= 10\left(1 + \frac{1}{10} - \frac{1}{2}\right)$$

$$= 10 \left(\frac{10+1-5}{10} \right) \\ = 6$$

Question 198

Let $f(x)$ and $g(x)$ be two functions satisfying $f(x^2) + g(4-x) = 4x^3$ and $g(4-x) + g(x) = 0$ then the value of $\int_{-4}^4 (x)^2 dx$ is

[18 Mar 2021 Shift 1]

Answer: 512

Solution:

Given, $f(x^2) + g(4-x) = 4x^3$
and $g(4-x) + g(x) = 0$

$$\text{Let } I = \int_{-4}^4 f(x^2) dx$$

$$= 2 \int_0^4 f(x^2) dx$$

$$\Rightarrow I = 2 \cdot \int_0^4 [4x^3 - g(4-x)] dx$$

$$= 8 \int_0^4 x^3 dx - 2 \int_0^4 g(4-x) dx$$

$$= 8 \left(\frac{x^4}{4} \right)_0^4 - 2I_1 = 2(4^4 - 0^4) - 2I_1$$

$$= 2^9 - 2I_1$$

$$\text{where, } I_1 = \int_0^4 g(4-x) dx$$

$$\text{Now, } I_1 = \int_0^4 g(4-x) dx \dots (i)$$

$$\Rightarrow I_1 = \int_0^4 g \left[4 - (0 + 4 - x) \right] dx \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\Rightarrow I_1 = \int_0^4 g(x) dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$2I_1 = \int_0^4 [g(x) + g(4-x)] dx$$

$$\Rightarrow 2I_1 = 0$$

$$\Rightarrow I_1 = 0 \quad (\because g(x) + g(4-x) = 0, \text{ given})$$

$$\therefore I = 2^9 - 2I_1$$

$$\Rightarrow I = 2^9 = 512$$

Question 199

Let $P(x)$ be a real polynomial of degree 3 which vanishes at $x = -3$. Let $P(x)$ have local minima at $x = 1$, local maxima at $x = -1$ and $\int_{-1}^1 P(x) dx = 18$, then the sum of all the coefficients of the polynomial $P(x)$ is equal to

[18 Mar 2021 Shift 2]

Answer: 8

Solution:

$$\text{Let } P'(x) = a(x-1)(x+1)$$

$$\Rightarrow P'(x) = a(x^2 - 1)$$

$$\therefore P(x) = a \int (x^2 - 1) dx \Rightarrow P(x) = a \left(\frac{x^3}{3} - x \right) + C$$

According to the question, $P(-3) = 0$

$$a \left(-\frac{27}{3} + 3 \right) + C = 0$$

$$\Rightarrow -6a + C = 0 \dots (i)$$

$$\text{Now, } \int_{-1}^1 \left(a \left(\frac{x^3}{3} - x \right) + C \right) dx = 18 \quad (\text{given})$$

$$\Rightarrow 2C = 18$$

$$\Rightarrow C = 9 \dots (ii)$$

From Eqs. (i) and (ii),

$$-6a + 9 = 0 \Rightarrow a = \frac{3}{2}$$

$$\therefore P(x) = \frac{3}{2} \left(\frac{x^3}{3} - x \right) + 9$$

$$\therefore \text{Sum of the all coefficient} = \frac{1}{2} - \frac{3}{2} + 9 = 8$$

Question 200

Which of the following statements is correct for the function $g(\alpha)$ for $\alpha \in \mathbb{R}$, such that

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

[17 Mar 2021 Shift 1]

Options:

A. $g(\alpha)$ is a strictly increasing function

B. $g(\alpha)$ has an inflection point at $\alpha = -\frac{1}{2}$

C. $g(\alpha)$ is a strictly decreasing function

D. $g(\alpha)$ is an even function

Answer: D

Solution:

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx \dots (i)$$

$$\text{Applying } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha(\pi/2 - x)}{\cos^\alpha\left(\frac{\pi}{2} - x\right) + \sin^\alpha\left(\frac{\pi}{2} - x\right)} dx$$

$$g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx \dots (ii)$$

Adding Eqs. (i) and (ii),

$$2g(\alpha) = \int_{\pi/6}^{\pi/3} \frac{\sin^\alpha x + \cos^\alpha x}{\sin^\alpha x + \cos^\alpha x} dx$$

$$2g(\alpha) = \int_{\pi/6}^{\pi/3} 1 \cdot dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore g(\alpha) = \frac{\pi}{12}$$

$g(\alpha)$ is constant function.

$\therefore$ It is even function.

Question 201

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = e^{-x} \sin x$. If $F : [0, 1] \rightarrow \mathbb{R}$ is a differentiable function, such that $F(x) = \int_0^x f(t) dt$, then the value of $\int_0^1 [F'(x) + f(x)] e^x dx$ lies in the interval [17 Mar 2021 Shift 2]

Options:

A. $\left[\frac{327}{360}, \frac{329}{360} \right]$

B. $\left[\frac{330}{360}, \frac{331}{360} \right]$

C. $\left[\frac{331}{360}, \frac{334}{360} \right]$

D. $\left[\frac{335}{360}, \frac{336}{360} \right]$

Answer: B

Solution:

Given, $f(x) = e^{-x} \cdot \sin x$

and $F(x) = \int_0^x f(t) dt$

$\therefore F(x)$ is differentiable function.

$\therefore F'(x) = f(x) \times 1 - f(0) \times 0$ (using Newton-Leibnitz rule)

$\Rightarrow F'(x) = f(x) \dots (i)$

Let $I = \int_0^1 [F'(x) + f(x)] e^x dx$

$$= \int_0^1 [f(x) + f(x)] e^x dx = \int_0^1 2 \cdot f(x) \cdot e^x dx \text{ [from Eq. (i)]}$$

$$\Rightarrow I = 2 \cdot \int_0^1 f(x) \cdot e^x dx$$

$$= 2 \cdot \int_0^1 e^{-x} \sin x \cdot e^x dx$$

$$= 2 \int_0^1 \sin x dx = 2[-\cos x]_0^1$$

$$= 2[-(\cos 1 - \cos 0)] = 2(1 - \cos 1)$$

$$\Rightarrow I = 2 \cdot \left[1 - \left(1 - \frac{(1)^2}{2!} + \frac{(1)^4}{4!} - \frac{(1)^6}{6!} + \frac{(1)^8}{8!} - \dots \right) \right]$$

[using expansion of $\cos x$ i.e.,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$I = 2 \left[1 - 1 + \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \dots \right]$$

$$\Rightarrow I = 2 \left(\frac{1}{2} - \frac{1}{24} + \frac{1}{720} - \dots \right)$$

$$\text{Now, } I < 2 \left(\frac{1}{2} - \frac{1}{24} + \frac{1}{720} \right)$$

$$\Rightarrow I < \left(1 - \frac{1}{12} + \frac{1}{360} \right) \Rightarrow I < \frac{360 - 30 + 1}{360}$$

$$\Rightarrow I < \frac{331}{360} \dots (ii)$$

$$\text{Also, } I > 2 \left(\frac{1}{2} - \frac{1}{24} \right) \Rightarrow I > \left(1 - \frac{1}{12} \right)$$

$$\Rightarrow I > \frac{11}{12} \Rightarrow I > \frac{11 \times 30}{12 \times 30}$$

$$\Rightarrow I > \frac{330}{360} \dots (iii)$$

From Eqs. (ii) and (iii), we get

$$\frac{330}{360} < I < \frac{331}{360}$$

Question 202

If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and $[x]$ denotes the greatest integer less than or equal to x , then the value of $\alpha + \beta + \gamma$ is equal to
[17 Mar 2021 Shift 2]

Options:

- A. 0
- B. 20
- C. 25
- D. 10

Answer: A

Solution:

$$\text{Let } I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx$$

$$\Rightarrow I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{[x]}} dx \quad [\because x - [x] = \{x\}]$$

From above integrand, we observe that $\frac{[\sin 2\pi x]}{e^{[x]}}$ is a periodic function with period '1'.

$$\therefore I = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{[x]}} dx$$

[by the property of definite integral,

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx$$

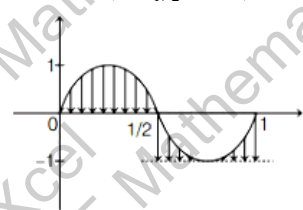
where $f(x)$ is a periodic function with period = T]

$$\Rightarrow I = 10 \cdot \int_0^1 \frac{[\sin 2\pi x]}{e^x} dx \quad [\because \{x\} = x, 0 \leq x < 1]$$

$$\Rightarrow I = 10 \left(\int_0^{1/2} \frac{[\sin 2\pi x]}{e^x} dx + \int_{1/2}^1 \frac{[\sin 2\pi x]}{e^x} dx \right)$$

$$\Rightarrow I = 10 \left(\int_0^{1/2} \frac{0}{e^x} dx + \int_{1/2}^1 \frac{(-1)}{e^x} dx \right)$$

$$\Rightarrow I = 10 \left(0 - \int_{1/2}^1 e^{-x} dx \right) \Rightarrow I = -10 \left[\frac{e^{-x}}{-1} \right]_{1/2}^1$$



$$\Rightarrow I = 10[e^{-1} - e^{-1/2}] \Rightarrow I = 10e^{-1} - 10e^{-\frac{1}{2}}$$

$$\Rightarrow I = 10e^{-1} + (-10) \cdot e^{-\frac{1}{2}} + 0 \dots (i)$$

Comparing Eq. (i) by $\alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, we get

$$\alpha = 10, \beta = -10 \text{ and } \gamma = 0$$

$$\text{Hence, } \alpha + \beta + \gamma = 10 - 10 + 0$$

$$\Rightarrow \alpha + \beta + \gamma = 0$$

Question 203

Let $I_n = \int_1^e x^{19} (\log |x|)^n dx$, where $n \in \mathbb{N}$. If $(20) I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β , then $\alpha - \beta$ is equal to
[17 Mar 2021 Shift 2]

Answer: 1

Solution:

$$\text{Given, } I_n = \int_1^e x^{19} (\log |x|)^n dx$$

$$\Rightarrow I_n = \left[\frac{x^{20}}{20} (\ln |x|)^n \right]_1^e - \int_1^e n \cdot \frac{(\ln |x|)^{n-1}}{x} \cdot x^{20} dx$$

(using integration by parts)

$$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20} \int_1^e (\ln |x|)^{n-1} \cdot x^{19} dx$$

$$\Rightarrow I_n = \frac{e^{20}}{20} - \frac{n}{20} \cdot I_{n-1} \Rightarrow 20I_n + nI_{n-1} = e^{20}$$

Put $n = 10$ and $n = 9$, we get

$$20I_{10} + 10I_9 = e^{20} \dots (i)$$

$$\text{and } 20I_9 + 9I_8 = e^{20} \dots (ii)$$

From Eqs. (i) and (ii),

$$20I_{10} - 10I_9 - 9I_8 = 0$$

$$\Rightarrow 20I_{10} = 10I_9 + 9I_8 \text{ comparing this to}$$

$$20(I_{10}) = \alpha I_9 + \beta I_8, \text{ we get}$$

$$\alpha = 10, \beta = 9$$

$$\therefore \alpha - \beta = 1$$

Question 204

Let $f : (0, 2) \rightarrow \mathbb{R}$ be defined as $f(x) = \log_2 \left[1 + \tan \left(\frac{\pi x}{4} \right) \right]$. Then,

$\lim_{n \rightarrow \infty} \frac{2}{n} \left[f \left(\frac{1}{n} \right) + f \left(\frac{2}{n} \right) + \dots + f(1) \right]$ is equal to.....

