



Differential Equations

Question 1

Let f be a twice differentiable non-negative function such that $(f(x))^2 = 25 + \int_0^x ((f(t))^2 + (f'(t))^2) dt$. Then the mean of $f(\log_e(1)), f(\log_e(2)), \dots, f(\log_e(625))$ is equal to _____.

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Answer: 1565

Solution:

Let $y = f(x)$

$$y^2 = 25 + \int_0^x (f(t)^2 + f'(t)^2) dt$$

Differentiating w.r.t. x

$$2y \frac{dy}{dx} = y^2 + \left(\frac{dy}{dx} \right)^2$$

$$\Rightarrow \left(\frac{dy}{dx} - y \right)^2 = 0 \Rightarrow \frac{dy}{y} = dx$$

$$\Rightarrow \ln|y| = x + c$$

Such that

$$f(0)^2 = 25 + \int_0^0 (f^2(t) + f'(t)^2) dt$$

$$\Rightarrow f(0) = 5$$

$$\Rightarrow \ln 5 = c$$

$$\Rightarrow \ln|y| = x + \ln 5$$

$$\Rightarrow y = 5e^x$$

$$f(\ln k) = 5e^{\ln k} = 5k$$

Mean of $\{5, 10, \dots, 625 \times 5\}$

$$= \frac{(5 + 10 + \dots + 5 \times 625)}{625} = \frac{5(625)(626)}{625 \times 2}$$

$$= 5 \times 313$$

$$= 1565$$

Question 2

If the solution curve $y = f(x)$ of the differential equation

$$(x^2 - 4)y' - 2xy + 2x(4 - x^2)^2 = 0, x > 2,$$

passes through the point $(3, 15)$, then the local maximum value of f is _____

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Answer: 16

Solution:

$$(x^2 - 4)y' - 2xy + 2x(4 - x^2)^2 = 0, x > 2$$

rearranging terms.

$$(x^2 - 4) \frac{dy}{dx} - 2xy = -2x(x^2 - 4)^2$$

$$\frac{dy}{dx} - \frac{2x}{x^2 - 4}y = -2x(x^2 - 4)$$

this is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

$$P = -\frac{2x}{x^2 - 4}, Q = -2x(x^2 - 4)$$

finding integrating factor (I.F.).

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{2x}{x^2 - 4} dx}$$

$$\text{I.F.} = e^{-\ln(x^2 - 4)} = \frac{1}{x^2 - 4}$$

the solution is $y \cdot (I.F.) = \int Q \cdot (I.F.) dx + C$.

$$y \cdot \frac{1}{x^2 - 4} = \int -2x(x^2 - 4) \cdot \frac{1}{x^2 - 4} dx + C$$

$$\frac{y}{x^2 - 4} = \int -2x dx + C$$

$$\frac{y}{x^2 - 4} = -x^2 + C \Rightarrow y = (x^2 - 4)(C - x^2)$$

substitute point $(3, 15)$

$$15 = (3^2 - 4)(C - 3^2)$$

$$15 = 5(C - 9) \Rightarrow 3 = C - 9 \Rightarrow C = 12$$

$$f(x) = (x^2 - 4)(12 - x^2)$$

$$f(x) = -x^4 + 16x^2 - 48$$

finding critical points for local maximum.

$$f'(x) = -4x^3 + 32x$$

$$f'(x) = 0 \Rightarrow -4x(x^2 - 8) = 0$$

$$\text{since } x > 2, x^2 = 8 \Rightarrow x = \sqrt{8} = 2\sqrt{2}$$

$$f'((2\sqrt{2})^-) = \text{positive}, f'((2\sqrt{2})^+) = \text{negative}$$

so sign of $f'(x)$ changing positive to negative about $x = 2\sqrt{2}$ so $x = 2\sqrt{2}$ is point of local maxima

calculating local maximum value.

$$f(2\sqrt{2}) = ((2\sqrt{2})^2 - 4)(12 - (2\sqrt{2})^2)$$

$$= (8 - 4)(12 - 8)$$

$$= 4 \times 4 = 16$$

the local maximum value is 16.

Question3

Let f be a twice differentiable function such that

$$f(x) = \int_0^x \tan(t - x) dt - \int_0^x f(t) \tan t dt, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Then $f''\left(\frac{\pi}{6}\right) + 12f'\left(-\frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right)$ is equal to _____.

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Answer: 5

Solution:

Let

$$f(x) = \int_0^x \tan(t - x) dt - \int_0^x f(t) \tan t dt, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We have to find

$$f''\left(\frac{\pi}{6}\right) + 12f'\left(-\frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right).$$

Step 1: Simplify the first integral

Consider

$$\int_0^x \tan(t - x) dt.$$

Put

$$u = t - x \implies du = dt.$$

When $t = 0$, $u = -x$, and when $t = x$, $u = 0$.

So,

$$\int_0^x \tan(t - x) dt = \int_{-x}^0 \tan u du.$$

Now,

$$\int \tan u du = -\ln(\cos u).$$

Hence,

$$\int_{-x}^0 \tan u du = [-\ln(\cos u)]_{-x}^0 = -\ln(\cos 0) + \ln(\cos(-x)).$$

Since $\cos 0 = 1$ and $\cos(-x) = \cos x$,

$$\int_0^x \tan(t - x) dt = \ln(\cos x).$$

Therefore the given equation becomes

$$f(x) = \ln(\cos x) - \int_0^x f(t) \tan t dt.$$

So,

$$f(x) + \int_0^x f(t) \tan t dt = \ln(\cos x).$$

Step 2: Differentiate the equation

Differentiate both sides with respect to x .

Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx} \left(\int_0^x f(t) \tan t dt \right) = f(x) \tan x.$$

Also,

$$\frac{d}{dx} (\ln(\cos x)) = -\tan x.$$

Thus,

$$f'(x) + f(x) \tan x = -\tan x.$$

So,

$$f'(x) = -(f(x) + 1) \tan x.$$

Step 3: Find $f(x)$ explicitly

Now observe that $f(x) = -1$ satisfies the differential equation:

$$f'(x) = 0, \quad -(f(x) + 1) \tan x = 0.$$

Also from the original equation, put $x = 0$:

$$f(0) = \int_0^0 \tan(t - 0) dt - \int_0^0 f(t) \tan t dt = 0.$$

So,

$$f(0) = 0.$$

Let us solve the differential equation properly.

From

$$f'(x) + (f(x) + 1) \tan x = 0,$$

put

$$y(x) = f(x) + 1.$$

Then

$$y'(x) + y(x) \tan x = 0.$$

This gives

$$\frac{y'}{y} = -\tan x.$$

Integrating,

$$\ln |y| = \int -\tan x \, dx = \ln(\cos x) + C.$$

Hence,

$$y = C \cos x.$$

Therefore,

$$f(x) + 1 = C \cos x.$$

Using $f(0) = 0$,

$$0 + 1 = C \cos 0 = C.$$

So $C = 1$, and hence

$$f(x) + 1 = \cos x$$

which gives

$$f(x) = \cos x - 1.$$

Step 4: Find derivatives

Since

$$f(x) = \cos x - 1,$$

we get

$$f'(x) = -\sin x,$$

and

$$f''(x) = -\cos x.$$

Now compute required values:

1. $f''\left(\frac{\pi}{6}\right)$

$$f''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

2. $f'\left(-\frac{\pi}{6}\right)$

$$f'\left(-\frac{\pi}{6}\right) = -\sin\left(-\frac{\pi}{6}\right) = \frac{1}{2}.$$

So,

$$12f' \left(-\frac{\pi}{6} \right) = 12 \cdot \frac{1}{2} = 6,$$

$$3. f \left(\frac{\pi}{6} \right)$$

$$f \left(\frac{\pi}{6} \right) = \cos \left(\frac{\pi}{6} \right) - 1 = \frac{\sqrt{3}}{2} - 1.$$

Step 5: Add them

$$f'' \left(\frac{\pi}{6} \right) + 12f' \left(-\frac{\pi}{6} \right) + f \left(\frac{\pi}{6} \right)$$

$$= -\frac{\sqrt{3}}{2} + 6 + \left(\frac{\sqrt{3}}{2} - 1 \right)$$

$$= 6 - 1 = 5.$$

Hence, the required value is

5.

Question4

Let $y = y(x)$ be the solution of the differential equation

$$x \sin \left(\frac{y}{x} \right) dy = \left(y \sin \left(\frac{y}{x} \right) - x \right) dx, y(1) = \frac{\pi}{2} \text{ and let } \alpha = \cos \left(\frac{y(e^{12})}{e^{12}} \right). \text{ Then}$$

the number of integral value of p , for which the equation

$$x^2 + y^2 - 2px + 2py + \alpha + 2 = 0 \text{ represents a circle of radius } r \leq 6, \text{ is } \underline{\hspace{2cm}}$$

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Answer: 11

Solution:

We are given the differential equation

$$x \sin \left(\frac{y}{x} \right) dy = \left(y \sin \left(\frac{y}{x} \right) - x \right) dx$$

with initial condition $y(1) = \frac{\pi}{2}$.

Step 1: Express the differential equation in standard form

$$x \sin \left(\frac{y}{x} \right) \frac{dy}{dx} = y \sin \left(\frac{y}{x} \right) - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sin \left(\frac{y}{x} \right) - x}{x \sin \left(\frac{y}{x} \right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{\sin\left(\frac{y}{x}\right)}$$

Step 2: Substitution $y = vx$

Let $y = vx$, where v is a function of x .

Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substitute in the differential equation:

$$v + x \frac{dv}{dx} = v - \frac{1}{\sin v}$$

Simplify:

$$x \frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \sin v \, dv = -\frac{dx}{x}$$

Step 3: Integrate both sides

$$\int \sin v \, dv = -\int \frac{dx}{x}$$

$$-\cos v = -\ln|x| + C$$

$$\Rightarrow \cos v = \ln|x| + C$$

Now recall $v = \frac{y}{x}$, so:

$$\cos\left(\frac{y}{x}\right) = \ln|x| + C$$

Step 4: Apply the initial condition $y(1) = \frac{\pi}{2}$

$$x = 1, y = \frac{\pi}{2} \Rightarrow \frac{y}{x} = \frac{\pi}{2}$$

$$\cos\left(\frac{\pi}{2}\right) = 0 = \ln 1 + C \Rightarrow C = 0$$

So the particular solution is:

$$\cos\left(\frac{y}{x}\right) = \ln x$$

Step 5: Find α

We are told that

$$\alpha = \cos\left(\frac{y(e^{12})}{e^{12}}\right)$$

From the equation above:

$$\cos\left(\frac{y}{x}\right) = \ln x$$

Thus at $x = e^{12}$:

$$\alpha = \ln(e^{12}) = 12$$

$$\alpha = 12$$

Step 6: Equation of circle

Given:

$$x^2 + y^2 - 2px + 2py + \alpha + 2 = 0$$

Substitute $\alpha = 12$:

$$x^2 + y^2 - 2px + 2py + 14 = 0$$

Step 7: Write in standard form of circle

We know:

$$x^2 + y^2 - 2gx - 2fy + c = 0$$

represents a circle with center (g, f) and radius $r = \sqrt{g^2 + f^2 - c}$.

Compare with the given equation:

$$x^2 + y^2 - 2px + 2py + 14 = 0$$

We can write $+2py = -2fy \Rightarrow f = -p$.

Hence:

$$g = p, \quad f = -p, \quad c = 14$$

Radius:

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{p^2 + (-p)^2 - 14} = \sqrt{2p^2 - 14}$$

Step 8: Condition for $r \leq 6$

$$\sqrt{2p^2 - 14} \leq 6$$

$$\Rightarrow 2p^2 - 14 \leq 36$$

$$\Rightarrow 2p^2 \leq 50$$

$$\Rightarrow p^2 \leq 25$$

$$\Rightarrow -5 \leq p \leq 5$$

Step 9: Count the integral values

Integral values of p are:

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$$

Hence, total number of integral values = 11.

11

Question 5

Let $y = y(x)$ be the solution of the differential equation

$$(\tan x)^{1/2} dy = (\sec^3 x - (\tan x)^{3/2} y) dx, \quad 0 < x < \frac{\pi}{2}, \quad y\left(\frac{\pi}{4}\right) = \frac{6\sqrt{2}}{5}. \text{ If}$$

$$y\left(\frac{\pi}{3}\right) = \frac{4}{5}\alpha, \text{ then } \alpha^4 \text{ equals}$$

_____.

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Answer: 48

Solution:

$$\frac{dy}{dx} + y \tan x = \frac{\sec^3 x}{\sqrt{\tan x}}$$

$$\text{IF} = e^{\int \tan x \, dx} = \sec x$$

$$y \sec x = \int \frac{\sec^4 x}{\sqrt{\tan x}} \, dx$$

$$y \sec x = \int \frac{(1+\tan^2 x) \sec^2 x}{\sqrt{\tan x}} \, dx$$

$$\tan x = t$$

$$y \sec x = 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + c$$

$$y\left(\frac{\pi}{4}\right) = \frac{6\sqrt{2}}{5}$$

$$\frac{6\sqrt{2}}{5} \times \sqrt{2} = 2 + \frac{2}{5} + c \rightarrow \frac{12}{5} = \frac{12}{5} + c$$

$$\rightarrow c = 0$$

$$y\left(\frac{\pi}{3}\right) = \frac{8}{5} \cdot 3^{1/4} \rightarrow \alpha = 2 \times 3^{1/4}$$

$$\therefore \alpha^4 = 48$$

Question 6

Let $y = y(x)$ be the solution of the differential equation

$\left(x^2 - x\sqrt{x^2 - 1}\right)dy + \left(y\left(x - \sqrt{x^2 - 1}\right) - x\right)dx = 0, x \geq 1$. If $y(1) = 1$, then the greatest integer less than $y(\sqrt{5})$ is _____.

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Answer: 3

Solution:

First write the given differential equation in the standard linear form.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x - \sqrt{x^2 - 1}}$$

Now simplify the right side by rationalising:

$$\frac{1}{x - \sqrt{x^2 - 1}} = \frac{x + \sqrt{x^2 - 1}}{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})} = \frac{x + \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)} = x + \sqrt{x^2 - 1}$$

So the equation becomes:

$$\frac{dy}{dx} + \frac{y}{x} = x + \sqrt{x^2 - 1}$$

This is a linear differential equation with integrating factor (I.F.):

$$\text{I.f} = e^{\int \frac{1}{x} dx} = x$$

Multiply the whole equation by x :

$$y \cdot x = \int (x + \sqrt{x^2 - 1}) x dx$$

Now integrate term by term:

$$= \frac{x^2}{2} + \frac{(x^2 - 1)^{3/2}}{3} + C$$

Use the initial condition $y(1) = 1$ to find C .

$$\text{Given } y(1) = 1$$

Substitute $x = 1$ in the expression for yx :

$$1 = \frac{1}{2} + C \rightarrow C = \frac{1}{2}$$

So,

$$yx = \frac{x^2}{2} + \frac{(x^2 - 1)^{3/2}}{3} + \frac{1}{2}$$

Divide by x :

$$y = \frac{x}{2} + \frac{(x^2 - 1)^{3/2}}{3x} + \frac{1}{2x}$$

Now put $x = \sqrt{5}$:

$$\left[y(\sqrt{5}) \right] = \left[\frac{\sqrt{5}}{2} + \frac{8}{3} + \frac{1}{2\sqrt{5}} \right] = 3$$

Question7

Let $y = y(x)$ be the solution curve of the differential equation $(1 + x^2)dy + (y - \tan^{-1} x)dx = 0, y(0) = 1$. Then the value of $y(1)$ is :

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Options:

A.

$$\frac{4}{e^{\pi/4}} - \frac{\pi}{2} - 1$$

B.

$$\frac{2}{e^{\pi/4}} + \frac{\pi}{4} - 1$$

C.

$$\frac{4}{e^{\pi/4}} + \frac{\pi}{2} - 1$$

D.

$$\frac{2}{e^{\pi/4}} - \frac{\pi}{4} - 1$$

Answer: B

Solution:

Given

$$(1+x^2)dy + (y - \tan^{-1}x)dx = 0, \quad y(0) = 1,$$

divide by dx :

$$(1+x^2)\frac{dy}{dx} + y - \tan^{-1}x = 0$$

$$\frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{\tan^{-1}x}{1+x^2}.$$

This is linear. Integrating factor:

$$\mu(x) = \exp\left(\int \frac{1}{1+x^2} dx\right) = e^{\tan^{-1}x}.$$

So

$$\frac{d}{dx}\left(ye^{\tan^{-1}x}\right) = e^{\tan^{-1}x} \cdot \frac{\tan^{-1}x}{1+x^2}.$$

Let $t = \tan^{-1}x$, then $dt = \frac{dx}{1+x^2}$. Hence

$$ye^t = \int te^t dt + C = e^t(t-1) + C.$$

Therefore

$$y = t - 1 + Ce^{-t} = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}.$$

Use $y(0) = 1$ and $\tan^{-1}0 = 0$:

$$1 = 0 - 1 + C \implies C = 2.$$

So

$$y(1) = \frac{\pi}{4} - 1 + 2e^{-\pi/4} = \frac{2}{e^{\pi/4}} + \frac{\pi}{4} - 1.$$

Correct option: B.