[16 Mar 2021 Shift 1]

Answer: 1

Solution:

$$f(x) = \log_2 \left[1 + \tan \left(\frac{\pi x}{4} \right) \right]$$

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{2}{n} \left[f \left(\frac{1}{n} \right) + f \left(\frac{2}{n} \right) + \dots + f(1) \right] = 2 \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{n} \right) f \left(\frac{r}{n} \right)$$

$$\text{Let } I = \frac{2}{\log_2 2} \int_0^1 \log_2 \left[1 + \tan \left(\frac{\pi x}{4} \right) \right] dx \dots (i)$$

$$\text{as, } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\text{So, } x \rightarrow 1-x$$

$$I = \frac{2}{\log_2 2} \int_0^1 \log_2 \left[1 + \tan \frac{\pi}{4} (1-x) \right] dx$$

$$\begin{aligned}
 &= \frac{2}{\log_n 2} \int_0^1 \log_n \left[1 + \tan \left(\frac{\pi}{4} - \frac{\pi x}{4} \right) \right] dx \\
 &= \frac{2}{\log_n 2} \int_0^1 \log_n \left[1 + \left(\frac{1 - \tan \pi x/4}{1 + \tan \pi x/4} \right) \right] dx \\
 &= \frac{2}{\log_n 2} \int_0^1 \log_n \left(\frac{2}{1 + \tan \frac{\pi x}{4}} \right) dx \\
 &= \frac{2}{\log_n 2} \int_0^1 \log_n 2 - \log_n \left(1 + \tan \frac{\pi x}{4} \right) dx \dots (ii)
 \end{aligned}$$

Adding Eqs. (i) and (ii), we get

$$\begin{aligned}
 2I &= \frac{2}{\log_n 2} \int_0^1 \log_n 2 dx \\
 I &= 1
 \end{aligned}$$

Question 205

If the normal to the curve $y(x) = \int_0^x (2t^2 - 15t + 10) dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to.....
[16 Mar 2021 Shift 1]

Answer: 406

Solution:

$$\text{Given, } y(x) = \int_0^x (2t^2 - 15t + 10) dt$$

$$\Rightarrow y'(x) = 2x^2 - 15x + 10$$

Since equation of normal is parallel to $x + 3y = -5$

$\therefore$ Slope of normal to $y(x)$ = Slope of line

$$\Rightarrow \frac{-1}{[y'(x)]_{a,b}} = \frac{-1}{3}$$

$$\text{or } [y'(x)]_{a,b} = 3$$

$$2a^2 - 15a + 10 = 3$$

$$\Rightarrow 2a^2 - 15a + 7 = 0$$

$$\Rightarrow (2a - 1)(a - 7) = 0$$

$$a = \frac{1}{2} \text{ or } 7$$

As, $a > 1$, so, $a = 7$

Now, $(7, b)$ lies on $y(x)$,

$$\therefore b = \int_0^7 (2t^2 - 15t + 10) dt$$

$$\Rightarrow b = \frac{2}{3}a^3 - \frac{15}{2}a^2 + 10a$$

$$\Rightarrow b = \frac{2}{3}(7)^3 - \frac{15}{2}(7)^2 + 10(7)$$

$$\Rightarrow b = \frac{-413}{6}$$

$$\text{So, } a + 6b = 7 - 6 \left(\frac{413}{6} \right) = -406$$

$$\therefore |a + 6b| = 406$$

Question 206

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(x) + f(x+1) = 2$, for all $x \in \mathbb{R}$. If

$I_1 = \int_0^8 f(x) dx$ and $I_2 = \int_{-1}^3 f(x) dx$, then the value of $I_1 + 2I_2$ is equal to.....

[16 Mar 2021 Shift 1]

Answer: 16

Solution:

Given, $f(x) + f(x+1) = 2 \dots (i)$

$$I_1 = \int_0^8 f(x) dx$$

$$\text{and } I_2 = \int_{-1}^3 f(x) dx$$

Let $f(0) = a$

Put $x = 0$ in Eq. (i)

$$f(0) + f(1) = 2$$

$$f(1) = 2 - a$$

Put $x = 1$ in Eq. (i)

$$f(1) + f(2) = 2$$

$$f(2) = a \text{ and so on}$$

$$\text{So, } f(0) = f(2) = f(4) \dots = a$$

$$f(1) = f(3) = f(5) \dots = 2 - a$$

Clearly, $f(x)$ is periodic with its period 2 units.

$$\text{So, } I_1 = \int_0^{24} f(x) dx$$

$$\Rightarrow I_1 = 4 \int_0^2 f(x) dx$$

$$\text{Now, } I_2 = \int_{-1}^3 f(x) dx$$

$$x \rightarrow x+1$$

$$I_2 = \int_0^4 f(x+1) dx = \int_0^4 [2 - f(x)] dx$$

$$\Rightarrow I_2 = 8 - 2 \int_0^2 f(x) dx$$

$$\Rightarrow 2I_2 = 16 - 4 \int_0^2 f(x) dx$$

$$\Rightarrow 2I_2 = 16 - I_1$$

$$\therefore I_1 + 2I_2 = 16$$

Question 207

Consider the integral $I = \int_0^{10} \frac{[x]e^{[x]}}{e^x - 1} dx$ where $[x]$ denotes the greatest integer less than or equal to x .

Then, the value of I is equal to

[16 Mar 2021 Shift 2]

Options:

A. $9(e - 1)$

B. $45(e + 1)$

C. $45(e - 1)$

D. $9(e + 1)$

Answer: C

Solution:

$$\text{We have, } \int_0^{10} \frac{[x]e^{[x]}}{e^x - 1} dx$$

$$= e \int_0^{10} \frac{[x]e^{[x]}}{e^x} dx$$

$$\begin{aligned}
 &= e \int_0^1 \frac{0}{e^x} dx + e \int_1^2 \frac{e}{e^x} dx + e \int_2^3 \frac{2e^2}{e^x} dx + \dots \\
 &\Rightarrow \int_a^b e^{-x} dx = \frac{e^{-x}}{-1} \Big|_a^b \Rightarrow (e^{-a} - e^{-b}) \\
 &\Rightarrow e^2 \left(\frac{1}{e} - \frac{1}{e^2} \right) + 2e^3 \left(\frac{1}{e^2} - \frac{1}{e^3} \right) + 3e^4 \left(\frac{1}{e^3} - \frac{1}{e^4} \right) + \dots + 9e^{10} \left(\frac{1}{e^9} - \frac{1}{e^{10}} \right) \\
 &= (e-1) + 2(e-1) + 3(e-1) + \dots + 9(e-1) \\
 &= (1+2+3+\dots+9)(e-1) = \left(\frac{9 \times 10}{2} \right) (e-1) \\
 &= 45(e-1)
 \end{aligned}$$

Question 208

Let the domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$ be (a, b) . Then the value of the integral $\int_a^b \frac{\sin^3 x}{(\sin^3 x + \sin^3(a+b-x))} dx$ is equal to _____.

[27 Jul 2021 Shift 1]

Answer: 1

Solution:

For domain

$$\log_5(\log_3(18x - x^2 - 77)) > 0$$

$$\log_3(18x - x^2 - 77) > 1$$

$$18x - x^2 - 77 > 3$$

$$x^2 - 18x + 80 < 0$$

$$x \in (8, 10)$$

$$\Rightarrow a = 8 \text{ and } b = 10$$

$$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$I = \int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$2I = (b-a) \Rightarrow I = \frac{b-a}{2} \quad (\because a = 8 \text{ and } b = 10)$$

$$I = \frac{10-8}{2} = 1$$

Question 209

The value of $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to :

[27 Jul 2021 Shift 1]

Options:

A. $5 + \log_e \left(\frac{3}{2} \right)$

B. $2 - \log_e \left(\frac{2}{3} \right)$

C. $3 + 2\log_e \left(\frac{2}{3} \right)$

D. $1 + 2\log_e\left(\frac{3}{2}\right)$

Answer: D

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left(\frac{2j}{n} - \frac{1}{n} + 8 \right) &= \int_0^1 \left(2x - \frac{1}{2} + 8 \right) dx \\ &= \int_0^1 \left(2x + \frac{15}{2} \right) dx \\ &= \left[x^2 + \frac{15}{2}x \right]_0^1 \\ &= 1 + \frac{15}{2} = 1 + 2\log_e\left(\frac{3}{2}\right) \end{aligned}$$

Question 210

The value of the definite integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)}$$

is equal to:

[27 Jul 2021 Shift 1]

Options:

A. $-\frac{\pi}{2}$

B. $\frac{\pi}{2\sqrt{2}}$

C. $-\frac{\pi}{4}$

D. $\frac{\pi}{\sqrt{2}}$

Answer: B

Solution:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{x \cos x})(\sin^4 x + \cos^4 x)} \dots\dots(1)$$

Using $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1 + e^{-x \cos x})(\sin^4 x + \cos^4 x)} \dots\dots(2)$$

Add (1) and (2)

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$$

$$2I = 2 \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$$

$$I = \int_0^{\frac{\pi}{4}} \frac{(1 + \tan^2 x) \sec^2 x}{\tan^4 x + 1} dx$$

$$I = \int_0^{\frac{\pi}{4}} \left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x \frac{x}{\left(\tan x - \frac{1}{\tan x}\right)^2 + 2} dx$$

$$\tan x - \frac{1}{\tan x} = t$$

$$\left(1 + \frac{1}{\tan^2 x}\right) \sec^2 x dx = dt$$

$$I = \int_{-\infty}^0 \frac{dt}{t^2 + 2} = \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) \right]_{-\infty}^0$$

$$I = 0 - \frac{1}{\sqrt{2}} \left(-\frac{\pi}{2} \right) = \frac{\pi}{2\sqrt{2}}$$

Question 211

The value of the definite integral $\int_{\pi/24}^{5\pi/24} \frac{dx}{1 + \sqrt[3]{\tan 2x}}$ is
[25 Jul 2021 Shift 1]

Options:

- A. $\frac{\pi}{3}$
- B. $\frac{\pi}{6}$
- C. $\frac{\pi}{12}$
- D. $\frac{\pi}{18}$

Answer: C

Solution:

$$\text{Let } I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx \dots\dots(i)$$

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos \left\{ 2 \left(\frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}}}{\left(\cos \left\{ 2 \left(\frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}} + \left(\sin \left\{ 2 \left(\frac{\pi}{4} - x \right) \right\} \right)^{\frac{1}{3}}} dx$$

$$\left\{ \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right\}$$

$$\text{So } I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx \dots\dots(ii)$$

$$\text{Hence } 2I = \int_{\pi/24}^{5\pi/24} dx$$

$$[(i) + (ii)]$$

$$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow I = \frac{\pi}{12}$$

Question 212

The value of the integral $\int_{-1}^1 \log(x + \sqrt{x^2 + 1}) dx$ is:
[25 Jul 2021 Shift 2]

Options:

- A. 2
- B. 0
- C. -1
- D. 1

Answer: B

Solution:

$$\text{Let } I = \int_{-1}^1 \log(x + \sqrt{x^2 + 1}) dx$$

$\therefore \log(x + \sqrt{x^2 + 1})$ is an odd function

$$\therefore I = 0$$

Question 213

If $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} \left[\frac{x}{\pi} \right] \right)}} dx = \frac{a\pi^3}{1 + 4\pi^2}$, $a \in \mathbb{R}$ where $[x]$ is the greatest integer less than or equal to x , then the value of a is :
[22 Jul 2021 Shift 2]

Options:

- A. $200(1 - e^{-1})$
- B. $100(1 - e)$
- C. $50(e - 1)$
- D. $150(e^{-1} - 1)$

Answer: A

Solution:

$$I = \int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} \left[\frac{x}{\pi} \right] \right)}} dx = 100 \int_0^{\pi} \frac{\sin^2 x}{e^{x/\pi}} dx$$

$$100 \int_0^{\pi} \frac{e^{-x/\pi} (1 - \cos 2x)}{2} dx$$

$$= 50 \left\{ \int_0^{\pi} e^{-x/\pi} dx - \int_0^{\pi} e^{-x/\pi} \cos 2x dx \right\}$$

$$I_1 = \int_0^{\pi} e^{-x/\pi} dx = [-\pi e^{-x/\pi}]_0^{\pi} = \pi(1 - e^{-1})$$

$$I_2 = \int_0^{\pi} e^{-x/\pi} \cos 2x dx$$

$$= -\pi e^{-x/\pi} \cos 2x \Big|_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} (-2 \sin 2x) dx$$

$$= \pi(1 - e^{-1}) - 2\pi \int_0^{\pi} e^{-x/\pi} \sin 2x dx$$

$$= \pi(1 - e^{-1}) - 2\pi \left\{ -\pi e^{-x/\pi} \sin 2x \Big|_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} 2 \cos 2x dx \right\} = \pi(1 - e^{-1}) - 4\pi^2 I_2$$

$$\Rightarrow I_2 = \frac{\pi(1 - e^{-1})}{1 + 4\pi^2}$$

$$\therefore I = 50 \left\{ \pi(1 - e^{-1}) - \frac{\pi(1 - e^{-1})}{1 + 4\pi^2} \right\}$$

$$= \frac{200(1 - e^{-1})\pi^3}{1 + 4\pi^2}$$

Question 214

The value of the integral $\int_{-1}^1 \log_e(\sqrt{1-x} + \sqrt{1+x}) dx$ is equal to :
[20 Jul 2021 Shift 1]

Options:

A. $\frac{1}{2} \log_e 2 + \frac{\pi}{4} - \frac{3}{2}$

B. $2 \log_e 2 + \frac{\pi}{4} - 1$

C. $\log_e 2 + \frac{\pi}{2} - 1$

D. $2 \log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

Answer: B

Solution:

Let $I = 2 \int_0^1 \ln(\sqrt{1-x} + \sqrt{1+x}) dx$

$$\therefore I = 2 \left[(x \cdot \ln(\sqrt{1-x} + \sqrt{1+x}))_0^1 - \int_0^1 x \cdot \left(\frac{1}{\sqrt{1-x} + \sqrt{1+x}} \right) \cdot \left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} \right) dx \right]$$

$$= 2(\ln \sqrt{2} - 0) - \frac{2}{2} \int_0^1 \frac{x\sqrt{1-x} - \sqrt{1+x} dx}{(\sqrt{1-x} + \sqrt{1+x})\sqrt{1-x^2}}$$

$$= (\log_e 2) - \int_0^1 \frac{x \cdot (2 - 2\sqrt{1-x^2})}{-2x\sqrt{1-x^2}} dx \text{ (After rationalisation)}$$

$$= (\log_e 2) + \int_0^1 \left(\frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} \right) dx$$

$$= (\log_e 2) + (\sin^{-1} x)_0^1 - 1$$

$$= \log_e 2 + \left(\frac{\pi}{2} - 0 \right) - 1$$

$$\therefore I = (\log_e 2) + \frac{\pi}{2} - 1$$

$\Rightarrow$ Option (3) is correct.

Question 215

Let a be a positive real number such that $\int_0^a e^{x-[x]} dx = 10e - 9$ where $[x]$ is the greatest integer less than or equal to x . Then a is equal to :
[20 Jul 2021 Shift 1]

Options:

A. $10 - \log_e(1 + e)$

B. $10 + \log_e 2$

C. $10 + \log_e 3$

D. $10 + \log_e(1 + e)$

Answer: B

Solution:

$$\begin{aligned}
 a &> 0 \\
 \text{Let } n \leq a < n+1, n \in \mathbb{W} \\
 \therefore a &= [a] + \{a\} \\
 &\Downarrow \qquad \qquad \Downarrow \\
 &\text{G. I. F. Fractional part} \\
 \text{Here } [a] &= n \text{ Now, } \int_0^a e^{x-[x]} dx = 10e - 9 \\
 \Rightarrow \int_0^n e^{\{x\}} dx + \int_n^a e^{x-[x]} dx &= 10e - 9 \\
 \therefore n \int_0^1 e^x dx + \int_n^a e^{x-n} dx &= 10e - 9 \\
 \Rightarrow n(e - 1) + (e^{a-n} - 1) &= 10e - 9 \\
 \therefore n = 0 \text{ and } \{a\} &= \log_e 2 \\
 \text{So, } a &= [a] + \{a\} = (10 + \log_e 2) \\
 \Rightarrow \text{Option (2) is correct.}
 \end{aligned}$$

Question 216

If $[x]$ denotes the greatest integer less than or equal to x , then the value of the integral

$$\int_{-\pi/2}^{\pi/2} [[x] - \sin x] dx \text{ is equal to :}$$

[20 Jul 2021 Shift 2]

Options:

- A. $-\pi$
- B. π
- C. 0
- D. 1

Answer: A

Solution:

$$I = \int_{-\pi/2}^{\pi/2} ([x] + [-\sin x]) dx \dots\dots(1)$$

$$I = \int_{-\pi/2}^{\pi/2} ([-x] + [\sin x]) dx \dots\dots(2)$$

(King property)

$$2I = \int_{-\pi/2}^{\pi/2} ([x] + [-x]) + ([\sin x] + [-\sin x]) dx$$

$$2I = \int_{-\pi/2}^{\pi/2} (-2) dx = -2(\pi)$$

$$I = -\pi$$

Question 217

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x + 1$, then the value of $\lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + \dots + f\left(\frac{5(n-1)}{n}\right) \right]$ is:

[20 Jul 2021 Shift 2]

Options:

A. $\frac{3}{2}$

B. $\frac{5}{2}$

C. $\frac{1}{2}$

D. $\frac{7}{2}$

Answer: D

Solution:

$$I = \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right) \frac{1}{n}$$

$$I = \int_0^1 f(5x) dx$$

$$I = \int_0^1 (5x + 1) dx$$

$$I = \left[\frac{5x^2}{2} + x \right]_0^1$$

$$I = \frac{5}{2} + 1 = \frac{7}{2}$$

Question 218

Let $g(t) = \int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t + f(x)\right) dx$, where $f(x) = \log_e(x + \sqrt{x^2 + 1})$, $x \in \mathbb{R}$. Then which one of the following is correct?
[20 Jul 2021 Shift 2]

Options:

A. $g(1) = g(0)$

B. $\sqrt{2}g(1) = g(0)$

C. $g(1) = \sqrt{2}g(0)$

D. $g(1) + g(0) = 0$

Answer: B

Solution:

$$g(t) = \int_{-\pi/2}^{\pi/2} \left(\cos \frac{\pi}{4} t + f(x) \right) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t + \int_{-\pi/2}^{\pi/2} f(x) dx$$

$$g(t) = \pi \cos \frac{\pi}{4} t$$

$$g(1) = \frac{\pi}{\sqrt{2}}, g(0) = \pi$$

Question 219

Let $F : [3, 5] \rightarrow \mathbb{R}$ be a twice differentiable function on $(3, 5)$ such that

$$F(x) = e^{-x} \int_3^x (3t^2 + 2t + 4F'(t)) dt$$

If $F'(4) = \frac{\alpha e^{\beta} - 224}{(e^{\beta} - 4)^2}$, then $\alpha + \beta$ is equal to _____.

[27 Jul 2021 Shift 1]

Answer: 16

Solution:

$$F(3) = 0$$

$$e^x F(x) = \int_3^x (3t^2 + 2t + 4F'(t)) dt$$

$$e^x F(x) + e^x F'(x) = 3x^2 + 2x + 4F'(x)$$

$$(e^x - 4) \frac{dy}{dx} + e^x y = (3x^2 + 2x)$$

$$\frac{dy}{dx} + \frac{e^x}{(e^x - 4)} y = \frac{(3x^2 + 2x)}{(e^x - 4)}$$

$$\int \frac{e^x}{(e^x - 4)} dx = \int \frac{(3x^2 + 2x)}{(e^x - 4)} e^{\frac{x}{e^x - 4}} dx$$

$$y \cdot (e^x - 4) = \int (3x^2 + 2x) dx + c$$

$$y(e^x - 4) = x^3 + x^2 + c$$

$$\text{Put } x = 3 \Rightarrow c = -36$$

$$F(x) = \frac{(x^3 + x^2 - 36)}{(e^x - 4)}$$

$$F'(x) = \frac{(3x^2 + 2x)(e^x - 4) - (x^3 + x^2 - 36)e^x}{(e^x - 4)^2}$$

Now put value of $x = 4$ we will get $\alpha = 12$ & $\beta = 4$

Question 220

If $\int_0^{\pi} (\sin^3 x) e^{-\sin^2 x} dx = \alpha - \frac{\beta}{e} \int_0^1 \sqrt{t} e^t dt$, then $\alpha + \beta$ is equal to _____.