Question 8

Let $y = y(x)$ be the solution of the differential equation

$$\sec x \frac{dy}{dx} - 2y = 2 + 3 \sin x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$y(0) = -\frac{7}{4}$. Then $y\left(\frac{\pi}{6}\right)$ is equal to :

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Options:

A.

$$-\frac{5}{2}$$

B.

$$-3\sqrt{2} - 7$$

C.

$$-\frac{5}{4}$$

D.

$$-3\sqrt{3} - 7$$

Answer: A

Solution:

$$\frac{dy}{dx} - 2 \cos x \cdot y = \cos x (2 + 3 \sin x)$$

$$I \cdot F = e^{-2 \sin x}$$

$$G \cdot Sy \cdot e^{-2 \sin x} = \int e^{-2 \sin x} \cos x (2 + 3 \sin x) dx$$

$$\text{Put } \sin x = t \\ \cos x \cdot dx = dt$$

$$y = e^{-2 \sin x} \left(\frac{2+3 \sin x}{-2} - \frac{3}{4} \right) + c$$

$$\text{When } x = 0, c = 0$$

$$\text{When } x = \frac{\pi}{6}, y = -\frac{5}{2}$$

Question9

Let the solution curve of the differential equation

$xdy - ydx = \sqrt{x^2 + y^2}dx, x > 0, y(1) = 0$, be $y = y(x)$. Then $y(3)$ is equal to

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Options:

A.

6

B.

4

C.

1

D.

2

Answer: B

Solution:

$$\begin{aligned} xdy - ydx &= \sqrt{x^2 + y^2}dx \\ \frac{xdy - ydx}{x^2} &= \frac{\sqrt{x^2 + y^2}}{x^2}dx \end{aligned}$$

$$\int \frac{d(y/x)}{\sqrt{1 + (y/x)^2}} = \int \frac{1}{x}dx$$

$$\log \left(\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x} \right)^2} \right) = \log x + \log C$$

$$y + \sqrt{x^2 + y^2} = x^2$$

$$y - \sqrt{x^2 + y^2} = -1$$

$$2y = x^2 - 1$$

$$y(3) = 4$$

Question 10

If $y = y(x)$ satisfies the differential equation

$$16(\sqrt{x + 9\sqrt{x}})(4 + \sqrt{9 + \sqrt{x}}) \cos y \, dy = (1 + 2 \sin y) \, dx, \, x > 0 \text{ and } y(256) = \frac{\pi}{2}, \, y(49) = \alpha, \text{ then } 2 \sin \alpha \text{ is equal to :}$$

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Options:

A.

$$2\sqrt{2} - 1$$

B.

$$\sqrt{2} - 1$$

C.

$$2(\sqrt{2} - 1)$$

D.

$$3(\sqrt{2} - 1)$$

Answer: A

Solution:

$$\begin{aligned} \int \frac{\cos y}{1 + 2 \sin y} \, dy &= \int \frac{1}{16(\sqrt{x + 9\sqrt{x}})(4 + \sqrt{9 + \sqrt{x}})} \, dx \\ \Rightarrow \frac{1}{2} \int \frac{2 \cos y}{1 + 2 \sin y} \, dy &= \int \frac{1}{4t} \, dt \end{aligned}$$

$$\text{Put } 4 + \sqrt{9 + \sqrt{x}} = t$$

$$\Rightarrow \frac{1}{2\sqrt{9 + \sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{1}{4\sqrt{x + 9\sqrt{x}}} dx = dt$$

$$\Rightarrow \frac{1}{2} \log |1 + 2 \sin y| = \frac{1}{4} \log |t| + \log c \Rightarrow \sqrt{1 + 2 \sin y} = ct^{1/4} \Rightarrow \frac{1}{4} \log |t| + \log c$$

$$\Rightarrow \sqrt{1 + 2 \sin y} = ct^{1/4} \Rightarrow \sqrt{1 + 2 \sin y} = c(4 + \sqrt{9 + \sqrt{x}})^{1/4}$$

$$\left(256, \frac{\pi}{2}\right) \Rightarrow \sqrt{3} = c(\sqrt{3}) \Rightarrow c = 1$$

$$(49, \alpha) \Rightarrow \sqrt{1 + 2 \sin \alpha} = (1)8^{1/4}$$

$$\Rightarrow 1 + 2 \sin \alpha = 2\sqrt{2} \Rightarrow 2 \sin \alpha = 2\sqrt{2} - 1$$

Question 11

Let $y = y(x)$ be the solution of the differential equation $x^4 dy + (4x^3 y + 2 \sin x) dx = 0, x > 0, y\left(\frac{\pi}{2}\right) = 0$.

Then $\pi^4 y\left(\frac{\pi}{3}\right)$ is equal to :

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Options:

A.

92

B.

72

C.

64

D.

81

Answer: D

Solution:

$$x^4 dy + (4x^3 y + 2 \sin x) dx = 0$$

$$\frac{dy}{dx} + \frac{4x^3}{x^4} y = \frac{-2 \sin x}{x^4}$$

$$\frac{dy}{dx} + \frac{4}{x} y = \frac{-2 \sin x}{x^4}$$

$$\text{IF} = e^{\int \frac{4}{x} dx} = e^{\ln x^4} = x^4$$

$$x^4 \frac{dy}{dx} + 4x^3 y = -2 \sin x$$

$$\frac{d}{dx} (x^4 y) = -2 \sin x$$

$$\Rightarrow x^4 y = 2 \cos x + c$$

$$y \left(\frac{\pi}{2} \right) = 0$$

$$\Rightarrow c = 0$$

$$\therefore x^4 y = \cos x$$

$$\Rightarrow y = \frac{2 \cos x}{x^4}$$

$$y \left(\frac{\pi}{3} \right) = \frac{2 \cos \left(\frac{\pi}{3} \right)}{\left(\frac{\pi}{3} \right)^4} = \frac{1}{\left(\frac{\pi}{3} \right)^4} = \frac{81}{\pi^4}$$

$$\pi^4 y \left(\frac{\pi}{3} \right) = 81$$

Question 12

Let $y = y(x)$ be the solution of the differential equation

$$x \frac{dy}{dx} - \sin 2y = x^3 (2 - x^3) \cos^2 y, x \neq 0.$$

If $y(2) = 0$, then $\tan(y(1))$ is equal to

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Options:

A.

$$-\frac{7}{4}$$

B.

$$-\frac{3}{4}$$

C.

$$\frac{3}{4}$$

D.

$$\frac{7}{4}$$

Answer: D

Solution:

Given equation is :

$$x \frac{dy}{dx} - \sin 2y = x^3 (2 - x^3) \cos^2 y, \quad x \neq 0$$

Divide by $x \cos^2 y$

$$\sec^2 y \frac{dy}{dx} - \frac{\sin 2y}{x \cos^2 y} = x^2 (2 - x^3)$$

$$\sec^2 y \frac{dy}{dx} - \frac{2 \tan y}{x} = 2x^2 - x^5$$

Substitute $\tan y = t$ and differentiate w.r.t x

$$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

Equation (1) becomes linear differential equation.

$$\frac{dt}{dx} - \frac{2}{x} t = 2x^2 - x^5$$

$$\text{Integrating factor } IF = e^{\int (-\frac{2}{x}) dx} = e^{-2 \ln x} = e^{\ln(\frac{1}{x^2})} = \frac{1}{x^2}$$

Solution of linear differential equation :

$$t \cdot IF = \int (IF) (2x^2 - x^5) dx$$

$$\Rightarrow t \cdot \frac{1}{x^2} = \int \frac{1}{x^2} (2x^2 - x^5) dx$$

$$\Rightarrow \frac{t}{x^2} = \int (2 - x^3) dx = 2x - \frac{x^4}{4} + C$$

put value of $t = \tan y$

$$\frac{\tan y}{x^2} = 2x - \frac{x^4}{4} + C$$

using condition $y(2) = 0$ to find C .

$$\frac{\tan(0)}{2^2} = 2 \times 2 - \frac{2^4}{4} + C \Rightarrow C = 0$$

So, solution of differential equation is

$$\tan y = 2x^3 - \frac{x^6}{4}$$

to calculate $\tan(y(1))$, put $x = 1$ in the solution.

$$\tan(y(1)) = 2(1)^3 - \frac{(1)^6}{4}$$

$$= 2 - \frac{1}{4} = \frac{7}{4}$$

Question 13

Let $y = y(x)$ be the solution of the differential equation $x \frac{dy}{dx} - y = x^2 \cot x$, $x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, then

$6y\left(\frac{\pi}{6}\right) - 8y\left(\frac{\pi}{4}\right)$ is equal to :

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Options:

A.

$$-3\pi$$

B.

$$3\pi$$

C.

$$-\pi$$

D.

$$\pi$$

Answer: C

Solution:

The given equation is:

$$x \frac{dy}{dx} - y = x^2 \cot x$$

To get it into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$, we divide the entire equation by x :

$$\frac{dy}{dx} - \frac{1}{x}y = x \cot x$$

Here, $P(x) = -\frac{1}{x}$ and $Q(x) = x \cot x$.

Finding the Integrating Factor (I.F.)

The integrating factor is calculated as:

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$$

The solution to the differential equation is:

$$y \cdot (I.F.) = \int Q(x) \cdot (I.F.) dx$$

$$\frac{y}{x} = \int (x \cot x) \cdot \frac{1}{x} dx$$

$$\frac{y}{x} = \int \cot x dx$$

$$\frac{y}{x} = \ln(\sin x) + C$$

$$y = x \ln(\sin x) + Cx$$

Applying the Initial Condition

We are given $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$. Plugging these values in:

$$\frac{\pi}{2} = \frac{\pi}{2} \ln\left(\sin \frac{\pi}{2}\right) + C\left(\frac{\pi}{2}\right)$$

So, the particular solution is $y = x \ln(\sin x) + x$.

We need to find the value of $6y\left(\frac{\pi}{6}\right) - 8y\left(\frac{\pi}{4}\right)$.

- Calculate $y\left(\frac{\pi}{6}\right)$:

$$y\left(\frac{\pi}{6}\right) = \frac{\pi}{6} \ln\left(\sin \frac{\pi}{6}\right) + \frac{\pi}{6} = \frac{\pi}{6} \ln\left(\frac{1}{2}\right) + \frac{\pi}{6} = \frac{\pi}{6}(1 - \ln 2)$$

- Calculate $y\left(\frac{\pi}{4}\right)$:

$$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \ln\left(\sin \frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{\pi}{4} \ln\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4} = \frac{\pi}{4} \ln\left(2^{-1/2}\right) + \frac{\pi}{4} = \frac{\pi}{4}\left(1 - \frac{1}{2} \ln 2\right)$$

- Calculate $y\left(\frac{\pi}{4}\right)$:

$$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \ln\left(\sin \frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{\pi}{4} \ln\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4} = \frac{\pi}{4} \ln\left(2^{-1/2}\right) + \frac{\pi}{4} = \frac{\pi}{4}\left(1 - \frac{1}{2} \ln 2\right)$$

Now, substitute these into the expression:

$$\begin{aligned} 6y\left(\frac{\pi}{6}\right) - 8y\left(\frac{\pi}{4}\right) &= 6\left[\frac{\pi}{6}(1 - \ln 2)\right] - 8\left[\frac{\pi}{4}\left(1 - \frac{1}{2} \ln 2\right)\right] = \pi(1 - \ln 2) - 2\pi\left(1 - \frac{1}{2} \ln 2\right) \\ &= \pi - \pi \ln 2 - 2\pi + \pi \ln 2 = -\pi \end{aligned}$$

Question 14

Let $y = y(x)$ be the solution curve of the differential equation

$(1 + \sin x) \frac{dy}{dx} + (y + 1) \cos x = 0$, $y(0) = 0$. If the curve $y = y(x)$ passes through the point $\left(\alpha, \frac{-1}{2}\right)$,

then a value of α is:

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Options:

A.

$$\frac{\pi}{6}$$

B.

$$\frac{\pi}{4}$$

C.

$$\frac{\pi}{3}$$

D.

$$\frac{\pi}{2}$$

Answer: D

Solution:

$$\int \frac{dy}{y+1} = - \int \frac{\cos x}{1+\sin x} dx$$

$$\text{On the left side, } \int \frac{dy}{y+1} = \ln(y+1).$$

$$\text{On the right side, take } 1 + \sin x \text{ as substitution: let } u = 1 + \sin x, \text{ then } du = \cos x dx. \text{ So,}$$
$$- \int \frac{\cos x}{1+\sin x} dx = - \int \frac{du}{u} = - \ln(1 + \sin x).$$

$$\text{So we get } \Rightarrow \ln(y+1) = -\ln(1 + \sin x) + C$$

Now use the initial condition $y(0) = 0$. Put $x = 0$ and $y = 0$ in the equation:

$$\therefore y(0) = 0 \rightarrow \ln(0+1) = -\ln(1 + \sin 0) + C$$

$$\text{That is } 0 = -\ln(1) + C \Rightarrow C = 0.$$

$$\text{So } \Rightarrow \ln(y+1) = -\ln(1 + \sin x)$$

$$\text{Taking antilog on both sides, } \Rightarrow y+1 = \frac{1}{1+\sin x}$$

Given that the curve passes through $(\alpha, -\frac{1}{2})$, substitute $y = -\frac{1}{2}$ and $x = \alpha$:

$$-\frac{1}{2} + 1 = \frac{1}{1+\sin \alpha}$$

$$\text{So } \frac{1}{2} = \frac{1}{1+\sin \alpha} \Rightarrow 1 + \sin \alpha = 2 \Rightarrow \sin \alpha = 1$$

$$\text{Now } \sin \alpha = 1 \text{ gives } \Rightarrow \alpha = \frac{\pi}{2} \text{ (as per options)}$$

Question 15

If the curve $y = f(x)$ passes through the point $(1, e)$ and satisfies the differential equation $dy = y(2 + \log_e x)dx$, $x > 0$, then $f(e)$ is equal to :

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Options:

A.

$$e^e$$

B.

$$e^{e^2}$$

C.

$$e^{2e}$$

D.

$$e^{2e}$$

Answer: C

Solution:

$$\frac{dy}{dx} = y(2 + \ln x)$$

To solve this, first separate the variables by dividing both sides by y :

$$\int \frac{dy}{y} = \int (2 + \ln x) dx$$

Now integrate both sides. The left side gives $\ln y$. For the right side, integrate term by term:

$$\ln y = 2x + x \ln x - x + C$$

Combine like terms $2x - x = x$:

$$\ln y = x + x \ln x + C$$

Since it passes through $(1, e)$, put $x = 1$ and $y = e$ in the equation. Then $\ln y = \ln e = 1$ and $\ln 1 = 0$:

$$1 = 1 + 0 + C \rightarrow C = 0$$

So the required relation becomes:

$$\ln y = x + x \ln x$$

Taking antilog (exponential) on both sides:

$$f(x) = y = e^{x+x \ln x}$$

Now substitute $x = e$. Also, $\ln e = 1$:

$$\Rightarrow f(e) = e^{e+e} = e^{2e}$$

Question 16

Let $x = x(y)$ be the solution of the differential equation $2y^2 \frac{dx}{dy} - 2xy + x^2 = 0, y > 1, x(e) = e$.

Then $x(e^2)$ is equal to:

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Options:

A.

$$\frac{3}{2}e^2$$

B.

$$\frac{2}{3}e^2$$

C.

$$e^2$$

D.

$$2e^2$$

Answer: B

Solution:

$$2y(ydx - xdy) + x^2dy = 0$$

$$\Rightarrow -2yx^2 d\left(\frac{y}{x}\right) + x^2dy = 0$$

$$\Rightarrow -2y d\left(\frac{y}{x}\right) + dy = 0$$

$$\Rightarrow -2d\left(\frac{y}{x}\right) + \frac{1}{y}dy = 0$$

$$\Rightarrow -\frac{2y}{x} + \log_e y = C$$

$$\text{Given } x(e) = e$$

$$\Rightarrow C = -1$$

$$\therefore -\frac{2y}{x} + \log_e y = -1$$

$$\Rightarrow \frac{2y}{x} - \log_e y = 1$$

Put $y = e^2$, we get

$$\Rightarrow \frac{2e^2}{x} - 2 = 1 \rightarrow \frac{2e^2}{x} = 3 \rightarrow x = \frac{2e^2}{3}$$

$$\Rightarrow x(e^2) = \frac{2e^2}{3}$$

Question 17

Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (1 + x + x^2)(1 - y + y^2)$, $y(0) = \frac{1}{2}$. Then $(2y(1) - 1)$ is equal to

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Options:

A.

$$\sqrt{3} \tan\left(\frac{11\sqrt{3}}{6}\right)$$

B.

$$\frac{\sqrt{3}}{2} \tan\left(\frac{11\sqrt{3}}{12}\right)$$

C.

$$\sqrt{3} \tan\left(\frac{11\sqrt{3}}{12}\right)$$

D.

$$\frac{\sqrt{3}}{2} \tan\left(\frac{11\sqrt{3}}{6}\right)$$

Answer: C

Solution:

Given

$$\frac{dy}{dx} = (1 + x + x^2)(1 - y + y^2), \quad y(0) = \frac{1}{2}$$

We will solve it by the method of separation of variables.

Since

$$\frac{dy}{1-y+y^2} = (1+x+x^2) dx$$

Now simplify the denominator on the left:

$$1 - y + y^2 = y^2 - y + 1 = \left(y - \frac{1}{2}\right)^2 + \frac{3}{4}$$

So,

$$\int \frac{dy}{\left(y - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \int (1 + x + x^2) dx$$

Using the standard result

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right)$$

we get

$$\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2y-1}{\sqrt{3}} \right) = x + \frac{x^2}{2} + \frac{x^3}{3} + C$$

Now use the initial condition $y(0) = \frac{1}{2}$.

Then

$$2y - 1 = 0$$

so

$$\frac{2}{\sqrt{3}} \tan^{-1}(0) = C \Rightarrow C = 0$$

Hence,

$$\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2y-1}{\sqrt{3}} \right) = x + \frac{x^2}{2} + \frac{x^3}{3}$$

At $x = 1$,

$$x + \frac{x^2}{2} + \frac{x^3}{3} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

Therefore,

$$\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2y(1)-1}{\sqrt{3}} \right) = \frac{11}{6}$$

So

$$\tan^{-1} \left(\frac{2y(1)-1}{\sqrt{3}} \right) = \frac{11\sqrt{3}}{12}$$

Taking tangent on both sides,

$$\frac{2y(1)-1}{\sqrt{3}} = \tan \left(\frac{11\sqrt{3}}{12} \right)$$

Hence,

$$2y(1) - 1 = \sqrt{3} \tan \left(\frac{11\sqrt{3}}{12} \right)$$

So the correct answer is

$$\boxed{\sqrt{3} \tan \left(\frac{11\sqrt{3}}{12} \right)}$$

Therefore, the correct option is ☐ C.