[27 Jul 2021 Shift 2]

Answer: 5

Solution:

$$\begin{aligned}
 I &= \int_0^{\pi} (\sin^3 x) e^{-\sin^2 x} dx \\
 &= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + \int_0^{\pi/2} \cos x e^{-\sin^2 x} (-\sin 2x) dx \\
 &= 2 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + \left[\cos x e^{-\sin^2 x} \right]_0^{\pi/2} + \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx \\
 &= 3 \int_0^{\pi/2} \sin x e^{-\sin^2 x} dx - 1 \\
 &= \frac{3}{2} \int_{-1}^0 \frac{e^{\alpha} d\alpha}{\sqrt{1+\alpha}} - 1 \quad (\text{Put } -\sin^2 x = t) \\
 &= \frac{3}{2} \int_0^1 \frac{e^{-x}}{\sqrt{x}} dx - 1 \quad (\text{put } 1+\alpha = x) \\
 &= \frac{3}{2} \left[e^{-x} \frac{1}{\sqrt{x}} \right]_0^1 dx - 1 \\
 &= 2 - \frac{3}{2} \left[e^{-x} \sqrt{x} \right]_0^1 dx \\
 \text{Hence, } \alpha + \beta &= 5
 \end{aligned}$$

Question 221

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \sin^2 x}{1 + \pi^{\sin x}} \right) dx$ is

[26 Aug 2021 Shift 2]

Options:

- A. $\frac{\pi}{2}$
- B. $\frac{5\pi}{2}$
- C. $\frac{3\pi}{4}$
- D. $\frac{3\pi}{2}$

Answer: C

Solution:

$$\begin{aligned}
 I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin^2 x}{1 + \pi^{\sin x}} dx \dots (i) \\
 I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin^2(-x)}{1 + \pi^{\sin(-x)}} \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \sin^2 x}{1 + \pi^{-\sin x}} dx \\
 I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^{\sin x} (1 + \sin^2 x)}{1 + \pi^{\sin x}} dx \dots (ii)
 \end{aligned}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + \pi^{\sin x})(1 + \sin^2 x)}{(1 + \pi^{\sin x})} dx$$

$$\Rightarrow 2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin^2 x) dx$$

$$\Rightarrow 2I = [x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$[\because \sin^2 x \text{ is an even function, so } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx = 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx]$$

$$\Rightarrow I = \frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) + \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx$$

$$\Rightarrow I = \frac{\pi}{2} + \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{2} + \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right]$$

$$I = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Question 222

If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a \sin^{-1} \left(\frac{\sin x + \cos x}{b} \right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to [2021]

Options:

A. $(-1, 3)$

B. $(3, 1)$

C. $(1, 3)$

D. $(1, -3)$

Answer: C

Solution:

$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$$

$$\int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}} dx$$

$$\text{Let } \sin x + \cos x = t$$

$$\int \frac{dt}{\sqrt{9 - t^2}} = \sin^{-1} \frac{t}{3} + c$$

$$= \sin^{-1} \left(\frac{\sin x + \cos x}{3} \right) + c$$

$$\text{So } a = 1, b = 3.$$

Question 223

The integral $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}} dx$ is equal to (where C is a constant of integration)

[31 Aug 2021 Shift 1]

Options:

A. $\frac{3}{4} \left(\frac{x+2}{x-1} \right)^{\frac{1}{4}} + C$

B. $\frac{3}{4} \left(\frac{x+2}{x-1} \right)^{\frac{5}{4}} + C$

C. $\frac{4}{3} \left(\frac{x-1}{x+2} \right)^{\frac{1}{4}} + C$

D. $\frac{4}{3} \left(\frac{x-1}{x+2} \right)^{\frac{5}{4}} + C$

Answer: C

Solution:

$$\int \frac{1}{(x-1)^{\frac{3}{4}}(x+2)^{\frac{5}{4}}} dx = \int \frac{dx}{\left(\frac{x+2}{x-1}\right)^{\frac{5}{4}}(x-1)^2}$$

$$\frac{x+2}{x-1} = t$$

$$\Rightarrow \left(\frac{(x-1)-(x+2)}{(x-1)^2} \right) dx = dt$$

$$\Rightarrow -\frac{3}{(x-1)^2} dx = dt$$

$$\Rightarrow -\frac{1}{3} \int \frac{dt}{t^{\frac{5}{4}}} = \frac{4}{3} \cdot \frac{1}{t^{\frac{1}{4}}} + C$$

$$= \frac{4}{3} \left(\frac{x-1}{x+2} \right)^{\frac{1}{4}} + C$$

Question 224

If

$$\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx = \alpha \log_e \left| 1 + \tan x \right| + \beta \log_e \left| 1 - \tan x + \tan^2 x \right| + \gamma \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + C,$$

when C is constant of integration, then the value of $18(\alpha + \beta + \gamma^2)$ is [31 Aug 2021 Shift 2]

Answer: 3

Solution:

$$\text{Let } I = \int \frac{\sin x}{\sin^3 x + \cos^3 x} dx =$$

$$= \frac{\tan x \sec^2 x}{\tan^3 x + 1} dx$$

$$\text{Put } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$I = \int \frac{tdt}{t^3 + 1} = \int \frac{t}{(t+1)(t^2 - t + 1)} dt$$

$$\text{Now, } \frac{t}{(t+1)(t^2 - t + 1)} = \frac{A}{t+1} + Bt + Ct^2 - t + 1$$

$$\Rightarrow t = A(t^2 - t + 1) + (Bt + C)(t + 1)$$

Comparing coefficients to both the sides and solving them for A, B, C, we have

$$A = -\frac{1}{3}, B = \frac{1}{3}, C = \frac{1}{3}$$

$$\text{Hence, } I = -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{3} \int \frac{t+1}{t^2-t+1} dt$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{3} \int \frac{\frac{1}{2}(2t-1) + \frac{3}{2}}{t^2-t+1} dt$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{6} \ln(t^2-t+1) + \frac{1}{2} \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3} \ln(t+1) + \frac{1}{6} \ln(t^2-t+1) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t-1}{\sqrt{3}} \right) + C$$

$$= -\frac{1}{3} \ln(\tan x + 1) + \frac{1}{6} \ln(\tan^2 x - \tan x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + C$$

$$\Rightarrow \alpha = -\frac{1}{3}, \beta = \frac{1}{6}, \gamma = \frac{1}{\sqrt{3}}$$

$$\text{So, } 18(\alpha + \beta + \gamma^2) = 18 \left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{3} \right) = 3$$

Question 225

If $\int \frac{dx}{(x^2+x+1)^2} = a \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + b \left(\frac{2x+1}{x^2+x+1} \right) + C$, $x > 0$ where C is the constant of integration, then the value of $9(\sqrt{3}a + b)$ is equal to
[27 Aug 2021 Shift 1]

Answer: 15

Solution:

$$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \right]^2}$$

$$\text{Let } x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$$

$$\Rightarrow dx = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$$

$$\therefore \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\frac{9}{16} (\tan^2 \theta + 1)^2} = \frac{8}{3\sqrt{3}} \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta}$$

$$= \frac{8}{3\sqrt{3}} \int \cos^2 \theta d\theta$$

$$= \frac{8}{3\sqrt{3}} \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{4}{3\sqrt{3}} (\theta + \sin 2\theta) + C$$

$$= \frac{4}{2\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + \frac{4}{3\sqrt{3}} \frac{\frac{2x+1}{\sqrt{3}}}{1 + \left(\frac{2x+1}{\sqrt{3}} \right)^2} + C$$

$$= \frac{4}{3\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + \frac{1}{3} \frac{2x+1}{(x^2+x+1)} + C$$

$$\therefore a = \frac{4}{3\sqrt{3}}, b = \frac{1}{3}$$

$$\text{Hence, } 9(\sqrt{3}a + b) = 9 \left(\frac{4}{3} + \frac{1}{3} \right) = 15$$

Question 226

If $\int \frac{2e^x + 3e^{-x}}{4e^x + 7e^{-x}} dx = \frac{1}{14}(ux + v \log_e(4e^x + 7e^{-x})) + C$,

where C is a constant of integration, then $u + v$ is equal to
[27 Aug 2021 Shift 2]

Answer: 7

Solution:

$$I = \int \frac{2e^x + 3e^{-x}}{4e^x + 7e^{-x}} dx = \int \frac{2e^{2x} + 3}{4e^{2x} + 7} dx$$

$$\text{Let } 2e^{2x} + 3 = A \frac{d}{dx}(4e^{2x} + 7) + B(4e^{2x} + 7)$$

$$\Rightarrow 2e^{2x} + 3 = (8A + 4B)e^{2x} + 7B$$

Comparing both sides

$$B = \frac{3}{7} \text{ and } A = \frac{1}{28}$$

$$\therefore I = \int \frac{\frac{1}{28}(8e^{2x}) + \frac{3}{7}(4e^{2x} + 7)}{4e^{2x} + 7} dx$$

$$= \frac{1}{28} \ln |4e^{2x} + 7| + \frac{3}{7}x + C$$

$$= \frac{1}{28} \ln |e^x(4e^x + 7e^{-x})| + \frac{3}{7}x + C$$

$$= \frac{1}{28}x + \frac{1}{28} \ln |4e^x + 7e^{-x}| + \frac{3}{7}x + C$$

$$= \frac{1}{14} \left(\frac{13}{2}x + \frac{1}{2} \ln |4e^x + 7e^{-x}| \right) + C$$

$$\Rightarrow u = \frac{13}{2} \text{ and } v = \frac{1}{2}$$

$$\therefore u + v = \frac{13}{2} + \frac{1}{2} = 7$$

Question 227

Section B : Numerical Type Questions

Let $[t]$ denote the greatest integer $\leq t$. Then the value of $8 \cdot \int_{-\frac{1}{2}}^1 ([2x] + |x|) dx$ is

[31 Aug 2021 Shift 1]

Answer: 5

Solution:

$$8 \int_{-\frac{1}{2}}^1 ([2x] + |x|) dx$$

$$= -\frac{1}{2} \leq x < 0$$

$$\Rightarrow [2x] = -1$$

$$0 \leq x < \frac{1}{2}$$

$$\Rightarrow [2x] = 0$$

$$\frac{1}{2} \leq x < 1$$

$$\Rightarrow [2x] = 1$$

$$I = \int_{-\frac{1}{2}}^0 (-1-x) dx + \int_0^{\frac{1}{2}} (0+x) dx + \int_{\frac{1}{2}}^1 (1+x) dx$$

$$= \left[-x - \frac{x^2}{2} \right]_{-\frac{1}{2}}^0 + \left[\frac{x^2}{2} \right]_0^{\frac{1}{2}} + \left[x + \frac{x^2}{2} \right]_{\frac{1}{2}}^1$$

$$= -\left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(1 + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{5}{8}$$

$$\therefore 8I = 8 \cdot \frac{5}{8} = 5$$

Question 228

If $x\phi(x) = \int_5^x (3t^2 - 2\phi'(t)) dt$, $x > -2$, and $\phi(0) = 4$, then $\phi(2)$ is
[31 Aug 2021 Shift 1]

Answer: 4

Solution:

$$x\phi(x) = \int_5^x (3t^2 - 2\phi'(t)) dt$$

$$\Rightarrow x\phi(x) = \left[t^3 - 2\phi(t) \right]_5^x$$

$$\Rightarrow x\phi(x) = (x^3 - 125) - 2[\phi(x) - \phi(5)]$$

$$\text{Now, } \phi(0) = 4$$

$$\Rightarrow 0 = -125 - 2[4 - \phi(5)]$$

$$\Rightarrow \phi(5) = \frac{133}{2}$$

$$\text{For } \phi(2),$$

$$\Rightarrow 2\phi(2) = (8 - 125) - 2\left[\phi(2) - \frac{133}{2}\right]$$

$$\Rightarrow 4\phi(2) = 16$$

$$\Rightarrow \phi(2) = 4$$

Question 229

If $[x]$ is the greatest integer $\leq x$, then $\pi^2 \int_0^{2\pi} \left(\sin \frac{\pi x}{2} \right) (x - [x])^{[x]} dx$ is equal to
[31 Aug 2021 Shift 2]

Options:

A. $2(\pi - 1)$

B. $4(\pi - 1)$

C. $4(\pi + 1)$

D. $2(\pi + 1)$

Answer: B

Solution:

$$\begin{aligned}
 & \pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right) (x - |x|)^{[x]} dx \\
 &= \pi^2 \int_0^1 \sin\left(\frac{\pi x}{2}\right) x^0 dx + \pi^2 \int_1^2 \sin\left(\frac{\pi x}{2}\right) (x-1) dx \\
 &= \pi^2 \left[\frac{-2}{\pi} \cos \frac{\pi x}{2} \right]_0^1 + \pi^2 \left[(x-1) \frac{2}{\pi} \left(-\cos \frac{\pi x}{2} \right) \right]_1^2 + \pi^2 \int_1^2 \frac{2}{\pi} \cos \frac{\pi x}{2} dx \\
 &= \pi^2 \left(\frac{2}{\pi} \right) + \frac{2\pi^2}{\pi} (1-0) + 2\pi \cdot \frac{2}{\pi} \left(\sin \frac{\pi x}{2} \right) \Big|_1^2 \\
 &= 2\pi + 2\pi + 4(0-1) = 4\pi - 4 = 4(\pi - 1)
 \end{aligned}$$

Question 230

If $U_n = \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n^2}{n}\right)$, then $\lim_{n \rightarrow \infty} (U_n)^{\frac{-4}{n^2}}$ is equal to

[27 Aug 2021 Shift 1]

Options:

A. $\frac{e^2}{16}$

B. $\frac{4}{e}$

C. $\frac{16}{e^2}$

D. $\frac{4}{e^2}$

Answer: A

Solution:

Let $y = \lim_{n \rightarrow \infty} (U_n)^{\frac{-4}{n^2}}$

$$y = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^{\frac{-4}{n^2}} \left(1 + \frac{2}{n}\right)^{\frac{-4}{n^2}} \left(1 + \frac{3}{n}\right)^{\frac{-4}{n^2}} \dots \right]$$

Taking log on both sides, we get

$$\ln y = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left[\frac{-4}{n^2} \cdot r \ln \left(1 + \frac{r}{n}\right) \right]$$

Now, replace $\lim_{n \rightarrow \infty} \sum \rightarrow \int$

$$\frac{r}{n} \rightarrow x, \frac{1}{n} \rightarrow dx$$

Lower limit = 0

Upper limit = 1

$$\therefore \ln y = \int_0^1 -4x \ln(1+x^2) dx$$

Let $1+x^2 = t$

$$\Rightarrow x dx = \frac{dt}{2}$$

When $x \rightarrow 0$, $t \rightarrow 1$

and $x \rightarrow 1$, $t \rightarrow 2$

$$\therefore \ln y = \int_1^2 -2 \ln t dt$$

$$= -2(t \ln t - t) \Big|_1^2$$

$$= -2(2 \ln 2 - 2 + 1)$$

$$= -2(2 \ln 2 - 1)$$

$$\Rightarrow \ln y = \ln \frac{1}{16} + 2$$

$$\Rightarrow y = \frac{1}{16} e^2$$

Question 231

$\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x^2 - 44x + 484)} dx$ is equal to

[27 Aug 2021 Shift 1]

Options:

- A. 6
- B. 8
- C. 5
- D. 10

Answer: C

Solution:

$$\text{Let } I = \int_6^{16} \frac{\ln_e (x^2)}{\ln_e (x^2) + \ln_e (484 - 44x + x^2)} dx$$

$$= \int_6^{16} \frac{\ln_e (x^2)}{\ln_e (x^2) + \ln_e (22 - x)^2} dx$$

$$= \int_6^{16} \frac{2 \ln_e x dx}{2 \ln_e x + 2 \ln_e (22 - x)}$$

$$I = \int_6^{16} \frac{\ln_e x dx}{\ln_e x + \ln_e (22 - x)} \dots (i)$$

$$\because \int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

$$\therefore I = \int_6^{16} \frac{\ln_e (22 - x)}{\ln_e (22 - x) + \ln_e x} dx \dots (ii)$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_6^{16} \frac{\ln_e x + \ln_e (22 - x)}{\ln_e x + \ln_e (22 - x)} dx$$

$$2I = \int_6^{16} dx = x \Big|_6^{16} = 10$$

$$\text{or } I = 5$$

Question 232

The value of the integral $\int_0^1 \frac{\sqrt{x} dx}{(1+x)(1+3x)(3+x)}$ is

[27 Aug 2021 Shift 2]

Options:

A. $\frac{\pi}{8} \left(1 - \frac{\sqrt{3}}{2} \right)$

B. $\frac{\pi}{4} \left(1 - \frac{\sqrt{3}}{6} \right)$

C. $\frac{\pi}{8} \left(1 - \frac{\sqrt{3}}{6} \right)$

D. $\frac{\pi}{4} \left(1 - \frac{\sqrt{3}}{2} \right)$

Answer: A

Solution:

$$\int_0^1 \frac{\sqrt{x} dx}{(1+x)(1+3x)(3+x)}$$

Put $\sqrt{x} = t$

$$\Rightarrow x = t^2$$

or $dx = 2t dt$

$$\therefore I = \int_0^1 \frac{2t^2 dt}{(t^2+1)(3t^2+1)(t^2+3)}$$

$$= \int_0^1 \frac{(3t^2+1) - (t^2+1)}{(t^2+1)(3t^2+1)(t^2+3)} dt$$

$$= \int_0^1 \left[\frac{1}{(t^2+3)(t^2+1)} - \frac{1}{(t^2+3)(3t^2+1)} \right] dt$$

$$= \int_0^1 \left[\frac{1}{2(t^2+1)} - \frac{1}{2(t^2+3)} - \frac{3}{8(3t^2+1)} + \frac{1}{8(t^2+3)} \right] dt$$

$$= \int_0^1 \frac{dt}{2(t^2+1)} - \int_0^1 \frac{3}{8(3t^2+1)} dt - \int_0^1 \frac{dt}{8(t^2+3)}$$

$$= \left(\frac{1}{2} \tan^{-1} t \right)_0^1 - \left(\frac{3\sqrt{3}}{8} \tan^{-1} \sqrt{3} t \right)_0^1 - \left(\frac{3}{8\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \right)_0^1$$

$$= \frac{\pi}{8} - \frac{\sqrt{3}\pi}{8 \cdot 3} - \frac{\sqrt{3}\pi}{8 \cdot 6} = \frac{\pi}{8} - \frac{\sqrt{3}\pi}{16} = \frac{\pi}{8} \left(1 - \frac{\sqrt{3}}{2} \right)$$

Question 233

The value of $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\left(\frac{x+1}{x-1} \right)^2 + \left(\frac{x-1}{x+1} \right)^2 - 2 \right)^{1/2} dx$ is

[26 Aug 2021 Shift 1]

Options:

A. $\log_e 4$

B. $\log_e 16$

C. $2\log_e 16$

D. $4\log_e(3 + 2\sqrt{2})$

Answer: B

Solution:

$$\text{Let } I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\left(\frac{x+1}{x-1} \right)^2 + \left(\frac{x-1}{x+1} \right)^2 - 2 \right)^{1/2} dx$$

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[\left(\frac{x+1}{x-1} - \frac{x-1}{x+1} \right)^2 \right]^{1/2} dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left| \frac{x+1}{x-1} - \frac{x-1}{x+1} \right| dx$$

$$\begin{aligned}
 &= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left| \frac{(x+1)^2 - (x-1)^2}{(x-1)(x+1)} \right| dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left| \frac{4x}{(x-1)(x+1)} \right| dx \\
 &= 2.4 \int_0^{\frac{1}{\sqrt{2}}} \left| \frac{x}{(x-1)(x+1)} \right| dx = 4 \int_0^{\frac{1}{\sqrt{2}}} \frac{-2x}{x^2-1} dx \\
 &= -4 [\log(x^2-1)]_0^{\frac{1}{\sqrt{2}}} \\
 &= -4 \left[\log\left(\frac{1}{2}-1\right) - \log(-1) \right] \\
 &= -4 \log\left(\frac{1}{2}\right) = 4 \ln 2 = \ln 16
 \end{aligned}$$

Question 234

The value of $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2}$ is
 [26 Aug 2021 Shift 1]

Options:

- A. $\frac{1}{2} \tan^{-1}(2)$
- B. $\frac{1}{2} \tan^{-1}(4)$
- C. $\tan^{-1}(4)$
- D. $\frac{1}{4} \tan^{-1}(4)$

Answer: B

Solution:

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2}{n^2 + 4r^2} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{1}{1 + 4\left(\frac{r}{n}\right)^2} = \int_0^2 \frac{1}{1 + 4x^2} dx \\
 &= \frac{1}{2} [\tan^{-1} 2x]_0^2 = \frac{1}{2} \tan^{-1} 4
 \end{aligned}$$

Question 235

If the value of the integral $\int_0^5 \frac{x + [x]}{e^{x-[x]}} dx = \alpha e^\beta + \beta$, where $\alpha, \beta \in \mathbb{R}$, $5\alpha + 6\beta = 0$ and $[x]$ denotes the greatest integer less than or equal to x , then the value of $(\alpha + \beta)^2$ is equal to:
 [26 Aug 2021 Shift 2]

Options:

- A. 100
- B. 25
- C. 16
- D. 36

Answer: B

Solution:

$$I = \int_0^5 \frac{x + [x]}{e^{x-[x]}} dx = \alpha e^{-\beta} + \beta,$$

$$I = \int_0^1 \frac{x}{e^x} dx + \int_1^2 \frac{x+1}{e^{x-1}} dx + \int_2^3 \frac{x+2}{e^{x-2}} dx + \int_3^4 \frac{x+3}{e^{x-3}} dx + \int_4^5 \frac{x+4}{e^{x-4}} dx$$

$$\text{Let } I = I_1 + I_2 + I_3 + I_4 + I_5$$

$$\text{Here, } I_2 = \int_1^2 \frac{x+1}{e^{x-1}} dx \text{ Put } x = t + 1$$

$$\Rightarrow dx = dt$$

$$= \int_0^1 \frac{t+2}{e^t} dt = \int_0^1 \frac{t}{e^t} dt + \int_0^1 \frac{2}{e^t} dt$$

$$I_2 = I_1 + 2 \int_0^1 e^{-t} dt = I_1 + 2(1 - e^{-1})$$

Similarly,

$$I_3 = I_1 + 4(1 - e^{-1})$$

$$I_4 = I_1 + 6(1 - e^{-1})$$

$$I_5 = I_1 + 8(1 - e^{-1})$$

$$I = I_1 + I_2 + I_3 + I_4 + I_5 = 5I_1 + (2 + 4 + 6 + 8)(1 - e^{-1})$$
$$= 5I_1 + 20(1 - e^{-1})$$

$$I_1 = \int_0^1 x e^{-1} dx = -[e^{-x}(x+1)]_0^1 = 1 - 2e^{-1}$$

$$\therefore 5I_1 + 20(1 - e^{-1}) = 5(1 - 2e^{-1}) + 20(1 - e^{-1}) = 25 - 30e^{-1}$$

$$\therefore \alpha = -30, \beta = 25$$

Also it satisfy $5\beta + 6\alpha = 0$

$$\text{Now, } (\alpha + \beta)^2 = (-30 + 25)^2 = (-5)^2 = 25$$

Question 236

Let $J_{n,m} = \int_0^{\frac{1}{2}} \frac{x^n}{x^m - 1} dx, \forall n > m \text{ and } n, m \in \mathbb{N}.$

Consider a matrix $A = [a_{ij}]_{3 \times 3}$ where

$$a_{ii} = \begin{cases} J_{6+i, 3-J_i+3}, & 3 \leq i \\ 0 & i > j \end{cases}$$

then $|\text{adj } A^{-1}|$ is
[1 Sep 2021 Shift 2]

Options:

A. $(15)^2 \times 2^{42}$

B. $(15)^2 \times 2^{34}$

C. $(105)^2 \times 2^{38}$

D. $(105)^2 \times 2^{36}$

Answer: C

Solution:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33}$$

$$\Rightarrow |A| = (J_{7,3} - J_{4,3}) (J_{8,3} - J_{5,3}) (J_{9,3} - J_{6,3})$$

$$= \int_0^{\frac{1}{2}} \frac{x^7 - x^4}{x^3 - 1} dx \cdot \int_0^{\frac{1}{2}} \frac{x^8 - x^5}{x^3 - 1} dx \cdot \int_0^{\frac{1}{2}} \frac{x^9 - x^6}{x^3 - 1} dx$$

$$= \int_0^{\frac{1}{2}} x^4 dx \cdot \int_0^{\frac{1}{2}} x^5 dx \cdot \int_0^{\frac{1}{2}} x^6 dx$$

$$= \left. \frac{x^5}{5} \right|_0^{\frac{1}{2}} \cdot \left. \frac{x^6}{6} \right|_0^{\frac{1}{2}} \cdot \left. \frac{x^7}{7} \right|_0^{\frac{1}{2}} = \frac{1}{(210)2^{18}}$$

$$\text{Now, } |\text{adj } A^{-1}| = \frac{1}{|A|^2} = ((210)2^{18})^2 = 105^2 \cdot 2^{38}$$

Question 237

The function $f(x)$, that satisfies the condition

$$f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y f(y) dy, \text{ is}$$

[1 Sep 2021 Shift 2]

Options:

A. $x + \frac{2}{3}(\pi - 2) \sin x$

B. $x + (\pi + 2) \sin x$

C. $x + \frac{\pi}{2} \sin x$

D. $x + (\pi - 2) \sin x$

Answer: D

Solution:

$$f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y f(y) dy$$

$$\text{Let } K = \int_0^{\pi/2} \cos y f(y) dy \dots (i)$$

$$\text{Then, } f(x) = x + K \sin x \dots (ii)$$

From Eqs. (i) and (ii),

$$f(x) = x + \int_0^{\pi/2} \sin x \cos y (y + k \sin y) dy$$

$$= x + \sin x \int_0^{\pi/2} y \cos y dy + \frac{k}{2} \sin x \int_0^{\pi/2} \sin 2y dy$$

$$f(x) = x + \sin x \cdot \frac{\pi-2}{2} + \frac{k \sin x}{2} \dots (iii)$$

$$k = \frac{\pi-2}{2} + \frac{k}{2}$$

$$\Rightarrow k = \pi - 2$$

$$\therefore f(x) = x + (\pi - 2) \sin x$$

Question 238

The integral $\int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$ is equal to: (where C is a constant of integration)

[Jan. 9, 2020 (I)]

Options:

A. $\left(\frac{x-3}{x+4}\right)^{1/7} + C$

B. $-\left(\frac{x-3}{x+4}\right)^{1/7} + C$

C. $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{3/7} + C$

D. $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-13/7} + C$

Answer: A

Solution:

$$I = \int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$$
$$= \int \left(\frac{x-3}{x+4}\right)^{-\frac{6}{7}} \frac{1}{(x+4)^2} dx$$

Let $\frac{x-3}{x+4} = t^7$

Differentiate on both sides, we get

$$\frac{7}{(x+4)^2} dx = 7t^6 dt$$

Hence, $I = \int t^{-6} dt = t + C = \left(\frac{x-3}{x+4}\right)^{\frac{1}{7}} + C$

Question 239

If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)} = \lambda \tan \theta + 2 \log_e |f(\theta)| + C$ where C is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to:
[Jan. 9, 2020 (II)]

Options:

A. $(1, 1 - \tan \theta)$

B. $(-1, 1 - \tan \theta)$

C. $(-1, 1 + \tan \theta)$

D. $(1, 1 + \tan \theta)$

Answer: C

Solution:

$$I = \int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)}$$
$$= \int \frac{\sec^2\theta}{\frac{1+\tan^2\theta}{1-\tan^2\theta} + \frac{2\tan\theta}{1-\tan^2\theta}} d\theta$$

$$= \int \frac{\sec^2 \theta (1 - \tan^2 \theta)}{(1 + \tan \theta)^2} d\theta$$

$$= \int \frac{\sec^2 \theta (1 - \tan \theta)}{1 + \tan \theta} d\theta$$

Let $\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$, then

$$I = \int \left(\frac{1-t}{1+t} \right) dt = \int \left(-1 + \frac{2}{1+t} \right) dt$$

$$= -t + 2 \log(1+t) + C$$

$$= -\tan \theta + 2 \log(1 + \tan \theta) + C$$

Hence, by comparison $\lambda = -1$ and $f(x) = 1 + \tan \theta$

Question 240

If $\int \frac{\cos x \, dx}{\sin^3 x (1 + \sin^6 x)^{2/3}} = f(x)(1 + \sin^6 x)^{1/\lambda} + c$ where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal

to:

[Jan. 8, 2020 (II)]

Options:

A. $-\frac{9}{8}$

B. 2

C. $\frac{9}{8}$

D. -2

Answer: D

Solution:

$$\text{Let } I = \int \frac{\cos x \, dx}{\sin^3 x (1 + \sin^6 x)^{2/3}}$$

$$= f(x)(1 + \sin^6 x)^{1/\lambda} + c \dots (i)$$

If $\sin x = t$

then, $\cos x \, dx = dt$

$$I = \int \frac{dt}{t^3 (1+t^6)^{2/3}} = \int \frac{dt}{t^7 \left(1 + \frac{1}{t^6}\right)^{2/3}}$$

$$\text{Put } 1 + \frac{1}{t^6} = r^3 \Rightarrow \frac{dt}{t^7} = \frac{-1}{2} r^2 dr = -\frac{1}{2} r + c$$

$$= -\frac{1}{2} \left(\frac{\sin^6 x + 1}{\sin^6 x} \right)^{1/3} + c = -\frac{1}{2 \sin^2 x} (1 + \sin^6 x)^{1/3} + c$$

$$f(x) = -\frac{1}{2} \operatorname{cosec}^2 x \text{ and } \lambda = 3 [\text{from eqn. (i)}]$$

$$\therefore \lambda f\left(\frac{\pi}{3}\right) = -2$$

Question 241

If for all real triplets (a, b, c) , $f(x) = a + bx + cx^2$; then $\int_0^1 f(x) \, dx$ is equal to:

[Jan. 9, 2020 (I)]

Options:

A. $2 \left\{ 3f(1) + 2f\left(\frac{1}{2}\right) \right\}$

B. $\frac{1}{2} \left\{ f(1) + 3f\left(\frac{1}{2}\right) \right\}$

C. $\frac{1}{3} \left\{ f(0) + f\left(\frac{1}{2}\right) \right\}$

D. $\frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$

Answer: D

Solution:

$$\int_0^1 (a + bx + cx^2) dx = ax + \frac{bx^2}{2} + \frac{cx^3}{3} \Big|_0^1 = a + \frac{b}{2} + \frac{c}{3}$$

Now, $f(1) = a + b + c$, $f(0) = a$ and $f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4}$

$$\begin{aligned} \text{Now, } \frac{1}{6} \left(f(1) + f(0) + 4f\left(\frac{1}{2}\right) \right) \\ = \frac{1}{6} \left(a + b + c + a + 4 \left(a + \frac{b}{2} + \frac{c}{4} \right) \right) \\ = \frac{1}{6} (6a + 3b + 2c) = a + \frac{b}{2} + \frac{c}{3} \end{aligned}$$

Hence, $\int_0^1 f(x) dx = \frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$

Question 242

The value of $\int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx$ is equal to:

[Jan. 9, 2020 (I)]

Options:

A. 2π

B. $2\pi^2$

C. π^2

D. 4π

Answer: C

Solution:

$$\begin{aligned} & \int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx \\ &= \int_0^{\pi} \left[\frac{x \sin^8 x}{\sin^8 x + \cos^8 x} + \frac{(2\pi - x) \sin^8 x}{\sin^8 x + \cos^8 x} \right] dx \left[\because \int_0^{2\pi} f(x) dx = \int_0^{\pi} f(x) dx + \int_0^{\pi} f(2\pi - x) dx \right] \\ &= \int_0^{\pi} \frac{2\pi \sin^8 x}{\sin^8 x + \cos^8 x} dx \\ &= 2\pi \int_0^{\pi/2} \left[\frac{\sin^8 x}{\sin^8 x + \cos^8 x} + \frac{\cos^8 x}{\sin^8 x + \cos^8 x} \right] dx \\ &= 2\pi \int_0^{\pi/2} 1 dx = 2\pi \times \frac{\pi}{2} = \pi^2 \end{aligned}$$

Question 243

If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then:

[Jan. 8, 2020 (II)]

Options:

A. $\frac{1}{8} < I^2 < \frac{1}{4}$

B. $\frac{1}{9} < I^2 < \frac{1}{8}$

C. $\frac{1}{16} < I^2 < \frac{1}{9}$

D. $\frac{1}{6} < I^2 < \frac{1}{2}$

Answer: B

Solution:

$$f(x) = \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$$

$$f'(x) = \frac{-1}{2} \left(\frac{(6x^2 - 18x + 12)}{(2x^3 - 9x^2 + 12x + 4)^{3/2}} \right)$$

$$= \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{3/2}}$$

$$f(1) = \frac{1}{3} \text{ and } f(2) = \frac{1}{\sqrt{8}}$$

It is increasing function

$$\frac{1}{3} < I < \frac{1}{\sqrt{8}}$$

$$\frac{1}{9} < I^2 < \frac{1}{8}$$

Question 244

If $f(a + b + 1 - x) = f(x)$, for all x , where a and b are fixed positive real numbers, then

$\frac{1}{a+b} \int_a^b x(f(x) + f(x+1))dx$ is equal to:

[Jan. 7, 2020 (I)]

Options:

A. $\int_{a+1}^{b+1} f(x)dx$

B. $\int_{a-1}^{b-1} f(x)dx$

C. $\int_{a-1}^{b-1} f(x+1)dx$

D. $\int_{a+1}^{b+1} f(x+1)dx$

Answer: C

Solution:

$$I = \frac{1}{(a+b)} \int_a^b x[f(x) + f(x+1)]dx \dots\dots(i)$$

$$x \rightarrow a+b-x$$

$$I = \frac{1}{(a+b)} \int_a^b (a+b-x) [f(a+b-x) + f(a+b+1-x)] dx$$

$$I = \frac{1}{(a+b)} \int_a^b (a+b-x) [f(x+1) + f(x)] dx \dots (ii)$$

$$[\because \text{put } x \rightarrow x+1 \text{ in } f(a+b+1-x) = f(x)]$$

Add (i) and (ii)

$$2I = \int_a^b f(x+1) + f(x) dx$$

$$2I = \int_a^b f(x+1) dx + \int_a^b f(x) dx$$

$$= \int_a^b f(a+b+1-x) dx + \int_a^b f(x) dx$$

$$2I = 2 \int_a^b f(x) dx$$

$$\therefore \int_{a-1}^{b-1} f(x+1) dx [\because \text{Put } x \rightarrow x+1]$$