Question 18

Let $y = y(x)$ be the solution of the differential equation :

$$\frac{dy}{dx} + \left(\frac{6x^2 + (3x^2 + 2x^3 + 4)e^{-2x}}{(x^3 + 2)(2 + e^{-2x})} \right) y = 2 + e^{-2x}$$

$x \in (-1, 2)$, satisfying $y(0) = \frac{3}{2}$. If $y(1) = \alpha(2 + e^{-2})$, then α is equal to :

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Options:

A.

$$\frac{13}{8}$$

B.

$$\frac{6}{13}$$

C.

$$\frac{12}{13}$$

D.

$$\frac{13}{12}$$

Answer: D

Solution:

Given differential equation is

$$\frac{dy}{dx} + \left(\frac{6x^2 + (3x^2 + 2x^3 + 4)e^{-2x}}{(x^3 + 2)(2 + e^{-2x})} \right) y = 2 + e^{-2x}$$

with initial condition

$$y(0) = \frac{3}{2}.$$

We have to find $y(1) = \alpha(2 + e^{-2})$.

Step 1: Compare with linear differential equation

This is of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where

$$P(x) = \frac{6x^2 + (3x^2 + 2x^3 + 4)e^{-2x}}{(x^3 + 2)(2 + e^{-2x})}, \quad Q(x) = 2 + e^{-2x}.$$

To solve, we find the integrating factor:

$$\text{I.F.} = e^{\int P(x) dx}.$$

Step 2: Simplify $P(x)$

Notice that

$$\frac{d}{dx}(x^3 + 2) = 3x^2,$$

and

$$\frac{d}{dx}(2 + e^{-2x}) = -2e^{-2x}.$$

Now consider

$$\frac{d}{dx}[(x^3 + 2)(2 + e^{-2x})].$$

Using product rule,

$$= (3x^2)(2 + e^{-2x}) + (x^3 + 2)(-2e^{-2x})$$

$$= 6x^2 + 3x^2e^{-2x} - 2x^3e^{-2x} - 4e^{-2x}.$$

This does not match directly. So let us try another observation.

Write numerator as

$$6x^2 + (3x^2 + 2x^3 + 4)e^{-2x}.$$

Now check

$$\frac{d}{dx}[(x^3 + 2)(2 + e^{-2x})] = 6x^2 + (3x^2 - 2x^3 - 4)e^{-2x}.$$

Hence if we take

$$\frac{d}{dx}[(x^3 + 2)(2 + e^{-2x})^{-1}]$$

it becomes complicated. So instead, let us look for a useful factorization.

Observe that

$$6x^2 + (3x^2 + 2x^3 + 4)e^{-2x} = 3x^2(2 + e^{-2x}) + (x^3 + 2) \cdot 2e^{-2x}.$$

Therefore

$$P(x) = \frac{3x^2(2 + e^{-2x}) + (x^3 + 2) \cdot 2e^{-2x}}{(x^3 + 2)(2 + e^{-2x})}$$

$$= \frac{3x^2}{x^3 + 2} + \frac{2e^{-2x}}{2 + e^{-2x}}.$$

So,

$$P(x) = \frac{3x^2}{x^3 + 2} + \frac{2e^{-2x}}{2 + e^{-2x}}.$$

Step 3: Find integrating factor

Now

$$\int P(x) dx = \int \frac{3x^2}{x^3 + 2} dx + \int \frac{2e^{-2x}}{2 + e^{-2x}} dx.$$

For the first integral:

Let $u = x^3 + 2$, then $du = 3x^2 dx$.

So,

$$\int \frac{3x^2}{x^3+2} dx = \ln(x^3 + 2).$$

For the second integral, note that

$$\frac{d}{dx}(2 + e^{-2x}) = -2e^{-2x}.$$

Hence

$$\int \frac{2e^{-2x}}{2+e^{-2x}} dx = -\ln(2 + e^{-2x}).$$

Therefore,

$$\int P(x) dx = \ln(x^3 + 2) - \ln(2 + e^{-2x}) = \ln\left(\frac{x^3+2}{2+e^{-2x}}\right).$$

So the integrating factor is

$$\text{I.F.} = e^{\int P(x) dx} = \frac{x^3+2}{2+e^{-2x}}.$$

Step 4: Multiply the equation by the integrating factor

Multiplying throughout by

$$\frac{x^3+2}{2+e^{-2x}},$$

we get

$$\frac{x^3+2}{2+e^{-2x}} \frac{dy}{dx} + \frac{x^3+2}{2+e^{-2x}} P(x)y = (x^3 + 2).$$

So the left side becomes

$$\frac{d}{dx} \left[y \cdot \frac{x^3+2}{2+e^{-2x}} \right].$$

Hence

$$\frac{d}{dx} \left[y \cdot \frac{x^3+2}{2+e^{-2x}} \right] = x^3 + 2.$$

Integrating both sides,

$$y \cdot \frac{x^3+2}{2+e^{-2x}} = \int (x^3 + 2) dx = \frac{x^4}{4} + 2x + C.$$

Thus,

$$y = \frac{2+e^{-2x}}{x^3+2} \left(\frac{x^4}{4} + 2x + C \right).$$

Step 5: Use initial condition

Given

$$y(0) = \frac{3}{2}.$$

Substitute $x = 0$:

$$\frac{3}{2} = \frac{2+e^0}{0^3+2} (0 + 0 + C) = \frac{3}{2} C.$$

So,

$$C = 1.$$

Hence solution is

$$y = \frac{2+e^{-2x}}{x^3+2} \left(\frac{x^4}{4} + 2x + 1 \right).$$

Step 6: Find $y(1)$

Substitute $x = 1$:

$$y(1) = \frac{2+e^{-2}}{1^3+2} \left(\frac{1}{4} + 2 + 1 \right).$$

$$y(1) = \frac{2+e^{-2}}{3} \left(\frac{1}{4} + 3 \right) = \frac{2+e^{-2}}{3} \cdot \frac{13}{4}$$

$$y(1) = \frac{13}{12} (2 + e^{-2}).$$

Given that

$$y(1) = \alpha(2 + e^{-2}),$$

so

$$\alpha = \frac{13}{12}.$$

Final Answer

$$\boxed{\frac{13}{12}}$$

So, the correct option is **D**.

Question 19

Let $f: [1, \infty) \rightarrow \mathbf{R}$ be a differentiable function defined as

$$f(x) = \int_1^x f(t) dt + (1-x)(\log_e x - 1) + e.$$

Then the value of $f(f(1))$ is :

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Options:

A.

$$(1 + e^e)$$

B.

$$(1 + e)$$

C.

$$(1 + e + e^e)$$

D.

$$1 + 2e$$

Answer: A

Solution:

Given

$$f(x) = \int_1^x f(t) dt + (1-x)(\ln x - 1) + e, \quad x \in [1, \infty)$$

We have to find $f(f(1))$.

First, put $x = 1$ in the given equation:

$$f(1) = \int_1^1 f(t) dt + (1-1)(\ln 1 - 1) + e$$

Now,

$$\int_1^1 f(t) dt = 0, \quad (1-1)(\ln 1 - 1) = 0$$

So,

$$f(1) = e$$

Hence,

$$f(f(1)) = f(e)$$

Now we find $f(x)$.

Differentiate both sides of

$$f(x) = \int_1^x f(t) dt + (1-x)(\ln x - 1) + e$$

Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx} \left(\int_1^x f(t) dt \right) = f(x)$$

So,

$$f'(x) = f(x) + \frac{d}{dx} [(1-x)(\ln x - 1)]$$

Now differentiate the second term carefully:

$$\frac{d}{dx} [(1-x)(\ln x - 1)] = -(\ln x - 1) + (1-x) \cdot \frac{1}{x}$$

Simplifying,

$$= -\ln x + 1 + \frac{1-x}{x}$$

$$= -\ln x + 1 + \frac{1}{x} - 1$$

$$= \frac{1}{x} - \ln x$$

Therefore,

$$f'(x) = f(x) + \frac{1}{x} - \ln x$$

So,

$$f'(x) - f(x) = \frac{1}{x} - \ln x$$

Now let us check whether

$$f(x) = \ln x + 1$$

satisfies this equation.

Then

$$f'(x) = \frac{1}{x}$$

So,

$$f'(x) - f(x) = \frac{1}{x} - (\ln x + 1)$$

This does not match directly. So instead, let us use the original equation to test $f(x) = \ln x + 1$.

Right-hand side:

$$\int_1^x (\ln t + 1) dt + (1 - x)(\ln x - 1) + e$$

Now,

$$\int (\ln t + 1) dt = t \ln t$$

Therefore,

$$\int_1^x (\ln t + 1) dt = x \ln x - 1 \cdot \ln 1 = x \ln x$$

since $\ln 1 = 0$.

Thus RHS becomes

$$x \ln x + (1 - x)(\ln x - 1) + e$$

Expand:

$$= x \ln x + (1 - x) \ln x - (1 - x) + e$$

$$= \ln x - 1 + x + e$$

This is not equal to $\ln x + 1$, so this trial is not correct.

Now solve the differential equation properly:

$$f'(x) - f(x) = \frac{1}{x} - \ln x$$

This is a linear differential equation.

Its integrating factor is

$$I.F. = e^{-x}$$

Multiplying both sides by e^{-x} ,

$$e^{-x} f'(x) - e^{-x} f(x) = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

So,

$$\frac{d}{dx} (f(x) e^{-x}) = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

This looks difficult, so let us instead use the original functional equation more intelligently.

Write

$$g(x) = f(x) - (\ln x + 1)$$

Then

$$f(x) = g(x) + \ln x + 1$$

Substitute into the original equation:

$$g(x) + \ln x + 1 = \int_1^x (g(t) + \ln t + 1) dt + (1-x)(\ln x - 1) + e$$

Now,

$$\int_1^x (\ln t + 1) dt = x \ln x$$

Hence,

$$g(x) + \ln x + 1 = \int_1^x g(t) dt + x \ln x + (1-x)(\ln x - 1) + e$$

Simplify the terms:

$$\begin{aligned} x \ln x + (1-x)(\ln x - 1) &= x \ln x + (1-x) \ln x - (1-x) \\ &= \ln x - 1 + x \end{aligned}$$

So,

$$g(x) + \ln x + 1 = \int_1^x g(t) dt + \ln x - 1 + x + e$$

Thus,

$$g(x) + 1 = \int_1^x g(t) dt - 1 + x + e$$

This is not simplifying nicely. Let us now directly guess from options.

Since

$$f(1) = e$$

we need $f(e)$.

Use the original equation at $x = e$:

$$f(e) = \int_1^e f(t) dt + (1-e)(\ln e - 1) + e$$

But

$$\ln e = 1$$

so

$$(1-e)(\ln e - 1) = 0$$

Hence

$$f(e) = \int_1^e f(t) dt + e$$

Now from the given equation,

$$f'(x) = f(x) + \frac{1}{x} - \ln x$$

Try

$$f(x) = \ln x + 1 + Ce^x$$

Then

$$f'(x) = \frac{1}{x} + Ce^x$$

So,

$$f'(x) - f(x) = \frac{1}{x} + Ce^x - (\ln x + 1 + Ce^x) = \frac{1}{x} - \ln x - 1$$

We are short by -1 . So modify the trial.

Try

$$f(x) = \ln x + A + Ce^x$$

Then

$$f'(x) = \frac{1}{x} + Ce^x$$

Hence

$$f'(x) - f(x) = \frac{1}{x} + Ce^x - (\ln x + A + Ce^x) = \frac{1}{x} - \ln x - A$$

Comparing with

$$\frac{1}{x} - \ln x$$

we get

$$A = 0$$

So,

$$f(x) = \ln x + Ce^x$$

Now use $f(1) = e$:

$$f(1) = \ln 1 + Ce = e$$

$$0 + Ce = e$$

So,

$$C = 1$$

Therefore,

$$f(x) = \ln x + e^x$$

Now,

$$f(e) = \ln e + e^e = 1 + e^e$$

Hence,

$$f(f(1)) = 1 + e^e$$

So the correct answer is:

$$\boxed{1 + e^e}$$

Therefore, Option A is correct.

Question 20

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(xy) = f(x)f(y)$, for all $x, y \in \mathbb{R}$ and $f(0) \neq 0$. Let $g : [1, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that

$$x^2 g(x) = \int_1^x (t^2 f(t) - t g(t)) dt$$

Then $g(2)$ is equal to :

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Options:

A.

$$\frac{13}{8}$$

B.

$$\frac{11}{16}$$

C.

$$\frac{15}{32}$$

D.

$$\frac{17}{64}$$

Answer: C

Solution:

We are given

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and

$$f(0) \neq 0.$$

Also,

$$x^2 g(x) = \int_1^x (t^2 f(t) - t g(t)) dt, \quad x \geq 1.$$

We have to find $g(2)$.

Step 1: Use the condition on f

Given

$$f(xy) = f(x)f(y)$$

for all real x, y .

Now put $x = 0$ and any real y . Then

$$f(0 \cdot y) = f(0)f(y)$$

that is,

$$f(0) = f(0)f(y).$$

Since $f(0) \neq 0$, we can divide by $f(0)$:

$$f(y) = 1 \quad \text{for all } y \in \mathbb{R}.$$

So,

$$f(t) = 1 \quad \text{for all } t.$$

Step 2: Substitute into the given integral equation

The equation becomes

$$x^2 g(x) = \int_1^x (t^2 - tg(t)) dt.$$

Step 3: Differentiate both sides

Differentiate with respect to x .

Left side:

$$\frac{d}{dx}[x^2 g(x)] = x^2 g'(x) + 2xg(x).$$

Right side, by the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_1^x (t^2 - tg(t)) dt = x^2 - xg(x).$$

Hence,

$$x^2 g'(x) + 2xg(x) = x^2 - xg(x).$$

So,

$$x^2 g'(x) + 3xg(x) = x^2.$$

Divide by x (since $x \geq 1$):

$$xg'(x) + 3g(x) = x.$$

Or,

$$g'(x) + \frac{3}{x}g(x) = 1.$$

Step 4: Solve the differential equation

This is a linear differential equation:

$$g'(x) + \frac{3}{x}g(x) = 1.$$

Its integrating factor is

$$\text{I. F.} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3.$$

Multiply both sides by x^3 :

$$x^3 g'(x) + 3x^2 g(x) = x^3.$$

The left side is

$$\frac{d}{dx}(x^3 g(x)).$$

So,

$$\frac{d}{dx}(x^3 g(x)) = x^3.$$

Integrate:

$$x^3 g(x) = \frac{x^4}{4} + C.$$

Thus,

$$g(x) = \frac{x}{4} + \frac{C}{x^3}.$$

Step 5: Find the constant using the original equation

Put $x = 1$ in

$$x^2 g(x) = \int_1^x (t^2 - tg(t)) dt.$$

Then

$$1^2 g(1) = \int_1^1 (t^2 - tg(t)) dt = 0.$$

So,

$$g(1) = 0.$$

Now use

$$g(1) = \frac{1}{4} + C = 0.$$

Hence,

$$C = -\frac{1}{4}.$$

Therefore,

$$g(x) = \frac{x}{4} - \frac{1}{4x^3}.$$

Step 6: Find $g(2)$

$$g(2) = \frac{2}{4} - \frac{1}{4(2^3)} = \frac{1}{2} - \frac{1}{32} = \frac{16-1}{32} = \frac{15}{32}.$$

So the correct answer is

$$\boxed{\frac{15}{32}}.$$

Hence, **Option C**.

Question 21

Let $y = y(x)$ be the solution of the differential equation

$$x\sqrt{1-x^2}dy + (y\sqrt{1-x^2} - x\cos^{-1}x)dx = 0, x \in (0, 1), \lim_{x \rightarrow 1^-} y(x) = 1.$$

Then $y\left(\frac{1}{2}\right)$ equals :

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Options:

A.

$$3 - \frac{\pi}{\sqrt{3}}$$

B.

$$4 - \sqrt{3}\pi$$

C.

$$4 - \frac{2\pi}{\sqrt{3}}$$

D.

$$3 - \frac{\pi}{2\sqrt{3}}$$

Answer: A

Solution:

$$(x\sqrt{1-x^2})dy + (y\sqrt{1-x^2} - x\cos^{-1}x)dx = 0$$

$$\Rightarrow (x\sqrt{1-x^2})\frac{dy}{dx} + (y\sqrt{1-x^2} - x\cos^{-1}x) = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{(y\sqrt{1-x^2} - x\cos^{-1}x)}{(x\sqrt{1-x^2})} = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{\cos^{-1}x}{\sqrt{1-x^2}} \dots (i)$$

Bernoulli's linear ODE of first order :

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \text{ having } I \cdot F = e^{\int P(x)dx}$$

$P(x)$	$Q(x)$
$\frac{1}{x}$	$\frac{\cos^{-1}x}{\sqrt{1-x^2}}$

$$\Rightarrow I \cdot F = e^{\int P(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x \dots (ii)$$

Apply the I.F from equation (ii) in ODE(i) :

$$-x \frac{dy}{dx} + y = \frac{x \cos^{-1} x}{\sqrt{1-x^2}} \Rightarrow x dy + y dx = \left(\frac{x \cos^{-1} x}{\sqrt{1-x^2}} \right) dx$$

$$\Rightarrow d(xy) = \left(\frac{x \cos^{-1} x}{\sqrt{1-x^2}} \right) dx \Rightarrow \int d(xy) = \int \left(\frac{x \cos^{-1} x}{\sqrt{1-x^2}} \right) dx \dots (iii)$$

We choose the first and second functions

u	v
$\cos^{-1} x$	$\frac{x}{\sqrt{1-x^2}}$

for the integrand in equation (iii) to proceed further:

$$xy = (\cos^{-1} x) \cdot \int \left(\frac{x}{\sqrt{1-x^2}} \right) dx + \int \left[\left(\frac{1}{\sqrt{1-x^2}} \right) \cdot \int \left(\frac{x}{\sqrt{1-x^2}} \right) dx \right] dx$$

$$= -(\cos^{-1} x) \cdot \sqrt{1-x^2} - \int \left(\frac{1}{\sqrt{1-x^2}} \right) \cdot (\sqrt{1-x^2}) dx$$

$$= -(\cos^{-1} x) \cdot \sqrt{1-x^2} - \int \left(\frac{1}{\sqrt{1-x^2}} \right) \cdot (\sqrt{1-x^2}) dx$$

$$= -(\cos^{-1} x) \cdot \sqrt{1-x^2} - \int dx$$

$$= -(\cos^{-1} x) \cdot \sqrt{1-x^2} - x + C$$

$$y(x) = \frac{C}{x} - 1 - \frac{(\cos^{-1} x) \cdot \sqrt{1-x^2}}{x} \dots (iv) \text{ where } C \text{ is the arbitrary constant}$$

$$\lim_{x \rightarrow 1^-} y(x) = \lim_{x \rightarrow 1^-} \left[\frac{C}{x} - 1 - \frac{(\cos^{-1} x) \cdot \sqrt{1-x^2}}{x} \right] = C - 1 - \frac{(0) \cdot (0)}{1} = C - 1 \dots (v)$$

We have the boundary condition

$$\lim_{x \rightarrow 1^-} y(x) = 1 \Rightarrow C = 2 \dots (vi) \leftarrow \text{From equation (v)}$$

$$\text{We have solved the ODE to reach a particular solution } (C = 2) \text{ from } (iv) y(x) = \frac{2}{x} - 1 - \frac{(\cos^{-1} x) \cdot \sqrt{1-x^2}}{x} \dots (vii)$$

$$\text{Substitute } x = \frac{1}{2} \text{ in equation (vii) to evaluate } y\left(\frac{1}{2}\right) = 4 - 1 - (2) \cdot \left(\frac{\pi}{3}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)$$

$$= 3 - \frac{\pi}{\sqrt{3}}$$

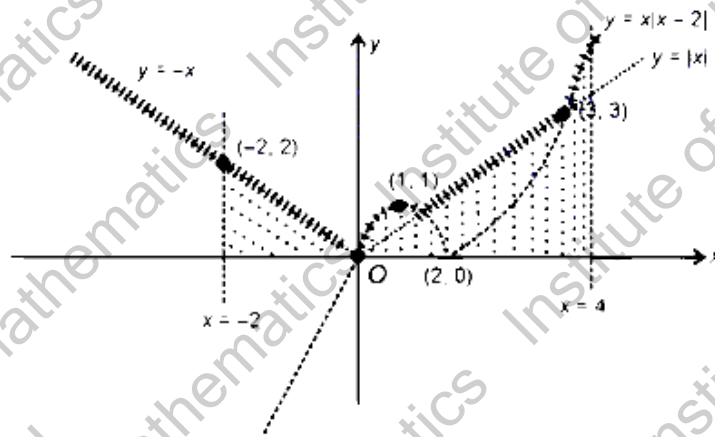
Question 22

The area of the region bounded by the curve $y = \max\{|x|, x|x-2|\}$, the x -axis and the lines $x = -2$ and $x = 4$ is equal to _____

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Answer: 12

Solution:



Area =

$$\begin{aligned} & \frac{1}{2} \times 2 \times 2 + \int_0^1 (2x - x^2) dx + \int_1^3 x dx + \int_3^4 (x^2 - 2x) dx \\ &= 2 + \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^2}{2} \right]_1^3 + \left[\frac{x^3}{3} - \frac{2x^2}{2} \right]_3^4 \\ &= 12 \text{ square units} \end{aligned}$$

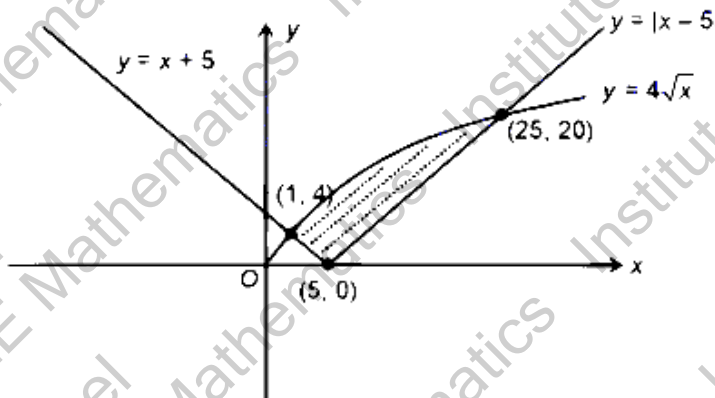
Question 23

If the area of the region $\{(x, y) : |x - 5| \leq y \leq 4\sqrt{x}\}$ is A , then $3A$ is equal to _____.

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Answer: 368

Solution:



$$\begin{aligned}\text{Area} &= \int_1^{25} 4\sqrt{x} dx - \frac{1}{2} \times (5-1) \cdot 4 - \frac{1}{2} \times (25-5) \times 20 \\ &= 4 \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^{25} - 8 - 200 = \frac{368}{3} \\ \Rightarrow 3A &= 368\end{aligned}$$

Question 24

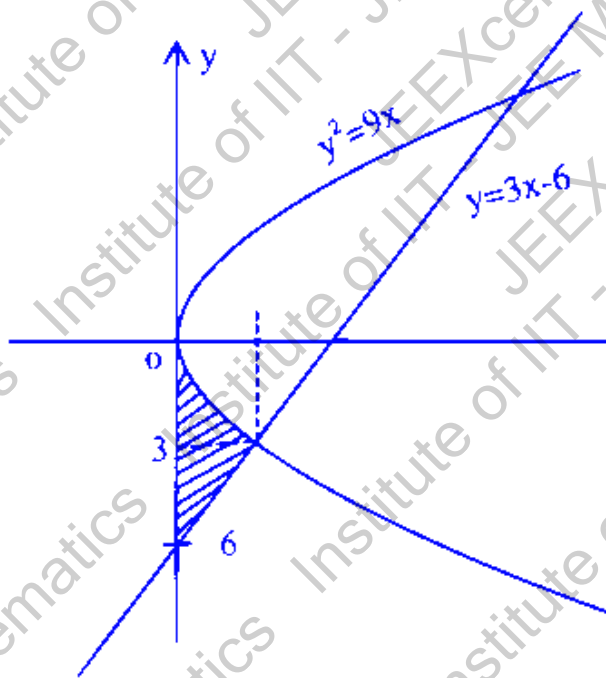
Let the area of the bounded region $\{(x, y) : 0 \leq 9x \leq y^2, y \geq 3x - 6\}$ be A . Then $6A$ is equal to _____.

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Answer: 15

Solution:

$$0 \leq 9x \leq y^2 \text{ \& } y \geq 3x - 6$$



$$A = \text{Required Area} = \left[\int_0^1 (-3\sqrt{x}) dx - \int_0^1 (3x - 6) dx \right]$$

$$A = -3 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) \Big|_0^1 - \left(\frac{3x^2}{2} - 6x \right) \Big|_0^1$$

$$A = -2[1 - 0] \left[\frac{3}{2} - 6 \right]$$

$$A = -2 - \frac{3}{2} + 6 = \frac{5}{2} \text{ Sq. unit}$$

$$\therefore 6A = 6 \times \frac{5}{2} = 15$$

Question 25

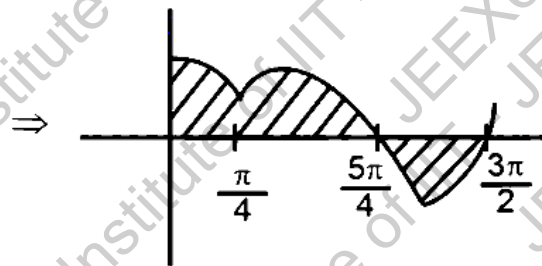
Let the area of the region bounded by the curve $y = \max\{\sin x, \cos x\}$, lines $x = 0$, $x = \frac{3\pi}{2}$, and the x -axis be A . Then, $A + A^2$ is equal to _____.

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Answer: 12

Solution:

$$f(x) = \max(\sin x, \cos x) \\ = \begin{cases} \cos x, & x \in (0, \frac{\pi}{4}) \\ \sin x, & x \in (\frac{\pi}{4}, \frac{5\pi}{4}) \\ \cos x, & x \in (\frac{5\pi}{4}, \frac{3\pi}{2}) \end{cases}$$



$$\Rightarrow \int_0^{\frac{3\pi}{2}} |f(x)| dx = \int_0^{\frac{\pi}{4}} (\cos x) dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x) dx + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\cos x) dx$$

$$= 3 \Rightarrow A = 3$$

$$A + A^2 = 12$$

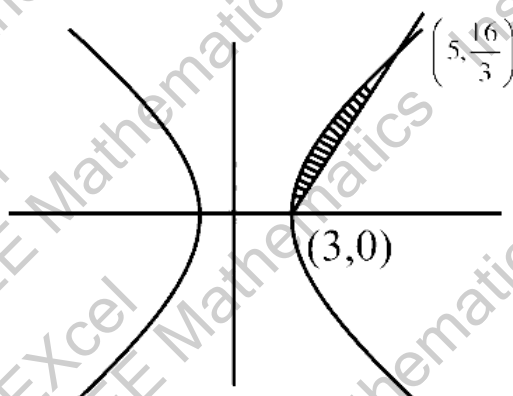
Question 26

If the area of the region bounded by $16x^2 - 9y^2 = 144$ and $8x - 3y = 24$ is A , then $3(A + 6 \log_e(3))$ is equal to _____.

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Answer: 24

Solution:



$$16x^2 - 9y^2 = 144$$

$$16x^2 - (8x - 24)^2 = 144$$

$$16x^2 - 64(x - 3)^2 = 144 \rightarrow x^2 - 4(x - 3)^2 = 9$$

$$3x^2 - 24x + 45 \rightarrow x^2 - 8x + 15 = 0$$

$$x = 3, 5$$

$$\text{Area} = \int_3^5 \sqrt{\frac{16x^2 - 144}{9}} - \frac{1}{2} \cdot 2 \cdot \frac{16}{3}$$

$$= \frac{4}{3} \int_3^5 \sqrt{x^2 - 9} - \frac{16}{3}$$

$$= \frac{4}{3} \left(\frac{x}{2} \sqrt{x^2 - 9} - \frac{9}{2} \log_e \left(x + \sqrt{x^2 - 9} \right) \right) \Big|_3^5 - \frac{16}{3}$$

$$= \frac{4}{3} \left(\frac{5}{2} \cdot 4 - \frac{9}{2} \log_e 9 - \frac{3}{2} \cdot 0 + \frac{9}{2} \log_e 3 \right) - \frac{16}{3}$$

$$\text{Area} = 8 - 6 \log_e 3 = A$$

$$\therefore A + 6 \ln 3 = 8$$

$$\Rightarrow 3(A + 6 \ln 3) = 24$$

Question 27

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function such that $f(x) + 3f\left(\frac{\pi}{2} - x\right) = \sin x, x \in \mathbf{R}$.

Let the maximum value of f on $\mathbf{R}$ be α . If the area of the region bounded by the curves $g(x) = x^2$ and $h(x) = \beta x^3, \beta > 0$, is α^2 , then $30\beta^3$ is equal to

_____.

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Answer: 16

Solution:

$$f(x) + 3f\left(\frac{\pi}{2} - x\right) = \sin x \dots (1)$$

Now replace x by $\frac{\pi}{2} - x$ in (1). This is a standard NCERT step to get a second equation in the same two terms.

$$\Rightarrow f\left(\frac{\pi}{2} - x\right) + 3f(x) = \cos x \dots (2)$$

Now we solve the two linear equations (1) and (2) to find $f(x)$.

Multiply (1) by 3:

$$3f(x) + 9f\left(\frac{\pi}{2} - x\right) = 3\sin x \dots (3)$$

Subtract (2) from (3):

$$(3f(x) - 3f(x)) + (9 - 1)f\left(\frac{\pi}{2} - x\right) = 3\sin x - \cos x$$

$$8f\left(\frac{\pi}{2} - x\right) = 3\sin x - \cos x$$

Now put $x \rightarrow \frac{\pi}{2} - x$ in this result to get $f(x)$:

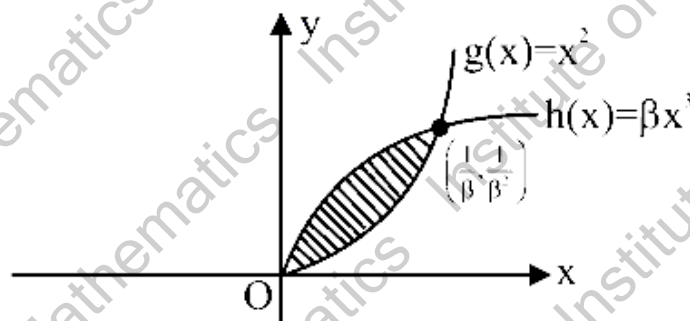
$$8f(x) = 3\cos x - \sin x$$

$$f(x) = \frac{1}{8}(3\cos x - \sin x)$$

To find the maximum value, write $3\cos x - \sin x$ in the form $R\cos(x + \phi)$, where $R = \sqrt{3^2 + (-1)^2} = \sqrt{10}$. So the maximum value of $3\cos x - \sin x$ is $\sqrt{10}$.

$$f_{\max} = \frac{\sqrt{10}}{8}$$

$y = g(x)$ & $y = h(x)$ intersect as shown in the figure



The curves are $g(x) = x^2$ and $h(x) = \beta x^3$ with $\beta > 0$. Their points of intersection come from $x^2 = \beta x^3$:

$$x^2 = \beta x^3 \Rightarrow x^2(1 - \beta x) = 0 \Rightarrow x = 0 \text{ and } x = \frac{1}{\beta}$$

For $0 < x < \frac{1}{\beta}$, so the area between the curves is the integral of $(x^2 - \beta x^3)$ from 0 to $\frac{1}{\beta}$.

$$\begin{aligned}
 \therefore \text{Area bounded} = \Delta &= \left| \int_0^{\frac{1}{\beta}} (\beta x^3 - x^2) dx \right| \\
 &= \left| \left[\frac{\beta x^4}{4} - \frac{x^3}{3} \right]_0^{\frac{1}{\beta}} \right| \\
 &= \left| \left(\frac{\beta}{4} \cdot \frac{1}{\beta^4} - \frac{1}{3} \cdot \frac{1}{\beta^3} \right) \right| \\
 &= \left| \left(\frac{1}{4\beta^3} - \frac{1}{3\beta^3} \right) \right| = \frac{1}{12\beta^3} = \alpha^2 \text{ (given)} \\
 \Rightarrow \frac{1}{12\beta^3} &= \left(\frac{\sqrt{10}}{8} \right)^2 = \frac{10}{64} = \frac{5}{32} \\
 \Rightarrow \beta^3 &= \frac{1}{12} \cdot \frac{32}{5} = \frac{8}{15} \\
 \Rightarrow 30\beta^3 &= 30 \cdot \frac{8}{15} = 16
 \end{aligned}$$

Question 28

If the area of the region $\{(x, y) : 1 - 2x \leq y \leq 4 - x^2, x \geq 0, y \geq 0\}$ is $\frac{\alpha}{\beta}$, $\alpha, \beta \in \mathbb{N}, \gcd(\alpha, \beta) = 1$, then the value of $(\alpha + \beta)$ is:

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Options:

A.

73

B.

85

C.

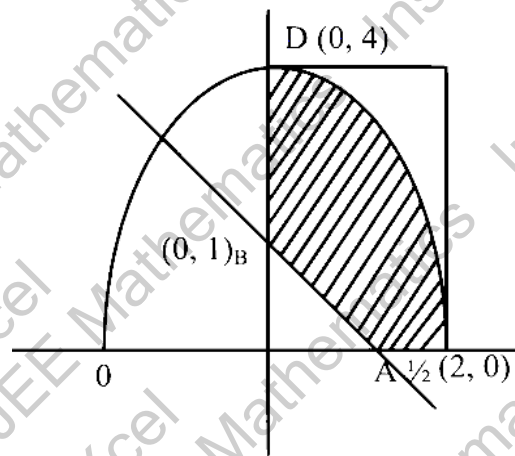
67

D.

91

Answer: A

Solution:



$$\int_0^2 (4 - x^2) dx - \frac{1}{2} \times 1 \times \frac{1}{2}$$

Question 29

Let the line $x = -1$ divide the area of the region $\{(x, y) : 1 + x^2 \leq y \leq 3 - x\}$ in the ratio $m : n$, $\gcd(m, n) = 1$. Then $m + n$ is equal to

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Options:

A.

27

B.

28

C.

25

D.

26

Answer: A

Solution:

$$m = \int_{-2}^{-1} (3-x) - (1+x^2) dx$$

$$m = 2 + \frac{3}{2} - \frac{7}{3} = \frac{7}{6} \quad M = \int_{-1}^1 (3-x) - (1+x^2) dx = 4 - \frac{1}{2}(0) - \frac{1}{3}(1+1) = 4 - \frac{2}{3} = \frac{10}{3}$$

$$m : M = \frac{7}{6} : \frac{10}{3} = \frac{7}{20}$$

$$m + n = 27$$

Question 30

The area of the region $A = \{(x, y) : 4x^2 + y^2 \leq 8 \text{ and } y^2 \leq 4x\}$ is:

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Options:

A.

$$\pi + \frac{2}{3}$$

B.

$$\frac{\pi}{2} + 2$$

C.