Question 245

The value of α for which $4\alpha \int_{-1}^2 e^{-\alpha|x|} dx = 5$, is :

[Jan. 7, 2020 (II)]

Options:

A. $\log_e 2$

B. $\log_e \left(\frac{3}{2} \right)$

C. $\log_e \sqrt{2}$

D. $\log_e \left(\frac{4}{3} \right)$

Answer: A

Solution:

$$4\alpha \left\{ \int_{-1}^0 e^{\alpha x} dx + \int_0^2 e^{-\alpha x} dx \right\} = 5$$

$$\Rightarrow 4\alpha \left\{ \frac{e^{\alpha x}}{\alpha} \Big|_{-1}^0 + \frac{e^{-\alpha x}}{-\alpha} \Big|_0^2 \right\} = 5$$

$$\Rightarrow 4\alpha \left\{ \left(\frac{1 - e^{-\alpha}}{\alpha} \right) - \left(\frac{e^{-2\alpha} - 1}{\alpha} \right) \right\} = 5$$

$$\Rightarrow 4(2 - e^{-\alpha} - e^{-2\alpha}) = 5$$

$$\text{Put } e^{-\alpha} = t$$

$$\Rightarrow 4t^2 + 4t - 3 = 0 \Rightarrow (2t+3)(2t-1) = 0$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = \log_e 2$$

Question 246

If θ_1 and θ_2 be respectively the smallest and the largest values of theta in $(0, 2\pi) - \{\pi\}$ which satisfy the equation, $2\cot^2 \theta - \frac{5}{\sin \theta} + 4 = 0$, then $\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$ is equal to:

[Jan. 7, 2020 (II)]

Options:

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{3} + \frac{1}{6}$

D. $\frac{\pi}{9}$

Answer: A

Solution:

$$2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$$

$$\frac{2\cos^2\theta}{\sin^2\theta} - \frac{5}{\sin\theta} + 4 = 0$$

$$\Rightarrow 2\cos^2\theta - 5\sin\theta + 4\sin^2\theta = 0, \sin\theta \neq 0$$

$$\Rightarrow 2\sin^2\theta - 5\sin\theta + 2 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) = 0$$

$$\therefore \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore \int_{\pi/6}^{5\pi/6} \cos^2 3\theta d\theta = \int_{\pi/6}^{5\pi/6} \frac{1 + \cos 6\theta}{2} d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 6\theta}{6} \right]_{\pi/6}^{5\pi/6} = \frac{1}{2} \left[\frac{5\pi}{6} - \frac{\pi}{6} + \frac{1}{6}(0 - 0) \right]$$

$$= \frac{1}{2} \cdot \frac{4\pi}{6} = \frac{\pi}{3}$$

Question 247

Let a function $f : [0, 5] \rightarrow \mathbb{R}$ be continuous, $f(1) = 3$ and F be defined as:

$$F(x) = \int_1^x t^2 g(t) dt, \text{ where } g(t) = \int_1^x f(u) du$$

Then for the function F , the point $x = 1$ is:

[Jan. 9, 2020 (H)]

Options:

A. a point of local minima.

B. not a critical point.

C. a point of local maxima.

D. a point of inflection.

Answer: A

Solution:

$$F(x) = \int_1^x t^2 g(t) dt$$

Differentiate by using Leibnitz's rule, we get

$$F'(x) = x^2 g(x) = x^2 \int_1^x f(u) du \dots\dots(i)$$

At $x = 1$

$$F'(1) = 1 \int_1^1 f(u) du = 0$$

Now, differentiate eqn (i)

$$F''(x) = x^2 f(x) - 2x \int_1^x f(u) du$$

At $x = 1$,

$$F''(1) = 1 \cdot f(1) - 2 \times 1 \cdot \int_1^1 f(u) du$$

$$= f(1) - 2 \times 0 = f(1)$$

$$F''(1) = 3$$

Then, for $F'(1) = 0$, $F''(1) = 3 > 0$

Hence, $x = 1$ is a point of local minima.

Question 248

$$\lim_{x \rightarrow 1} \left(\frac{\int_0^{(x-1)^2} t \cos(t^2) dt}{(x-1) \sin(x-1)} \right)$$

[Sep. 06, 2020 (I)]

Options:

A. is equal to $\frac{1}{2}$

B. is equal to 1

C. is equal to $-\frac{1}{2}$

D. (Bonus)

Answer: D

Solution:

$$\lim_{x \rightarrow 1} \frac{\frac{1}{2} \sin(x-1)^4}{(x-1) \sin(x-1)}$$

Let $x - 1 = h$ when $x \rightarrow 1$ then $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{\sin h^4}{h^4} \times \frac{h}{\sin h} \times h^2 = 1 \times 1 \times 0 = 0$$

(No any option is correct)

Question 249

If $\int (e^{2x} + 2e^x - e^{-x} - 1)e^{(e^x + e^{-x})} dx = g(x)e^{(e^x + e^{-x})} + c$, where c is a constant of integration, then $g(0)$ is equal to:

[Sep. 05, 2020 (I)]

Options:

A. e

B. e^2

C. 1

D. 2

Answer: D

Solution:

$$\int (e^{2x} + 2e^x - e^{-x} - 1) \cdot e^{(e^x + e^{-x})} dx$$

$$I = \int (e^{2x} + e^x - 1) \cdot e^{(e^x + e^{-x})} dx + \int (e^x - e^{-x}) e^{(e^x + e^{-x})} dx$$

$$= \int e^x (e^x + 1 - e^{-x}) \cdot e^{(e^x + e^{-x})} dx + e^{(e^x + e^{-x})}$$

$$= \int (e^x - e^{-x} + 1) e^{(e^x + e^{-x} + x)} dx + e^{(e^x + e^{-x})}$$

Let $e^x + e^{-x} + x = t \Rightarrow (e^x + e^{-x} + 1) dx = dt$

$$= \int e^t dt + e^{(e^x + e^{-x})} = e^t + e^{(e^x + e^{-x})} + C$$

$$= e^{(e^x + e^{-x} + x)} + e^{(e^x + e^{-x})} + C$$

$$= (e^x + 1) \cdot e^{(e^x + e^{-x})} + C$$

So, $g(x) = 1 + e^x$ and $g(0) = 2$

Question 250

If $\int \frac{\cos \theta}{5 + 7 \sin \theta - 2 \cos^2 \theta} d\theta = A \log_e |B(\theta)| + C$, where C is a constant of integration, then $\frac{B(\theta)}{A}$ can be:

[Sep. 05, 2020 (II)]

Options:

- A. $\frac{2 \sin \theta + 1}{\sin \theta + 3}$
- B. $\frac{2 \sin \theta + 1}{5(\sin \theta + 3)}$
- C. $\frac{5(\sin \theta + 3)}{2 \sin \theta + 1}$
- D. $\frac{5(2 \sin \theta + 1)}{\sin \theta + 3}$

Answer: D

Solution:

Let $\sin \theta = t \Rightarrow \cos \theta d\theta = dt$

$$\int \frac{\cos \theta}{5 + 7 \sin \theta - 2 \cos^2 \theta} d\theta = \frac{dt}{5 + 7t - 2 + 2t^2}$$

$$\Rightarrow \frac{1}{2} \int \frac{dt}{\left(t + \frac{7}{4}\right)^2 - \left(\frac{5}{4}\right)^2} = \frac{1}{5} \ln \left| \frac{t + \frac{1}{2}}{t + 3} \right| + C$$

$$= \frac{1}{5} \ln \left| \frac{2t + 1}{t + 3} \right| + C = \frac{1}{5} \ln \left| \frac{2 \sin \theta + 1}{\sin \theta + 3} \right| + C$$

$$\therefore B(\theta) = \frac{2 \sin \theta + 1}{2(\sin \theta + 3)} \text{ and } A = \frac{1}{5}$$

$$\Rightarrow \frac{B(\theta)}{A} = \frac{5(2 \sin \theta + 1)}{(\sin \theta + 3)}$$

Question 251

The integral $\int \left(\frac{x}{x \sin x + \cos x} \right)^2 dx$ is equal to (where C is a constant of integration):

[Sep. 04, 2020 (I)]

Options:

- A. $\tan x - \frac{x \sec x}{x \sin x + \cos x} + C$
- B. $\sec x + \frac{x \tan x}{x \sin x + \cos x} + C$

C. $\sec x - \frac{x \tan x}{x \sin x + \cos x} + C$

D. $\tan x + \frac{x \sec x}{x \sin x + \cos x} + C$

Answer: A

Solution:

$$\begin{aligned} & \int \frac{x^2}{(x \sin x + \cos x)^2} dx \\ & \because \frac{d}{dx}(x \sin x + \cos x) = x \cos x \\ & = \int \frac{x \cos x}{(x \sin x + \cos x)^2} \left(\frac{x}{\cos x} \right) dx \\ & = \frac{x}{\cos x} \left[\frac{-1}{x \sin x + \cos x} \right] - \int \frac{x \sin x + \cos x}{\cos^2 x} \left[\frac{-1}{x \sin x + \cos x} \right] dx \\ & = \frac{x}{\cos x} \left[\frac{-1}{x \sin x + \cos x} \right] + \int \sec^2 x dx \\ & = \frac{-x \sec x}{x \sin x + \cos x} + \tan x + C \end{aligned}$$

Question 252

Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2} dx$ ($x \geq 0$). Then $f(3) - f(1)$ is equal to :

[Sep. 04, 2020 (I)]

Options:

A. $-\frac{\pi}{12} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

B. $\frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

C. $-\frac{\pi}{6} + \frac{1}{2} + \frac{\sqrt{3}}{4}$

D. $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

Answer: D

Solution:

$$\begin{aligned} & \int \frac{\sqrt{x}}{(1+x)^2} dx \quad (x > 0) \\ & \text{Put } x = \tan^2 \theta \Rightarrow 2x dx = 2 \tan \theta \sec^2 \theta d\theta \\ & I = \int \frac{2 \tan^2 \theta \cdot \sec^2 \theta}{\sec^4 \theta} d\theta = \int 2 \sin^2 \theta d\theta \\ & = \theta - \frac{\sin 2\theta}{2} + C \\ & \Rightarrow f(x) = \theta - \frac{1}{2} \times \frac{2 \tan \theta}{1 + \tan^2 \theta} + C \\ & f(x) = \theta - \frac{\tan \theta}{1 + \tan^2 \theta} + C = \tan^{-1} \sqrt{x} - \frac{\sqrt{x}}{1+x} + C \end{aligned}$$

$$\begin{aligned} \text{Now } f(3) - f(1) &= \tan^{-1}(\sqrt{3}) - \frac{\sqrt{3}}{1+3} - \tan^{-1}(1) + \frac{1}{2} \\ &= \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4} \end{aligned}$$