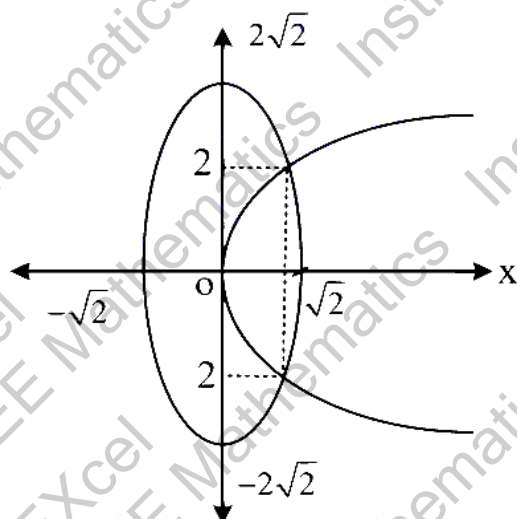
$$\pi + 4$$

D.

$$\frac{\pi}{2} + \frac{1}{3}$$

Answer: A

Solution:



$$A = 2 \int_0^1 2\sqrt{x} dx + 2 \int_1^{\sqrt{2}} \sqrt{8-4x^2} dx = \frac{8}{3} \left(X^{\frac{3}{2}} \right)_0^1 + 4 \int_1^{\sqrt{2}} \sqrt{2-x^2} dx$$

$$\frac{8}{3} + 4 \times \frac{1}{2} \left[x\sqrt{2-x^2} + 2\sin^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_1^{\sqrt{2}}$$

$$\frac{8}{3} + 2 \left[2 \times \frac{\pi}{2} - 1 - 2 \times \frac{\pi}{4} \right] = \frac{8}{3} + 2\pi - 2 - \pi = \pi + \frac{2}{3} \text{ Sq. unit}$$

Question31

The area of the region enclosed between the circles $x^2 + y^2 = 4$ and $x^2 + (y-2)^2 = 4$ is:

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Options:

A. $\frac{2}{3}(4\pi - 3\sqrt{3})$

B. $\frac{4}{3}(2\pi - \sqrt{3})$

C. $\frac{4}{3}(2\pi - 3\sqrt{3})$

D.

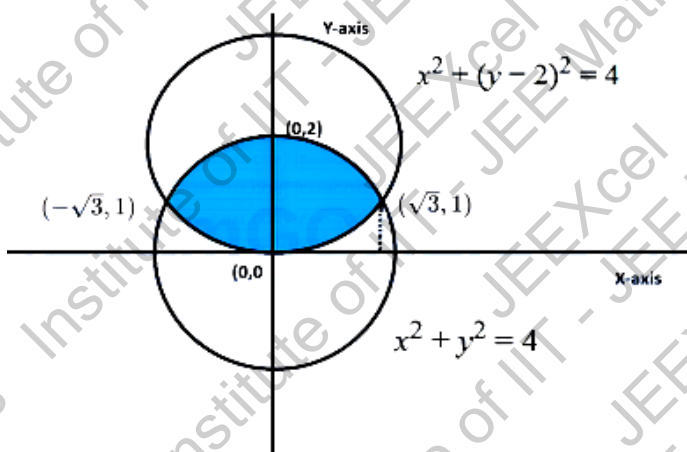
$$\frac{2}{3}(2\pi - 3\sqrt{3})$$

Answer: A

Solution:

$$x^2 + y^2 = 4 \implies C_1(0, 0), r = 2$$

$$x^2 + (y - 2)^2 = 4 \implies C_2(0, 2), r = 2$$



Intersection points:

$$x^2 + y^2 = x^2 + (y - 2)^2$$

$$y^2 = y^2 - 4y + 4 \implies 4y = 4 \implies y = 1$$

$$x^2 + 1 = 4 \implies x^2 = 3 \implies x = \pm\sqrt{3}$$

Points are $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$

For the region between $y = 0$ and $y = 2$:

$$y_{\text{upper}} = \sqrt{4 - x^2} \text{ (from } x^2 + y^2 = 4 \text{)}$$

$$y_{\text{lower}} = 2 - \sqrt{4 - x^2} \text{ (from } x^2 + (y - 2)^2 = 4 \text{)}$$

$$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} (y_{\text{upper}} - y_{\text{lower}}) dx$$

$$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\sqrt{4 - x^2} - \left(2 - \sqrt{4 - x^2} \right) \right) dx$$

$$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} \left(2\sqrt{4 - x^2} - 2 \right) dx$$

$$\text{Property: } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ (for even functions)}$$

$$\text{Area} = 2 \int_0^{\sqrt{3}} (2\sqrt{4-x^2} - 2) dx$$

$$\text{Area} = 4 \int_0^{\sqrt{3}} (\sqrt{4-x^2} - 1) dx$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

$$\text{Area} = 4 \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) - x \right]_0^{\sqrt{3}}$$

$$\text{Area} = 4 \left[\frac{\sqrt{3}}{2} (1) + 2 \left(\frac{\pi}{3} \right) - \sqrt{3} \right]$$

$$\text{Area} = 4 \left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right]$$

$$\text{Area} = \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3} (4\pi - 3\sqrt{3})$$

Question 32

Let A_1 be the bounded area enclosed by the curves $y = x^2 + 2$, $x + y = 8$ and y -axis that lies in the first quadrant. Let A_2 be the bounded area enclosed by the curves $y = x^2 + 2$, $y^2 = x$, $x = 2$, and y -axis that lies in the first quadrant. Then $A_1 - A_2$ is equal to

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Options:

A.

$$\frac{2}{3}(2\sqrt{2} + 1)$$

B.

$$\frac{2}{3}(3\sqrt{2} + 1)$$

C.

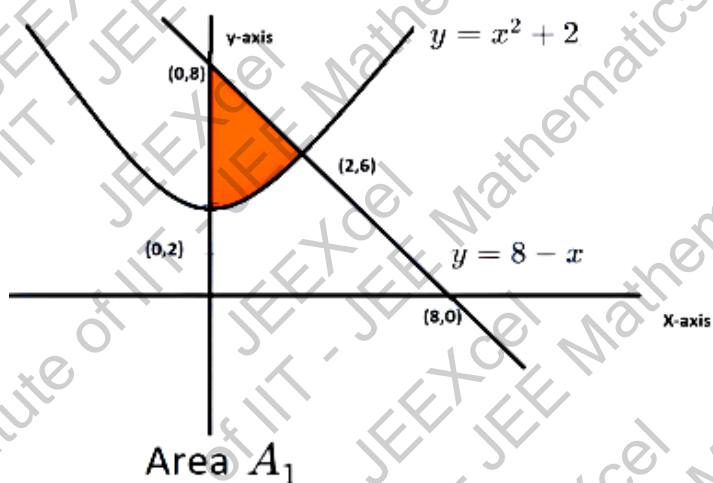
$$\frac{2}{3}(\sqrt{2} + 1)$$

D.

$$\frac{2}{3}(4\sqrt{2} + 1)$$

Answer: A

Solution:



Area A_1 enclosed by curves

$y = x^2 + 2, x + y = 8, y$ -axis in first quadrant as shown in graph.

Solve $y = x^2 + 2$ and $x + y = 8$ to find point of intersection.

Substituting $y = 8 - x$ in $y = x^2 + 2$

$$8 - x = x^2 + 2$$

$$x^2 + x - 6 = 0$$

$$x^2 + 3x - 2x - 6 = 0$$

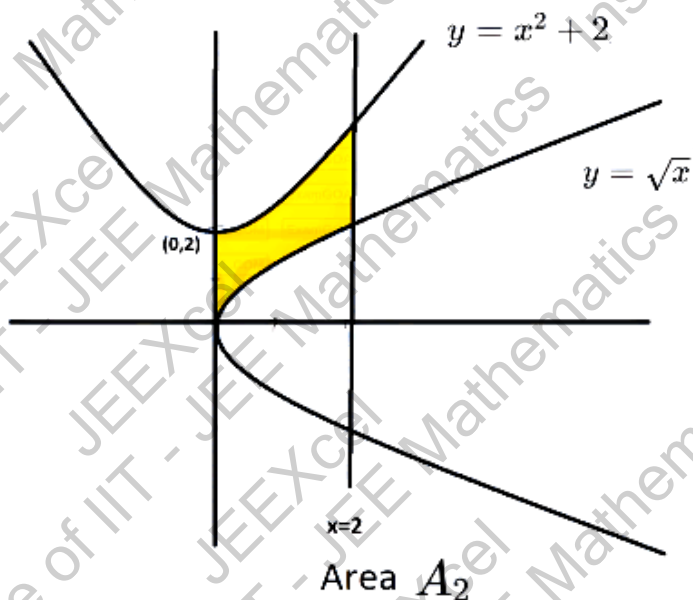
$$x(x + 3) - 2(x + 3) = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3, 2$$

Since area in first quadrant, point of intersection at $y = 8 - 2 = 6 \Rightarrow (2, 6)$.

$$\begin{aligned} A_1 &= \int_0^2 [(8 - x) - (x^2 + 2)] dx \\ &= \int_0^2 (-x^2 - x + 6) dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x \right]_0^2 \\ &= \left[-\frac{8}{3} - \frac{4}{2} + 12 - \{0 - 0 - 0\} \right] \\ &= \frac{-16 - 12 + 72}{6} = \frac{72 - 28}{6} = \frac{44}{6} \\ A_1 &= \frac{22}{3} \end{aligned}$$



Now, A_2 is area enclosed by curves

$$y = x^2 + 2, y^2 = x, x = 2$$

Solve $y = x^2 + 2$ and $y = \sqrt{x}$

$$y = y^4 + 2$$

$$y^4 - y + 2 = 0$$

$$D = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$$

discriminant (D) is negative. This means curves do not intersect.

So A_2 is bounded by curves y-axis, 2, $y = x^2 + 2$ and $y = \sqrt{x}$ in first quadrant is:

$$\begin{aligned} A_2 &= \int_0^2 (x^2 + 2 - \sqrt{x}) dx \\ &= \left[\frac{x^3}{3} + 2x - \frac{2x^{3/2}}{3} \right]_0^2 \\ &= \left[\frac{8}{3} + 2 \times 2 - \frac{2 \times 2\sqrt{2}}{3} - \{0 + 0 - 0\} \right] \\ &= \left[\frac{8}{3} + 4 - \frac{4\sqrt{2}}{3} \right] = \left[\frac{20}{3} - \frac{4\sqrt{2}}{3} \right] \\ A_1 - A_2 &= \frac{22}{3} - \left(\frac{20}{3} - \frac{4\sqrt{2}}{3} \right) \\ &= \frac{22}{3} - \frac{20}{3} + \frac{4\sqrt{2}}{3} = \frac{2}{3} + \frac{4\sqrt{2}}{3} \\ &= \frac{2}{3} (2\sqrt{2} + 1) \end{aligned}$$

Question 33

Let $f(\alpha)$ denote the area of the region in the first quadrant bounded by $x = 0, x = 1, y^2 = x$ and $y = |\alpha x - 5| - |1 - \alpha x| + \alpha x^2$. Then $(f(0) + f(1))$ is equal to

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Options:

A.

12

B.

14

C.

9

D.

7

Answer: D

Solution:

Given Curves $y^2 = x, y = |\alpha x - 5| - |1 - \alpha x| + \alpha x^2$ and $x = 0$ and $x = 1$

The area of region bounded by these curve is $f(\alpha)$.

To find $f(0) + f(1)$.

for $\alpha = 0$

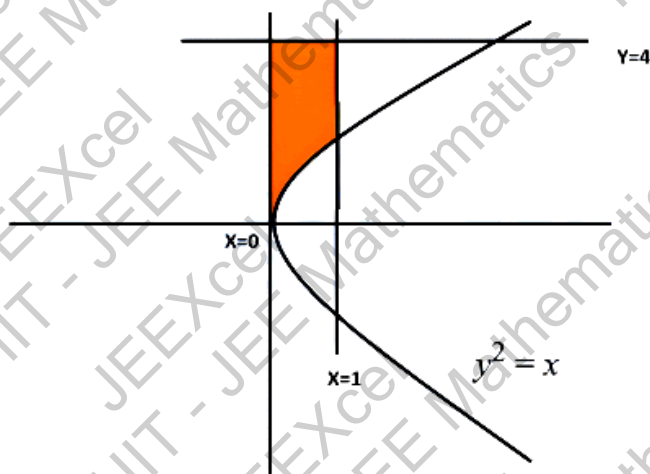
$$y = |0 \times x - 5| - |1 - 0 \times x| + 0 \times x^2$$
$$y = 5 - 1 = 4$$

so the region bounded by $x = 0, x = 1, y = \sqrt{x}$

and $y = 4$ is shown in graph.

$$f(0) = \int_0^1 (4 - \sqrt{x}) dx$$

$$f(0) = \left[4x - \frac{2}{3} x^{3/2} \right]_0^1 = 4 - \frac{2}{3} = \frac{10}{3}$$



If, When $\alpha = 1$, the second equation becomes:

$$y = |x - 5| - |1 - x| + x^2$$

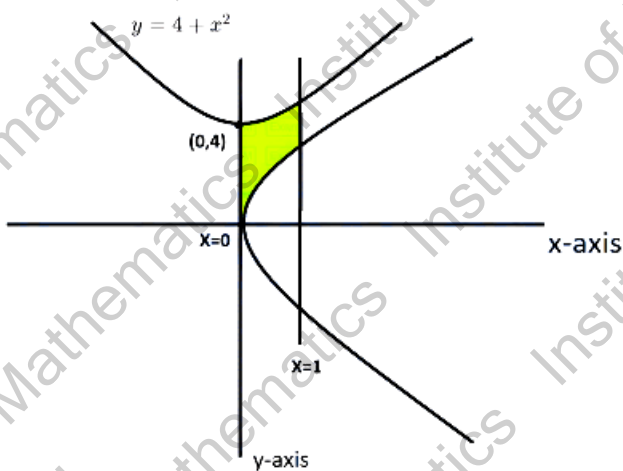
For the interval $0 \leq x \leq 1$:

$$|x - 5| = -(x - 5) = 5 - x$$

$$|1 - x| = 1 - x$$

$$y = (5 - x) - (1 - x) + x^2 = 4 + x^2$$

The region is bounded by $x = 0$, $x = 1$, $y = \sqrt{x}$, and $y = 4 + x^2$ as shown in graph is.



$$f(1) = \int_0^1 (4 + x^2 - \sqrt{x}) dx$$

$$f(1) = \left[4x + \frac{x^3}{3} - \frac{2}{3} x^{3/2} \right]_0^1 = 4 + \frac{1}{3} - \frac{2}{3} = 4 - \frac{1}{3} = \frac{11}{3}$$

$$\text{So, } (f(0) + f(1)) = \frac{10}{3} + \frac{11}{3} = \frac{21}{3} = 7$$

Question 34

The area of the region $R = \{(x, y) : xy \leq 8, 1 \leq y \leq x^2, x \geq 0\}$ is

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Options:

A.

$$\frac{2}{3}(20 \log_e(2) + 9)$$

B.

$$\frac{1}{3}(40 \log_e(2) + 27)$$

C.

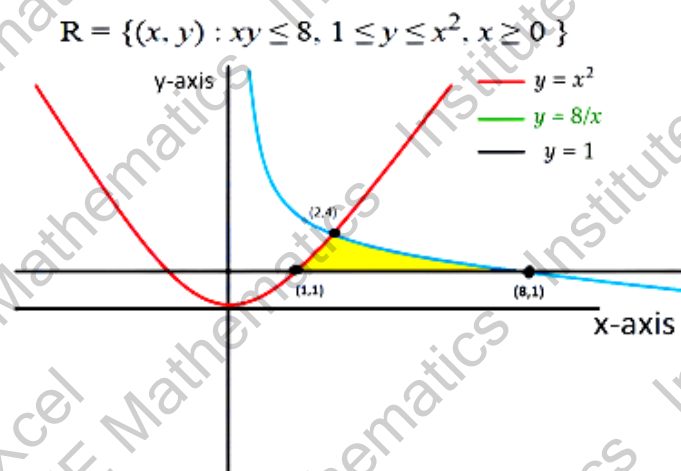
$$\frac{1}{3}(49 \log_e(2) - 15)$$

D.

$$\frac{2}{3}(24 \log_e(2) - 7)$$

Answer: D

Solution:



The region is bounded by three curves in the first quadrant ($x \geq 0$):

- 1. $y = 1$: A horizontal line.
- 2. $y = x^2$: A parabola opening upwards.
- 3. $y = \frac{8}{x}$: A rectangular hyperbola.

Find the Intersection Points

We need to know where these curves meet to set our integration limits:

- Parabola and Line ($y = x^2$ and $y = 1$) : $x^2 = 1 \implies x = 1$ (since $x \geq 0$). Point: $(1, 1)$.

- Parabola and Hyperbola ($y = x^2$ and $y = \frac{8}{x}$) : $x^2 = \frac{8}{x} \implies x^3 = 8 \implies x = 2$. Point: (2, 4).
- Hyperbola and Line ($y = \frac{8}{x}$ and $y = 1$) : $\frac{8}{x} = 1 \implies x = 8$. Point: (8, 1).

Understanding the Inequalities

- $x \geq 0$: The region is located in the first and fourth quadrants (but limited to the first quadrant by $1 \leq y$).
- $1 \leq y \leq x^2$: This gives us a lower bound of the horizontal line $y = 1$ and an upper bound of the parabola $y = x^2$.
- $xy \leq 8$: This can be rewritten as $y \leq \frac{8}{x}$. This represents the area below the rectangular hyperbola.

Setting up the Integrals

The region starts at $x = 1$ and ends at $x = 8$. Because the "upper" boundary changes at $x = 2$, we must split the area into two parts:

- Part 1 (from $x = 1$ to $x = 2$) : Bounded above by the parabola $y = x^2$ and below by $y = 1$.
- Part 2 (from $x = 2$ to $x = 8$) : Bounded above by the hyperbola $y = \frac{8}{x}$ and below by $y = 1$.

$$\begin{aligned}
 \text{Area} &= \int_1^2 (x^2 - 1) dx + \int_2^8 \left(\frac{8}{x} - 1\right) dx \\
 &= \left[\frac{x^3}{3} - x\right]_1^2 + [8 \ln x - x]_2^8 \\
 &= \left[\frac{8}{3} - 2 - \left\{\frac{1}{3} - 1\right\}\right] + [(8 \ln 8 - 8) - \{8 \ln 2 - 2\}] \\
 &= \frac{7}{3} - 1 + 8 \ln 2^3 - 8 \ln 2 - 6 \\
 &= \frac{7}{3} - 7 + 24 \ln 2 - 8 \ln 2 \quad \{\ln a^m = m \ln a\} \\
 &= -\frac{14}{3} + 16 \ln 2 \\
 &= \frac{2}{3}[24 \ln 2 - 7]
 \end{aligned}$$

Question 35

Let $P_1 : y = 4x^2$ and $P_2 : y = x^2 + 27$ be two parabolas. If the area of the bounded region enclosed between P_1 and P_2 is six times the area of the bounded region enclosed between the line $y = \alpha x$, $\alpha > 0$ and P_1 , then α is equal to :

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Options:

A.

12

B.

15

C.

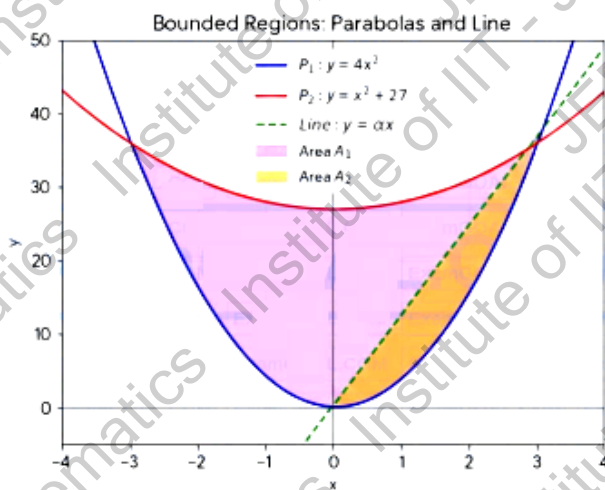
8

D.

6

Answer: A

Solution:



1. Calculate the Area A_1 between P_1 and P_2

The two parabolas are given by:

- $P_1 : y = 4x^2$
- $P_2 : y = x^2 + 27$

Find the intersection points: Setting the equations equal to each other:

$$4x^2 = x^2 + 27$$

$$\Rightarrow 3x^2 = 27$$

$$\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

Find the intersection points : Setting the equations equal to each other:

$$4x^2 = x^2 + 27$$

$$3x^2 = 27, \quad x^2 = 9 \Rightarrow x = \pm 3$$

Calculate the area: On the interval $[-3, 3]$, the parabola P_2 lies above P_1 . The area A_1 is:

$$A_1 = \int_{-3}^3 (x^2 + 27 - 4x^2) dx = \int_{-3}^3 (27 - 3x^2) dx$$

$$A_1 = [27x - x^3]_{-3}^3$$

$$A_1 = (27(3) - 3^3) - (27(-3) - (-3)^3)$$

$$A_1 = (81 - 27) - (-81 + 27) = 54 - (-54) = 108$$

Calculate the Area A_2 between P_1 and the line $y = \alpha x$

Find the intersection points:

$$4x^2 = \alpha x \implies x(4x - \alpha) = 0$$

The points are $x = 0$ and $x = \frac{\alpha}{4}$.

Calculate the area: On the interval $[0, \frac{\alpha}{4}]$, the line $y = \alpha x$ lies above $y = 4x^2$. The area A_2 is:

$$A_2 = \int_0^{\alpha/4} (\alpha x - 4x^2) dx$$

$$A_2 = \left[\frac{\alpha x^2}{2} - \frac{4x^3}{3} \right]_0^{\alpha/4}$$

$$A_2 = \frac{\alpha}{2} \left(\frac{\alpha}{4} \right)^2 - \frac{4}{3} \left(\frac{\alpha}{4} \right)^3$$

$$A_2 = \frac{\alpha^3}{32} - \frac{4\alpha^3}{192} = \frac{\alpha^3}{32} - \frac{\alpha^3}{48} = \frac{3\alpha^3 - 2\alpha^3}{96} = \frac{\alpha^3}{96}$$

The problem states that A_1 is six times A_2 :

$$A_1 = 6 \times A_2$$

$$108 = 6 \times \left(\frac{\alpha^3}{96} \right)$$

$$108 = \frac{\alpha^3}{16}$$

$$\alpha^3 = 108 \times 16 = 1728$$

$$\alpha = \sqrt[3]{1728} = 12$$

Question 36

The area of the region $\{(x, y) : y \leq \pi - |x|, y \leq |x \sin x|, y \geq 0\}$ is:

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Options:

A. $1 + \frac{\pi^2}{8}$

B.

$2 + \frac{\pi^2}{4}$

C.

$$\frac{\pi^2}{8} - 1$$

D.

$$4 + \frac{\pi^2}{2}$$

Answer: B

Solution:

We need the area of the region

$$\{(x, y) : y \leq \pi - |x|, y \leq |x \sin x|, y \geq 0\}.$$

This means the required area is the area under the curve

$$y = \min(\pi - |x|, |x \sin x|)$$

and above the x -axis.

So the area is

$$A = \int \min(\pi - |x|, |x \sin x|) dx$$

over the values of x where the region exists.

Since $y \geq 0$ and $y \leq \pi - |x|$, we must have

$$\pi - |x| \geq 0 \implies |x| \leq \pi.$$

Hence the region lies in $[-\pi, \pi]$.

Also, both functions $\pi - |x|$ and $|x \sin x|$ are even functions, so the region is symmetric about the y -axis. Therefore,

$$A = 2 \int_0^\pi \min(\pi - x, x \sin x) dx.$$

Now for $0 \leq x \leq \pi$, we have $\sin x \geq 0$, so

$$|x \sin x| = x \sin x.$$

Thus,

$$A = 2 \int_0^\pi \min(\pi - x, x \sin x) dx.$$

Now compare $\pi - x$ and $x \sin x$ on $[0, \pi]$.

At $x = 0$,

$$\pi - x = \pi, \quad x \sin x = 0.$$

At $x = \pi$,

$$\pi - x = 0, \quad x \sin x = 0.$$

Check where they meet:

$$x \sin x = \pi - x.$$

It is easy to see that $x = \frac{\pi}{2}$ satisfies this, because

$$\frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2} = \pi - \frac{\pi}{2}.$$

Also, on $[0, \pi/2]$, we have $x \sin x \leq \pi - x$, and on $[\pi/2, \pi]$, we have $\pi - x \leq x \sin x$.

So,

$$A = 2 \left[\int_0^{\pi/2} x \sin x \, dx + \int_{\pi/2}^{\pi} (\pi - x) \, dx \right].$$

Now evaluate each integral.

First,

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.$$

Therefore,

$$\int_0^{\pi/2} x \sin x \, dx = [-x \cos x + \sin x]_0^{\pi/2} = 1.$$

Next,

$$\int_{\pi/2}^{\pi} (\pi - x) \, dx.$$

Let $u = \pi - x$. Then as x goes from $\pi/2$ to π , u goes from $\pi/2$ to 0. Or directly,

$$\int_{\pi/2}^{\pi} (\pi - x) \, dx = \left[\pi x - \frac{x^2}{2} \right]_{\pi/2}^{\pi}.$$

Compute:

At $x = \pi$,

$$\pi^2 - \frac{\pi^2}{2} = \frac{\pi^2}{2}.$$

At $x = \frac{\pi}{2}$,

$$\pi \cdot \frac{\pi}{2} - \frac{1}{2} \left(\frac{\pi}{2} \right)^2 = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}.$$

So,

$$\int_{\pi/2}^{\pi} (\pi - x) \, dx = \frac{\pi^2}{2} - \frac{3\pi^2}{8} = \frac{\pi^2}{8}.$$

Hence,

$$A = 2 \left(1 + \frac{\pi^2}{8} \right) = 2 + \frac{\pi^2}{4}.$$

Therefore, the correct answer is

$$\boxed{2 + \frac{\pi^2}{4}}$$

So, Option B is correct.

Question 37

The area of the region bounded by the curves $x + 3y^2 = 0$ and $x + 4y^2 = 1$ is equal to :

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Options:

A.

$$\frac{1}{3}$$

B.

$$\frac{2}{3}$$

C.

$$\frac{4}{3}$$

D.

$$\frac{5}{3}$$

Answer: C

Solution:

Given curves are

$$x + 3y^2 = 0 \quad \text{and} \quad x + 4y^2 = 1$$

First, write them in the form $x =$:

$$x = -3y^2$$

and

$$x = 1 - 4y^2$$

To find the bounded area, we first find their points of intersection.

Set the two values of x equal:

$$-3y^2 = 1 - 4y^2$$

$$y^2 = 1$$

$$y = \pm 1$$

So the curves intersect at $y = -1$ and $y = 1$.

Now, for a fixed value of y , the right curve is

$$x = 1 - 4y^2$$

and the left curve is

$$x = -3y^2$$

So the horizontal width of the region is

$$(1 - 4y^2) - (-3y^2) = 1 - y^2$$

Hence, area of the enclosed region is

$$A = \int_{-1}^1 (1 - y^2) dy$$

Now evaluate:

$$A = \left[y - \frac{y^3}{3} \right]_{-1}^1$$

$$A = \left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right)$$

$$A = \frac{2}{3} - \left(-\frac{2}{3} \right)$$

$$A = \frac{4}{3}$$

Therefore, the required area is

$$\boxed{\frac{4}{3}}$$

So, the correct option is:

Option C

Question38

The area of the region $R = \{(x, y) : xy \leq 27, 1 \leq y \leq x^2\}$ is equal to :

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Options:

A.

$$78 \log_e 3 - \frac{52}{3}$$

B.

$$54 \log_e 3 - \frac{52}{3}$$

C.

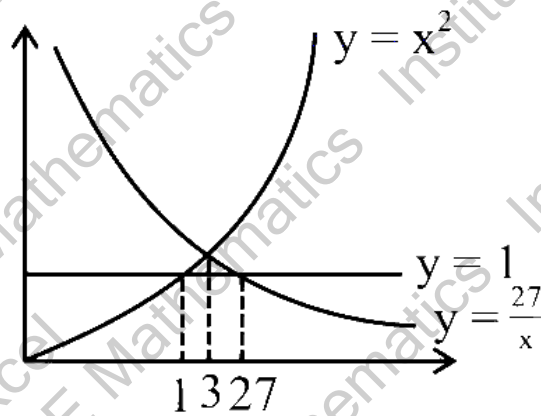
$$54 \log_e 3 - \frac{26}{3}$$

D.

$$54 \log_e 3 + \frac{26}{3}$$

Answer: B

Solution:



$$\begin{aligned}
 \text{Area} &= \int_1^3 (x^2 - 1) dx + \int_3^{27} \left(\frac{27}{x} - 1 \right) dx \\
 &= \left[\frac{x^3}{3} - x \right]_1^3 + [27 \ln x - x]_3^{27} \\
 &= \frac{26}{3} - 2 + 27 \ln 9 - 24 \\
 &= 27 \ln 9 - \frac{26 \times 2}{3} \\
 &= 54 \ln 3 - \frac{52}{3}
 \end{aligned}$$

Question 39

Let e be the base of natural logarithm and let $f : \{1, 2, 3, 4\} \rightarrow \{1, e, e^2, e^3\}$ and $g : \{1, e, e^2, e^3\} \rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ be two bijective functions such that f is strictly decreasing and g is strictly increasing. If $\phi(x) = [f^{-1} \{g^{-1}(\frac{1}{2})\}]^x$, then the area of the region $R = \{(x, y) : x^2 \leq y \leq \phi(x), 0 \leq x \leq 1\}$ is :

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Options:

A. $\frac{3 - \log_e(2)}{3 \log_e(2)}$

B.

C. $\frac{1}{3 \log_e(2)}$

$$3 + \log_e(2)$$

D.

$$\frac{3 + \log_e(2)}{2 + \log_e(3)}$$

Answer: A

Solution:

Since $f : \{1, 2, 3, 4\} \rightarrow \{1, e, e^2, e^3\}$ is bijective and strictly decreasing, we match the largest element of the domain to the smallest element of the codomain.

So,

$$f(1) = e^3, \quad f(2) = e^2, \quad f(3) = e, \quad f(4) = 1$$

Hence,

$$f^{-1}(1) = 4, \quad f^{-1}(e) = 3, \quad f^{-1}(e^2) = 2, \quad f^{-1}(e^3) = 1$$

Now $g : \{1, e, e^2, e^3\} \rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ is bijective and strictly increasing.

Since

$$1 < \frac{1}{2} < \frac{1}{3} < \frac{1}{4}$$

and

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1$$

for g to be strictly increasing, we must assign:

$$g(1) = \frac{1}{4}, \quad g(e) = \frac{1}{3}, \quad g(e^2) = \frac{1}{2}, \quad g(e^3) = 1$$

Therefore,

$$g^{-1}\left(\frac{1}{2}\right) = e^2$$

Now,

$$f^{-1}\left(g^{-1}\left(\frac{1}{2}\right)\right) = f^{-1}(e^2) = 2$$

So,

$$\phi(x) = \left[f^{-1}\left\{g^{-1}\left(\frac{1}{2}\right)\right\}\right]^x = 2^x$$

Thus the region is

$$R = \{(x, y) : x^2 \leq y \leq 2^x, 0 \leq x \leq 1\}$$

Its area is

$$\int_0^1 (2^x - x^2) dx$$

Now,

$$\int 2^x dx = \frac{2^x}{\ln 2}$$

So,

$$\int_0^1 2^x dx = \left[\frac{2^x}{\ln 2} \right]_0^1 = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$$

Also,

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

Hence area

$$= \frac{1}{\ln 2} - \frac{1}{3}$$

Taking LCM,

$$= \frac{3 - \ln 2}{3 \ln 2}$$

Therefore, the required area is

$$\boxed{\frac{3 - \log_e 2}{3 \log_e 2}}$$

So the correct option is A.

Question 40

The area of the region $\{(x, y) : 0 \leq y \leq 6 - x, y^2 \geq 4x - 3, x \geq 0\}$ is :

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Options:

A.

8

B.

9

C.

12

D.

15

Answer: B

Solution:

We need to find the area of the region

$$\{(x, y) : 0 \leq y \leq 6 - x, y^2 \geq 4x - 3, x \geq 0\}.$$

Let us understand each condition carefully.

1. From $0 \leq y \leq 6 - x$

This means:

- $y \geq 0$, so the region lies above the x -axis.
- $y \leq 6 - x$, or

$$x + y \leq 6,$$

so the region lies below the line

$$y = 6 - x.$$

2. From $y^2 \geq 4x - 3$

Rewrite it in terms of x :

$$4x - 3 \leq y^2$$

$$4x \leq y^2 + 3$$

$$x \leq \frac{y^2 + 3}{4}.$$

So the region lies to the left of the parabola

$$x = \frac{y^2 + 3}{4}.$$

Also, we are given

$$x \geq 0.$$

So for a fixed y , x lies between 0 and whichever is smaller:

$$x = \frac{y^2 + 3}{4} \quad \text{and} \quad x = 6 - y.$$

So the horizontal width is

$$0 \leq x \leq \min\left(\frac{y^2 + 3}{4}, 6 - y\right).$$

Now we find where these two curves intersect.

3. Point of intersection

Set

$$\frac{y^2 + 3}{4} = 6 - y.$$

Multiply by 4:

$$y^2 + 3 = 24 - 4y$$

$$y^2 + 4y - 21 = 0$$

$$(y + 7)(y - 3) = 0.$$

So,

$$y = 3 \quad \text{or} \quad y = -7.$$

Since $y \geq 0$, we take

$$y = 3.$$

Now compare the two right boundaries:

- For $0 \leq y \leq 3$, we have

$$\frac{y^2+3}{4} \leq 6-y,$$

so width is

$$\frac{y^2+3}{4}.$$

- For $3 \leq y \leq 6$, we have

$$6-y \leq \frac{y^2+3}{4},$$

so width is

$$6-y.$$

Hence area is

$$A = \int_0^3 \frac{y^2+3}{4} dy + \int_3^6 (6-y) dy.$$

4. Evaluate the integrals

First integral:

$$\begin{aligned} \int_0^3 \frac{y^2+3}{4} dy &= \frac{1}{4} \int_0^3 (y^2+3) dy \\ &= \frac{1}{4} \left[\frac{y^3}{3} + 3y \right]_0^3 = \frac{1}{4} \left(\frac{27}{3} + 9 \right) = \frac{1}{4} (9+9) = \frac{18}{4} = \frac{9}{2}. \end{aligned}$$

Second integral:

$$\begin{aligned} \int_3^6 (6-y) dy &= \left[6y - \frac{y^2}{2} \right]_3^6 \\ &= (36-18) - (18-\frac{9}{2}) = 18 - \frac{27}{2} = \frac{9}{2}. \end{aligned}$$

So total area is

$$A = \frac{9}{2} + \frac{9}{2} = 9.$$

Therefore, the correct answer is:

9

So, **Option B** is correct.