Question 253

If $\int \sin^{-1} \left(\sqrt{\frac{x}{1+x}} \right) dx = A(x) \tan^{-1}(\sqrt{x}) + B(x) + C$, where C is a constant of integration, then the ordered pair $(A(x), B(x))$ can be :
[Sep. 03, 2020 (II)]

Options:

- A. $(x+1, -\sqrt{x})$
- B. $(x+1, \sqrt{x})$
- C. $(x-1, -\sqrt{x})$
- D. $(x-1, \sqrt{x})$

Answer: A

Solution:

$$\begin{aligned} I &= \int \sin^{-1} \left(\frac{\sqrt{x}}{\sqrt{1+x}} \right) dx = \int \tan^{-1} \sqrt{x} \cdot 1 dx \\ &= x \tan^{-1} \sqrt{x} - \int \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \cdot x dx + C \\ &= x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{t \cdot 2t dt}{1+t^2} + C \quad (\text{Put } x = t^2 \Rightarrow dx = 2t dt) \\ &= x \tan^{-1} \sqrt{x} - \int \frac{t^2}{1+t^2} dt + C \\ &= x \tan^{-1} \sqrt{x} - t + \tan^{-1} t + C \\ &= x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} \sqrt{x} + C \\ &= (x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + C \\ &\Rightarrow A(x) = x+1 \Rightarrow B(x) = -\sqrt{x} \end{aligned}$$

Question 254

The integral $\int_1^2 e^x \cdot x^x (2 + \log_e x) dx$ equals:
[Sep. 06, 2020 (II)]

Options:

- A. $e(4e+1)$
- B. $4e^2-1$
- C. $e(4e-1)$
- D. $e(2e-1)$

Answer: C

Solution:

$$\begin{aligned} I &= \int_1^2 e^x x^x (2 + \log_e x) dx \\ I &= \int_1^2 e^x x^x [1 + (1 + \log_e x)] dx \end{aligned}$$

$$= \int_1^2 e^x [x^x + x^x (1 + \log_e x)] dx$$

$$\therefore \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

$$\therefore I = [e^x x^x]_1^2$$

$$= e^2 \times 4 - e \times 1 = 4e^2 - e = e(4e - 1)$$

Question 255

If $I_1 = \int_0^1 (1 - x^{50})^{100} dx$ and $I_2 = \int_0^1 (1 - x^{50})^{101} dx$ such that $I_2 = \alpha I_1$ then α equals to :

[Sep. 06, 2020 (I)]

Options:

A. $\frac{5049}{5050}$

B. $\frac{5050}{5049}$

C. $\frac{5050}{5051}$

D. $\frac{5051}{5050}$

Answer: C

Solution:

$$I_2 = \int_0^1 (1 - x^{50})^{101} dx = \int_0^1 (1 - x^{50})(1 - x^{50})^{100} dx$$

$$I_2 = \int_0^1 (1 - x^{50})^{100} dx - \int_0^1 x \cdot x^{49} (1 - x^{50})^{100} dx$$

$$I_2 = I_1 + \left[\frac{x}{5050} (1 - x^{50})^{101} \right]_0^1 - \int_0^1 \frac{(1 - x^{50})^{101}}{5050} dx$$

$$I_2 = I_1 + 0 - \frac{I_2}{5050}$$

$$\Rightarrow \frac{5051}{5050} I_2 = I_1 \Rightarrow I_2 = \frac{5050}{5051} I_1$$

$$\Rightarrow \alpha = \frac{5050}{5051}$$

Question 256

The value of $\int_{-\pi/2}^{\pi/2} \frac{1}{1 + e^{\sin x}} dx$ is:

[Sep. 05, 2020 (I)]

Options:

A. $\frac{\pi}{4}$

B. π

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{2}$

Answer: C

Solution:

$$\begin{aligned}
 I &= \int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}} dx \\
 &= \int_{-\pi/2}^0 \frac{1}{1+e^{\sin x}} dx + \int_0^{\pi/2} \frac{1}{1+e^{\sin x}} dx \\
 &= \int_0^{\pi/2} \left(\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}} \right) dx \\
 &= \int_0^{\pi/2} \frac{1+e^{\sin x}}{1+e^{\sin x}} dx = \frac{\pi}{2}
 \end{aligned}$$

Question 257

Let $f(x) = |x - 2|$ and $g(x) = f(f(x))$, $x \in [0, 4]$ Then $\int_0^3 (g(x) - f(x)) dx$ is equal to:

[Sep. 04, 2020 (I)]

Options:

- A. 1
- B. 0
- C. $\frac{1}{2}$
- D. $\frac{3}{2}$

Answer: A

Solution:

$$\begin{aligned}
 f(x) &= |x - 2| = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases} \\
 g(x) &= f(f(x)) = \begin{cases} 2 - f(x), & f(x) < 2 \\ f(x) - 2, & f(x) \geq 2 \end{cases} \\
 &= \begin{cases} 2 - (2 - x), & 2 - x < 2, \quad x < 2 \\ (2 - x) - 2, & 2 - x \geq 2, \quad x < 2 \\ 2 - (x - 2), & x - 2 < 2, \quad x \geq 2 \\ (x - 2) - 2, & x - 2 \geq 2, \quad x \geq 2 \end{cases} \\
 &= \begin{cases} -x, & 0 < x \leq 0 \\ x, & 0 < x < 2 \\ 4 - x, & 2 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases} \\
 \therefore \int_0^3 [g(x) - f(x)] dx &= \int_0^2 x dx + \int_2^3 (4 - x) dx - \int_0^3 |x - 2| dx = 1
 \end{aligned}$$

Question 258

The integral $\int_{\pi/6}^{\pi/3} \tan^3 x \cdot \sin^2 3x (2 \sec^2 x \cdot \sin^2 3x + 3 \tan x \cdot \sin 6x) dx$ is equal to :

[Sep. 04, 2020 (II)]

Options:

A. $\frac{7}{18}$

B. $-\frac{1}{9}$

C. $-\frac{1}{18}$

D. $\frac{9}{2}$

Answer: C

Solution:

$$\begin{aligned} & \int_{\pi/6}^{\pi/3} \left[\frac{1}{2} \frac{d(\tan^4 x)}{dx} \cdot \sin^4 3x + \frac{1}{2} \tan^4 x \cdot \frac{d(\sin^4 3x)}{dx} \right] dx \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/3} d(\tan^4 x \cdot \sin^4 3x) dx \\ &= \left[\frac{\tan^4 x \sin^4 3x}{2} \right]_{\pi/6}^{\pi/3} = \frac{9 \cdot 0}{2} - \frac{\frac{1}{9} \cdot 1}{2} = -\frac{1}{18} \end{aligned}$$

Question 259

Let $\{x\}$ and $[x]$ denote the fractional part of x and the greatest integer less than or equal to x respectively of a real number x . If $\int_0^n \{x\} dx$, $\int_0^n [x] dx$ and $10(n^2 - n)$, ($n \in \mathbb{N}$, $n > 1$) are three consecutive terms of a G.P., then n is equal to _____.

[NA Sep. 04, 2020 (II)]

Answer: 21

Solution:

$$\begin{aligned} \int_0^n \{x\} dx &= n \int_0^1 x \cdot dx = \frac{n}{2} \\ \Rightarrow \int_0^n [x] dx &= \int_0^n (x - \{x\}) dx = \frac{n^2}{2} - \frac{n}{2} \end{aligned}$$

According to the questions,

$$\frac{n}{2}, \frac{n^2}{2} - \frac{n}{2}, 10(n^2 - n) \text{ are in GP}$$

$$\therefore \left(\frac{n^2 - n}{2} \right)^2 = \frac{n}{2} \times 10(n^2 - n)$$

$$\Rightarrow n^2 = 21n \Rightarrow n = 21$$

Question 260

$\int_{-\pi}^{\pi} |\pi - |x|| dx$ is equal to :
[Sep. 03, 2020 (I)]

Options:

A. $\sqrt{2}\pi^2$

B. $2\pi^2$

C. π^2

D. $\frac{\pi^2}{2}$

Answer: C

Solution:

$$\begin{aligned} I &= \int_{-\pi}^{\pi} |\pi - |x|| dx \quad [\because |\pi - |x|| \text{ is even}] \\ &= 2 \int_0^{\pi} |\pi - |x|| dx \\ &= 2 \int_0^{\pi} (\pi - x) dx \\ &= 2 \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} = 2 \left(\pi^2 - \frac{\pi^2}{2} \right) = \pi^2. \end{aligned}$$

Question261

If the value of the integral $\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx$ is $\frac{k}{6}$, then k is equal to:

[Sep. 03, 2020 (II)]

Options:

A. $2\sqrt{3} - \pi$

B. $2\sqrt{3}\pi$

C. $3\sqrt{2} + \pi$

D. $3\sqrt{2} - \pi$

Answer: A

Solution:

$$\begin{aligned} \frac{k}{6} &= \int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx \\ \text{Let } x &= \sin \theta; dx = \cos \theta d\theta \\ \text{then } \int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}} dx &= \int_0^{\pi/6} \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta \\ \therefore \frac{k}{6} &= \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos^2 \theta} \cos \theta d\theta \\ &\Rightarrow \frac{k}{6} = \int_0^{\pi/6} \tan^2 \theta d\theta = \int_0^{\pi/6} (\sec^2 \theta - 1) d\theta \\ &\Rightarrow \frac{k}{6} = (\tan \theta - \theta)_0^{\pi/6} = \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} \right) = \frac{2\sqrt{3} - \pi}{6} \\ &\Rightarrow k = 2\sqrt{3} - \pi \end{aligned}$$

Question262

The integral $\int_0^2 ||x-1| - x| dx$ is equal to _____.
[NA Sep. 02, 2020 (I)]

Answer: 1.50

Solution:

$$\begin{aligned}\int_0^2 ||x-1| - x| dx &= \int_0^1 |1-x-x| dx + \int_1^2 |x-1-x| dx \\ &= \int_0^1 (1-2x) dx + \int_1^2 (2x-1) dx \\ &= \left[x - x^2 \right]_0^1 + \left[x^2 - x \right]_1^2 \\ &= \frac{1}{2} - \frac{1}{4} + (1-1) - \left(\frac{1}{4} - \frac{1}{2} \right) + 2 - 1 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}\end{aligned}$$

Question 263

Let $[t]$ denote the greatest integer less than or equal to t .
Then the value of $\int_1^2 |2x - [3x]| dx$ is _____.
[NA Sep. 02, 2020 (II)]

Answer: 1

Solution:

$$\begin{aligned}\int_1^2 |2x - [3x]| dx &= \int_1^2 |3x - [3x] - x| dx \\ &= \int_1^2 |\{3x\} - x| dx = \int_1^2 (x - \{3x\}) dx \\ &= \int_1^2 x dx - \int_1^2 \{3x\} dx \\ &= \left[\frac{x^2}{2} \right]_1^2 - 3 \int_0^1 3x dx \\ &= \frac{(4-1)}{2} - 9 \left[\frac{x^2}{2} \right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1\end{aligned}$$