Question 41

The area of the region $\{(x, y) : x^2 - 8x \leq y \leq -x\}$ is :

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Options:

A.

$$\frac{343}{6}$$

B.

$$\frac{637}{6}$$

C.

$$\frac{437}{6}$$

D.

$$\frac{523}{6}$$

Answer: A

Solution:

For the region

$$\{(x, y) : x^2 - 8x \leq y \leq -x\},$$

the lower curve is

$$y = x^2 - 8x$$

and the upper curve is

$$y = -x.$$

To find the area between the curves, we first find their points of intersection.

Set

$$x^2 - 8x = -x$$

$$x^2 - 7x = 0$$

$$x(x - 7) = 0$$

So, the curves intersect at

$$x = 0 \quad \text{and} \quad x = 7.$$

Now, on the interval $[0, 7]$, the line $y = -x$ lies above the parabola $y = x^2 - 8x$.

Hence, the required area is

$$\text{Area} = \int_0^7 [(-x) - (x^2 - 8x)] dx.$$

Simplify the integrand:

$$(-x) - (x^2 - 8x) = -x - x^2 + 8x = 7x - x^2.$$

Therefore,

$$\text{Area} = \int_0^7 (7x - x^2) dx.$$

Now integrate:

$$\int (7x - x^2) dx = \frac{7x^2}{2} - \frac{x^3}{3}.$$

Applying the limits from 0 to 7:

$$\text{Area} = \left[\frac{7x^2}{2} - \frac{x^3}{3} \right]_0^7$$

$$= \left(\frac{7 \cdot 49}{2} - \frac{343}{3} \right) - 0$$

$$= \frac{343}{2} - \frac{343}{3}$$

Taking LCM 6:

$$= \frac{1029 - 686}{6} = \frac{343}{6}.$$

Hence, the correct answer is

$$\boxed{\frac{343}{6}}$$

So, the correct option is A.

Question 42

Let $f : (1, \infty) \rightarrow \mathbf{R}$ be a function defined as $f(x) = \frac{x-1}{x+1}$. Let $f^{i+1}(x) = f(f^i(x))$, $i = 1, 2, \dots, 25$, where $f^1(x) = f(x)$. If $g(x) + f^{26}(x) = 0$, $x \in (1, \infty)$, then the area of the region bounded by the curves $y = g(x)$, $2y = 2x - 3$, $y = 0$ and $x = 4$ is :

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Options:

A.

$$\frac{1}{8} + \log_e 2$$

B.

$$\frac{1}{4} + \log_e 2$$

C.

$$\frac{5}{6} + 3 \log_e 2$$

$$\frac{5}{6} + \log_e 2$$

Solution:

$$f^1(x) = f(x) = \frac{x-1}{x+1}; x \in (1, \infty) \dots\dots (i)$$

$$f^2(x) = f\{f^1(x)\} = f\{f(x)\} = f\left(\frac{x-1}{x+1}\right) = \frac{\left(\frac{x-1}{x+1}\right)-1}{\left(\frac{x-1}{x+1}\right)+1} = -\frac{2}{2x} = -\frac{1}{x}; x \in (1, \infty) \dots\dots (ii)$$

$$\mathbf{f}^3(\mathbf{x}) = f\{\mathbf{f}^2(\mathbf{x})\} = f\left(-\frac{1}{\mathbf{x}}\right) = \frac{\left(-\frac{1}{\mathbf{x}}\right)-1}{\left(-\frac{1}{\mathbf{x}}\right)+1} = \frac{1+\mathbf{x}}{1-\mathbf{x}}; \mathbf{x} \in (1, \infty) \dots \dots \textbf{(iii)}$$

$$f^4(x) = f \{ f^3(x) \} = f \left(\frac{1+x}{1-x} \right) = \frac{\left(\frac{1+x}{1-x} \right) - 1}{\left(\frac{1+x}{1-x} \right) + 1} = \frac{2x}{2} = x; x \in (1, \infty) \dots \dots (iv)$$

$$f^5(x) = f\{f^4(x)\} = f(x) = \frac{x-1}{x+1}; x \in (1, \infty) \dots \dots (v)$$

[illegible]
$$f^{25}(x) = f(x) = f^1(x) \Rightarrow f^{26}(x) = f^2(x) = -\frac{1}{x} \dots \dots (vi)$$
$$\mathbf{g}(\mathbf{x}) + \mathbf{f}^{26}(\mathbf{x}) = 0 \Rightarrow \mathbf{g}(\mathbf{x}) = -\mathbf{f}^{26}(\mathbf{x}) = \frac{1}{\mathbf{x}} \dots\dots (vii)$$

$g(x)$	Slant line	Horizontal line	Vertical line
$\frac{1}{x}$	$y = x - \left(\frac{3}{2}\right)$	X -axis	$x = 4$

Pair	$g(x)$	Slant line	Feasible	Intersection Points
1	$\frac{1}{x}$	$y = x - \left(\frac{3}{2}\right)$	Yes	A, K

$$\Rightarrow \frac{1}{x} = x - \left(\frac{3}{2}\right) \Rightarrow 2x^2 - 3x - 2 = 0 \Rightarrow x = \left(2, -\frac{1}{2}\right) \Rightarrow y = \left(\frac{1}{2}, -2\right)$$

Intersections	$\mathbf{A}$	$\mathbf{K}$	Discarded point	Reason
Coordinates	$(2, \frac{1}{2})$	$(-\frac{1}{2}, -2)$	$\mathbf{K}$	Falls outside the scope of bounded area

Pair	$g(x)$	Vertical line	Feasible	Intersection Points
2	$\frac{1}{x}$	$x = 4$	Yes	B

$$\Rightarrow x = 4 \Rightarrow y = 0.25$$

Intersection	B
Coordinate	(4, 0.25)

Pair	Horizontal line	Vertical line	Feasible	Intersection point
3	X-axis	$x = 4$	Yes	C

$$\Rightarrow x = 4, y = 0$$

Intersection	C
Coordinate	(4, 0)

Pair	Slant line	Horizontal line	Feasible	Intersection point
4	$y = x - \left(\frac{3}{2}\right)$	X-axis	Yes	D

$$\Rightarrow y = 0 \Rightarrow x = \frac{3}{2}$$

Intersection	D
Coordinate	$\left(\frac{3}{2}, 0\right)$

g(x)	X-axis	Feasible Intersection point		Reason				
1/x	y = 0	No	None	X-axis is a horizontal asymptote to g(x)				
Slant line	Vertical line	Feasible	Intersection point	Reason				
y = x - (3/2)	x = 4	No	(4, 5/2)	Intersects much above B and outside the scope of bounded area				
Sides	Left boundary	Equation	Top boundary	Equation	Right boundary	Equation	Bottom boundary	Equation
Curves	Slant line	y = x - (3/2)	Rectangular Hyperbola	y = 1/x	Vertical line	x = 4	X-axis	y = 0
Vertices	Top left	Coordinates	Top right	Coordinates	Bottom right	Coordinates	Bottom left	Coordinates
Points	A	A (2, 1/2)	B	B (4, 1/4)	C	C (4, 0)	D	D (3/2, 0)

We can visualize that both **A** and **D** lie on the slant line but **A** is ahead of **D** along the **X**-axis. From **A** we can drop a $\perp$ on the **X**-axis at **M** to form a right angled $\triangle AMD$. From **A** and **D** the remaining area upto the vertical $x = 4$ lies under the rectangular hyperbola $g(x) = \frac{1}{x}$:

$$\begin{aligned} \text{Area} &= \left(\frac{1}{2}\right) \cdot (DM) \cdot (AM) + \int_2^4 g(x) dx = \left(\frac{1}{2}\right) \cdot (2 - 1 \cdot 5) \cdot \left(\frac{1}{2}\right) + \int_2^4 \left(\frac{1}{x}\right) dx \\ &= \frac{1}{8} + [\ln |x|]_2^4 \\ &= \frac{1}{8} + \ln 2 \end{aligned}$$

Question 43

Let $y = f(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^6 + 4x}{\sqrt{1 - x^2}}, \quad -1 < x < 1 \text{ such that } f(0) = 0. \text{ If}$$

$6 \int_{-1/2}^{1/2} f(x) dx = 2\pi - \alpha$ then α^2 is equal to _____.

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Answer: 27

Solution:

$$\text{I.F. } e^{-\frac{1}{2} \int \frac{2x}{1-x^2} dx} = e^{-\frac{1}{2} \ln(1-x^2)} = \sqrt{1-x^2}$$

$$y \times \sqrt{1-x^2} = \int (x^6 + 4x) dx = \frac{x^7}{7} + 2x^2 + c$$

$$\text{Given } y(0) = 0 \Rightarrow c = 0$$

$$y = \frac{\frac{x^7}{7} + 2x^2}{\sqrt{1-x^2}}$$

$$\text{Now, } 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{x^7}{7} + 2x^2}{\sqrt{1-x^2}} dx = 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2x^2}{\sqrt{1-x^2}} dx$$

$$= 24 \int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$$

$$\text{Put } x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$= 24 \int_0^{\frac{\pi}{6}} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta$$

$$= 24 \int_0^{\frac{\pi}{6}} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta = 12 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{6}}$$

$$= 12 \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right)$$

$$= 2\pi - 3\sqrt{3}$$

$$\alpha^2 = (3\sqrt{3})^2 = 27$$

Question 44

Let f be a differentiable function such that

$2(x+2)^2 f(x) - 3(x+2)^2 = 10 \int_0^x (t+2) f(t) dt, x \geq 0$. Then $f(2)$ is equal to _____.

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Answer: 19

Solution:

Differentiate both sides

$$4(x+2)f(x) + 2(x+2)^2 f'(x) - 6(x+2) = 10(x+2)f(x)$$

$$2(x+2)^2 f'(x) - 6(x+2)f(x) = 6(x+2)$$

$$(x+2) \frac{dy}{dx} - 3y = 3$$

$$\int \frac{dy}{dx} = 3 \int \frac{dx}{x+2}$$

$$\ln(y+1) = 3 \ln(x+2) + C$$

$$(y+1) = C(x+2)^3$$

$$f(0) = \frac{3}{2}$$

$$f(2) = 19$$

Question 45

Let $y = y(x)$ be the solution of the differential equation

$2 \cos x \frac{dy}{dx} = \sin 2x - 4y \sin x, x \in (0, \frac{\pi}{2})$. If $y(\frac{\pi}{3}) = 0$, then $y'(\frac{\pi}{4}) + y(\frac{\pi}{4})$ is equal to _____.

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Answer: 1

Solution:

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

$$\text{I.F.} = e^{\int 2 \tan x dx} = \sec^2 x$$

$$y \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \tan x \sec x dx$$

$$= \sec x + C$$

$$C = -2$$

$$y = \cos x - 2 \cos^2 x$$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - 1$$

$$y' = -\sin x + 4 \cos^2 x \sin x$$

$$y'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} + 2$$

$$y'\left(\frac{\pi}{4}\right) + y\left(\frac{\pi}{4}\right) = 1$$

Question 46

If $y = y(x)$ is the solution of the differential equation,

$$\sqrt{4-x^2} \frac{dy}{dx} = \left(\left(\sin^{-1} \left(\frac{x}{2} \right) \right)^2 - y \right) \sin^{-1} \left(\frac{x}{2} \right), \quad -2 \leq x \leq 2, \quad y(2) = \frac{\pi^2 - 8}{4},$$

then $y^2(0)$ is equal to _____.

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Answer: 4

Solution:

$$\frac{dy}{dx} + \frac{\left(\sin^{-1} \frac{x}{2} \right)}{\sqrt{4-x^2}} y = \frac{\left(\sin^{-1} \frac{x}{2} \right)^3}{\sqrt{4-x^2}}$$

$$y e^{\frac{\left(\sin^{-1} \frac{x}{2} \right)^2}{2}} = \int \frac{\left(\sin^{-1} \frac{x}{2} \right)^3}{4-x^2} e^{\frac{\left(\sin^{-1} \frac{x}{2} \right)^2}{2}} dx$$

$$y = \left(\sin^{-1} \frac{x}{2} \right)^2 - 2 + c \cdot e^{-\frac{\left(\sin^{-1} \frac{x}{2} \right)^2}{2}}$$

$$y(2) = \frac{\pi^2}{4} - 2 \Rightarrow c = 0$$

$$y(0) = -2$$

Question 47

Let $y = y(x)$ be the solution of the differential equation

$\frac{dy}{dx} + 2y \sec^2 x = 2 \sec^2 x + 3 \tan x \cdot \sec^2 x$ such that $y(0) = \frac{5}{4}$. Then

$12 \left(y \left(\frac{\pi}{4} \right) - e^{-2} \right)$ is equal to _____

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Answer: 21

Solution:

$$\frac{dy}{dx} + 2y \sec^2 x = 2 \sec^2 x + 3 \tan x \sec^2 x$$

$$\text{I.F.} = e^{\int 2 \sec^2 x dx}$$

$$\text{I.F.} = e^{2 \tan x}$$

$$y \cdot e^{2 \tan x} = \int e^{2 \tan x} (2 + 3 \tan x) \sec^2 x dx$$

$$\text{Put } \tan x = u$$

$$\sec^2 x dx = du$$

$$y \cdot e^{2u} = \int e^{2u} (2 + 3u) du$$

$$y \cdot e^{2u} \Rightarrow \frac{2e^{2u}}{2} + 3 \int e^{2u} \cdot u du$$

$$y \cdot e^{2u} = e^{2u} + 3 \left[\frac{ue^{2u}}{2} - \int \frac{e^{2u}}{2} \right]$$

$$ye^{2u} = e^{2u} + 3 \left[\frac{ue^{2u}}{2} - \frac{e^{2u}}{4} \right] + C$$

$$ye^{2 \tan x} = e^{2 \tan x} + 3 \left[\frac{\tan x e^{2 \tan x}}{2} - \frac{e^{2 \tan x}}{4} \right] + C$$

$$F(0) = \frac{5}{4}$$

$$\frac{5}{4} = 1 - \frac{3}{4} + C$$

$$\frac{5}{4} - \frac{1}{4} = C$$

$$1 = C$$

$$y = 1 + 3 \left(\frac{\tan x}{2} - \frac{1}{4} \right) + 1 \cdot e^{-2 \tan x}$$

$$y \left(\frac{\pi}{4} \right) = 1 + 3 \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{e^2}$$

$$y \left(\frac{\pi}{4} \right) = \frac{7}{4} + \frac{1}{e^2}$$

$$12 \left(y \left(\frac{x}{4} \right) - \frac{1}{e^2} \right) = 12 \left(\frac{7}{4} + \frac{1}{e^2} - \frac{1}{e^2} \right) = 21$$

Question 48

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a twice differentiable function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$. If $f'(0) = 4a$ and f satisfies $f''(x) - 3af'(x) - f(x) = 0$, $a > 0$, then the area of the region $R = \{(x, y) \mid 0 \leq y \leq f(ax), 0 \leq x \leq 2\}$ is :

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Options:

A. $e^2 - 1$

B. $e^4 + 1$

C. $e^2 + 1$

D. $e^4 - 1$

Answer: A

Solution:

$$f(x+y) = f(x)f(y)$$

$$\Rightarrow f(x) = e^{\lambda x} f'(0) = 4a$$

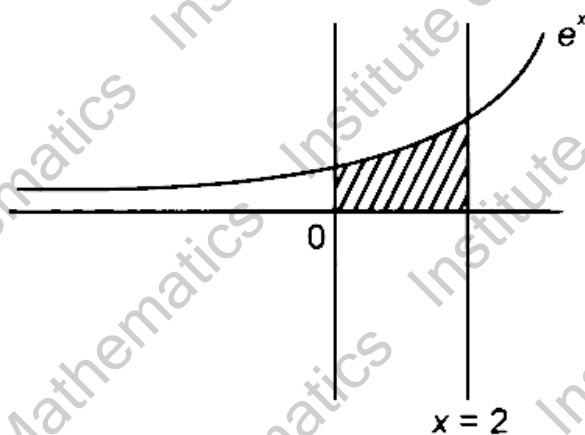
$$\Rightarrow f'(x) = \lambda e^{\lambda x} \Rightarrow \lambda = 4a$$

$$\text{So, } f(x) = e^{4ax}$$

$$f''(x) - 3af'(x) - f(x) = 0$$

$$\Rightarrow \lambda^2 - 3a\lambda - 1 = 0$$

$$\Rightarrow 16a^2 - 12a^2 - 1 = 0 \Rightarrow 4a^2 = 1 \Rightarrow a = \frac{1}{2}$$



$$F(x) = e^{2x}$$

$$\text{Area} = \int_0^2 e^x dx = e^2 - 1$$

Question 49

Let $f(x)$ be a real differentiable function such that $f(0) = 1$ and $f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in \mathbf{R}$. Then $\sum_{n=1}^{100} \log_e f(n)$ is equal to :

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Options:

A. 2406

B. 5220

C. 2525

D. 2384

Answer: C

Solution:

$$f(x+y) = f(x)f'(y) + f'(x)f(y)$$

$$\text{Put } x = y = 0$$

$$f(0) = f(0)f'(0) + f'(0)f(0)$$

$$f'(0) = \frac{1}{2}$$

$$\text{Put } y = 0$$

$$f(x) = f(x)f'(0) + f'(x)f(0)$$

$$f(x) = \frac{1}{2}f(x) + f'(x)$$

$$f'(x) = \frac{f(x)}{2}$$

$$\frac{dy}{dx} = \frac{y}{2} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{2}$$

$$\Rightarrow \ln y = \frac{x}{2} + c$$

$$\because f(0) = 1 \Rightarrow C = 0$$

$$\ln y = \frac{x}{2} \Rightarrow f(x) = e^{x/2}$$

$$\ln f(n) = \frac{n}{2}$$

$$\sum_{n=1}^{100} \ln f(n) = \frac{1}{2} \sum_{n=1}^{100} n = \frac{5050}{2} = 2525$$

Question 50

Let $x = x(y)$ be the solution of the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If $x(1) = 1$, then $x\left(\frac{1}{2}\right)$ is :

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Options:

A. $\frac{3}{2} + e$

B. $\frac{1}{2} + e$

C. $3 + e$

D. $3 - e$

Answer: D

Solution:

$$y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$$

$$y^2 dx = \left(\frac{1}{y} - x\right) dy$$

$$\Rightarrow y^2 \frac{dx}{dy} = \frac{1}{y} - x$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

$$\text{I.F.} = e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$$

$\therefore$ Solution is

$$xe^{-\frac{1}{y}} = \int e^{-\frac{1}{y}} \times \frac{1}{y^3} dy + C$$

$$\text{Let } \frac{-1}{y} = t$$

$$\Rightarrow \frac{1}{y^2} dy = dt$$

$$\Rightarrow xe^{-\frac{1}{y}} = -\int e^t t dt + C$$

$$\Rightarrow xe^{-\frac{1}{y}} = -e^t(t - 1) + C$$

$$\Rightarrow xe^{-\frac{1}{y}} = -e^{-\frac{1}{y}} \left(\frac{-1}{y} - 1\right) + C$$

$$\begin{aligned}
 x(1) &= 1 \\
 \Rightarrow e^{-1} &= -e^{-1}(-2) + C \\
 \Rightarrow C &= -e^{-1} \\
 \Rightarrow x &= \frac{1}{y} + 1 - e^{-1+\frac{1}{y}} \\
 x\left(\frac{1}{2}\right) &= 3 - e
 \end{aligned}$$

Question 51

If $x = f(y)$ is the solution of the differential equation $(1 + y^2) + (x - 2e^{\tan^{-1}y}) \frac{dy}{dx} = 0, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $f(0) = 1$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is equal to:

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Options:

A. $e^{\pi/4}$

B. $e^{\pi/12}$

C. $e^{\pi/6}$

D. $e^{\pi/3}$

Answer: C

Solution:

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{2e^{\tan^{-1}y}}{1+y^2}$$

$$\text{I.F.} = e^{\tan^{-1}y}$$

$$x^{\tan^{-1}y} = \int \frac{2(e^{\tan^{-1}y})^2 dy}{1+y^2}$$

$$\text{Put } \tan^{-1}y = t, \frac{dy}{1+y^2} = dt$$

$$xe^{\tan^{-1}y} = \int 2e^{2t} dt$$

$$xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$$

$$x = e^{\tan^{-1}y} + ce^{-\tan^{-1}y}$$

$$\because y = 0, x = 1$$

$$1 = 1 + c \Rightarrow c = 0$$

$$y = \frac{1}{\sqrt{3}}, x = e^{\pi/6}$$

Question 52

Let a curve $y = f(x)$ pass through the points $(0, 5)$ and $(\log_e 2, k)$. If the curve satisfies the differential equation $2(3 + y)e^{2x} dx - (7 + e^{2x}) dy = 0$, then k is equal to

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Options:

A. 32

B. 8

C. 4

D. 16

Answer: B

Solution:

$$\frac{dy}{dx} = \frac{2(3 + y) \cdot e^{2x}}{7 + e^{2x}}$$

$$\frac{dy}{dx} - \frac{2y \cdot e^{2x}}{7 + e^{2x}} = \frac{6 \cdot e^{2x}}{7 + e^{2x}}$$

$$\text{I.F.} = e^{-\int \frac{2e^{2x}}{7 + e^{2x}} dx} = \frac{1}{7 + e^{2x}}$$

$$\therefore y \cdot \frac{1}{7 + e^{2x}} = \int \frac{6e^{2x}}{(7 + e^{2x})^2} dx$$

$$\frac{y}{7+e^{2x}} = \frac{-3}{7+e^{2x}} + C$$

$$(0, 5) \Rightarrow \frac{5}{8} = \frac{-3}{8} + C \Rightarrow C = 1$$

$$\therefore y = -3 + 7 + e^{2x}$$

$$y = e^{2x} + 4$$

$$\therefore k = 8$$

Question 53

Let $x = x(y)$ be the solution of the differential equation $y = \left(x - y \frac{dx}{dy}\right) \sin\left(\frac{x}{y}\right)$, $y > 0$ and $x(1) = \frac{\pi}{2}$. Then $\cos(x(2))$ is equal to :

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Options:

- A. $2(\log_e 2) - 1$
- B. $1 - 2(\log_e 2)^2$
- C. $1 - 2(\log_e 2)$
- D. $2(\log_e 2)^2 - 1$

Answer: D

Solution:

$$y dy = (x dy - y dx) \sin\left(\frac{x}{y}\right)$$

$$\frac{dy}{y} = \left(\frac{x dy - y dx}{y^2}\right) \sin\left(\frac{x}{y}\right)$$

$$\frac{dy}{y} = \sin\left(\frac{x}{y}\right) d\left(-\frac{x}{y}\right)$$

$$\Rightarrow \ln y = \cos \frac{x}{y} + C$$

$$x(1) = \frac{\pi}{2} \Rightarrow 0 = \cos \frac{\pi}{2} + C \Rightarrow C = 0$$

$$\ln y = \cos \frac{x}{y}$$

$$\begin{aligned}\text{but } y = 2 &\Rightarrow \cos \frac{x}{2} = \ln 2 \\ \cos x &= 2 \cos^2 \frac{x}{2} - 1 \\ &= 2(\ln 2)^2 - 1\end{aligned}$$

Question 54

Let $y = y(x)$ be the solution of the differential equation $(xy - 5x^2\sqrt{1+x^2})dx + (1+x^2)dy = 0, y(0) = 0$. Then $y(\sqrt{3})$ is equal to

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Options:

A. $\frac{5\sqrt{3}}{2}$

B. $\sqrt{\frac{15}{2}}$

C. $\sqrt{\frac{14}{3}}$

D. $2\sqrt{2}$

Answer: A

Solution:

$$\begin{aligned}(1+x^2)\frac{dy}{dx} + xy &= 5x^2\sqrt{1+x^2} \\ \frac{dy}{dx} + \frac{xy}{1+x^2} &= \frac{5x^2}{\sqrt{1+x^2}} \\ \therefore \text{I.F.} &= e^{\int \frac{x}{1+x^2} dx} = e^{\frac{\ln(1+x^2)}{2}} = \sqrt{1+x^2} \\ \therefore y\sqrt{1+x^2} &= \int \frac{5x^2}{\sqrt{1+x^2}} \cdot \sqrt{1+x^2} dx \\ \therefore y\sqrt{1+x^2} &= \int \frac{5x^2}{\sqrt{1+x^2}} \cdot \sqrt{1+x^2} dx \\ y\sqrt{1+x^2} &= \frac{5x^3}{3} + C\end{aligned}$$

$$\because y(0) = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0$$

$$\therefore y = \frac{5x^3}{3\sqrt{1+x^2}}$$

$$y(\sqrt{3}) = \frac{15\sqrt{3}}{32} = \frac{5\sqrt{3}}{2}$$

Question 55

Let for some function $y = f(x)$, $\int_0^x t f(t) dt = x^2 f(x)$, $x > 0$ and $f(2) = 3$. Then $f(6)$ is equal to

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Options:

- A. 1
- B. 6
- C. 2
- D. 3

Answer: A

Solution:

$$\int_0^x t f(t) dt = x^2 + (x), x > 0$$

Diff. both side w.r. to x

$$x f(x) = x^2 f'(x) + 2x f(x)$$

$$-x f(x) = x^2 f'(x)$$

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{-1}{x} dx$$

$$\log f(x) = -\log x + \log c$$

$$f(x) = \frac{c}{x}$$

$$f(2) = 3 \Rightarrow 3 = \frac{c}{2} \Rightarrow c = 6$$

$$f(x) = \frac{6}{x}$$

$$f(6) = 1 \quad \therefore (1)$$

Question 56

Let $y = y(x)$ be the solution of the differential equation :

$$\cos x (\log_e (\cos x))^2 dy + (\sin x - 3y \sin x \log_e (\cos x)) dx = 0, \quad x \in (0, \frac{\pi}{2}).$$

If $y(\frac{\pi}{4}) = -\frac{1}{\log_e 2}$, then $y(\frac{\pi}{6})$ is equal to :

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Options:

A.

$$\frac{2}{\log_e(3) - \log_e(4)}$$

B.

$$-\frac{1}{\log_e(4)}$$

C.

$$\frac{1}{\log_e(4) - \log_e(3)}$$

D.

$$\frac{1}{\log_e(3) - \log_e(4)}$$

Answer: D

Solution:

$$\cos x (\ln(\cos x))^2 dy + (\sin x - 3y(\sin x) \ln(\cos x)) dx = 0$$

$$\cos x (\ln(\cos x))^2 \frac{dy}{dx} - 3 \sin x \cdot \ln(\cos x) y = -\sin x$$

$$\frac{dy}{dx} - \frac{3 \tan x}{\ln(\cos x)} y = \frac{-\tan x}{(\ln(\cos x))^2}$$

$$\frac{dy}{dx} + \frac{3 \tan x}{\ln(\sec x)} y = \frac{-\tan x}{(\ln(\sec x))^2}$$

$$\text{I.F.} = e^{\int \frac{3 \tan x}{\ln(\sec x)} dx} = (\ln(\sec x))^3$$

$$y \times (\ln(\sec x))^3 = - \int \frac{\tan x}{(\ln(\sec x))^2} (\ln(\sec x))^3 dx$$

$$y \times (\ln(\sec x))^3 = -\frac{1}{2} (\ln(\sec x))^2 + C$$

$$\text{Given : } x = \frac{\pi}{4}, y = -\frac{1}{\ln 2}$$

$$\frac{-1}{\ln 2} \times (\ln \sqrt{2})^3 = -\frac{1}{2} \times (\ln \sqrt{2})^2 + C$$

$$\Rightarrow \frac{-1}{8 \ln 2} \times (\ln 2)^3 = -\frac{1}{2} \times \frac{1}{4} (\ln 2)^2 + C$$

$$-\frac{1}{8} (\ln 2)^2 = -\frac{1}{8} (\ln 2)^2 + C$$

$$\Rightarrow C = 0$$

$$\therefore y(\ln(\sec x))^3 = -\frac{1}{2} (\ln(\sec x))^2 + 0$$

$$y = \frac{-1}{2 \ln(\sec x)}$$

$$y = \frac{1}{2 \ln(\cos x)}$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{1}{2 \ln\left(\cos \frac{\pi}{6}\right)}$$

$$= \frac{1}{2 \ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{2\left(\frac{1}{2} \ln 3 - \ln 2\right)}$$

$$= \frac{1}{\ln 3 - \ln 4}$$

Question 57

If for the solution curve $y = f(x)$ of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \frac{2 + \sec x}{(1 + 2 \sec x)^2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{10}, \text{ then } f\left(\frac{\pi}{4}\right) \text{ is equal}$$

to:

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Options:

A. $\frac{5-\sqrt{3}}{2\sqrt{2}}$

B.

$\frac{4-\sqrt{2}}{14}$

C.

$\frac{9\sqrt{3}+3}{10(4+\sqrt{3})}$

D.

$\frac{\sqrt{3}+1}{10(4+\sqrt{3})}$

Answer: B

Solution:

$$\text{If } e^{\int \tan x dx} = e^{\ln(\sec x)} = \sec x$$

$$\therefore y \cdot \sec x = \int \left\{ \frac{2 + \sec x}{(1 + 2 \sec x)^2} \right\} \sec x dx$$

$$= \int \frac{2 \cos x + 1}{(\cos x + 2)^2} dx \text{ Let } \cos x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{2 \left(\frac{1-t^2}{1+t^2} \right) + 1}{\left(\frac{1-t^2}{1+t^2} + 2 \right)^2} 2dt$$

$$= \int \frac{2 - 2t^2 + 1 + t^2}{(1 - t^2 + 2 + 2t^2)^2} \times 2dt$$

$$= 2 \int \frac{3 - t^2}{(t^2 + 3)^2} dt$$

$$\text{Let } t + \frac{3}{t} = u$$

$$\left(1 - \frac{3}{t^2} \right) dt = du$$

$$= -2 \int \frac{du}{u^2}$$

$$y \cdot (\sec x) = \frac{2}{u} + c$$

$$y \cdot \sec x = \frac{2}{t + \frac{3}{t}} + c \dots (I)$$

$$\text{At } x = \frac{\pi}{3}, t = \tan \frac{x}{2} = \frac{1}{\sqrt{3}}$$

$$2 \cdot \frac{\sqrt{3}}{10} = \frac{2}{\frac{1}{\sqrt{3}} + 3\sqrt{3}} + c$$

$$2 \cdot \frac{\sqrt{3}}{10} = \frac{2\sqrt{3}}{10} + c \Rightarrow C = 0$$

$$\text{At } x = \frac{\pi}{4}, t = \tan \frac{x}{2} = \sqrt{2} - 1$$

$$\begin{aligned}\therefore y \cdot \sqrt{2} &= \frac{2}{\sqrt{2}-1 + \frac{3}{\sqrt{2}-1}} \\ y \cdot \sqrt{2} &= \frac{2(\sqrt{2}-1)}{6-2\sqrt{2}} \\ y &= \frac{\sqrt{2}(\sqrt{2}-1)}{2(3-\sqrt{2})} = \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}-1}{7} \\ &= \frac{4-\sqrt{2}}{14}\end{aligned}$$

Question 58

Let g be a differentiable function such that

$\int_0^x g(t) dt = x - \int_0^x t g(t) dt, x \geq 0$ and let $y = y(x)$ satisfy the differential equation $\frac{dy}{dx} - y \tan x = 2(x+1) \sec x g(x), x \in [0, \frac{\pi}{2})$. If $y(0) = 0$, then $y(\frac{\pi}{3})$ is equal to

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

A. $\frac{4\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{2\pi}{3\sqrt{3}}$

D. $\frac{4\pi}{3\sqrt{3}}$

Answer: A

Solution:

To solve the given problem, let's start by considering the equation:

$$\int_0^x g(t) dt = x - \int_0^x t g(t) dt$$

Differentiate both sides with respect to x :

$$g(x) = 1 - xg(x)$$

Rearranging gives:

$$g(x)(1+x) = 1 \implies g(x) = \frac{1}{1+x}$$

Now, consider the differential equation:

$$\frac{dy}{dx} - y \tan x = 2(x+1) \sec x g(x)$$

Substitute $g(x) = \frac{1}{1+x}$:

$$\frac{dy}{dx} - y \tan x = 2(x+1) \sec x \cdot \frac{1}{1+x}$$

This simplifies to:

$$\frac{dy}{dx} - y \tan x = 2 \sec x$$

To solve this, we use an Integrating Factor (I.F):

$$\text{I.F} = e^{\int \tan x \, dx} = e^{-\ln \cos x} = \cos x$$

Multiply the entire differential equation by the Integrating Factor:

$$\cos x \cdot \frac{dy}{dx} - y \cdot \sin x = 2$$

This implies:

$$\frac{d}{dx}(y \cos x) = 2$$

Integrate both sides with respect to x :

$$y \cos x = \int 2 \, dx = 2x + c$$

Given the initial condition $y(0) = 0$:

$$0 \cdot 1 = 2 \cdot 0 + c \implies c = 0$$

Thus:

$$y \cos x = 2x$$

Evaluate at $x = \frac{\pi}{3}$:

$$y \cdot \frac{1}{2} = 2 \cdot \frac{\pi}{3}$$

Solving for y :

$$y = \frac{4\pi}{3}$$

Therefore, $y\left(\frac{\pi}{3}\right) = \frac{4\pi}{3}$.

Question 59

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 3(\tan^2 x)y + 3y = \sec^2 x, y(0) = \frac{1}{3} + e^3. \text{ Then } y\left(\frac{\pi}{4}\right) \text{ is equal to :}$$

JEE Main 2025 (Online) 3rd April Evening Shift

Options:

A. $\frac{4}{3}$

B. $\frac{2}{3} + e^3$

C. $\frac{4}{3} + e^3$

D. $\frac{2}{3}$

Answer: A

Solution:

$$\frac{dy}{dx} + 3(\tan^2 x)y + 3y = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + 3\sec^2 xy = \sec^2 x$$

$$\begin{aligned} \text{I.F} &= e^{\int 3\sec^2 x dx} \\ &= e^{3\tan x} \end{aligned}$$

$$y \cdot e^{\tan x} = \int e^{3\tan x} \cdot \sec^2 x dx + c$$

$$y \cdot e^{3\tan x} = \frac{e^{3\tan x}}{3} + c$$

$$\text{Also } f(0) = \frac{1}{3} + e^3$$

$$\Rightarrow \left(\frac{1}{3} + e^3 \right) = \frac{1}{3} + c$$

$$\Rightarrow c = e^3$$

$$\therefore y \cdot e^{3\tan x} = \frac{e^{3\tan x}}{3} + e^3$$

$$\text{Put } x = \frac{\pi}{4}$$

$$ye^3 = \frac{e^3}{3} + e^3 \Rightarrow y = \frac{4}{3}$$

Question 60

If a curve $y = y(x)$ passes through the point $(1, \frac{\pi}{2})$ and satisfies the differential equation $(7x^4 \cot y - e^x \operatorname{cosec} y) \frac{dx}{dy} = x^5, x \geq 1$, then at $x = 2$, the value of $\cos y$ is :

JEE Main 2025 (Online) 4th April Evening Shift

Options:

A. $\frac{2e^2 + e}{64}$

B. $\frac{2e^2 - e}{64}$

C. $\frac{2e^2 - e}{128}$

D. $\frac{2e^2+e}{128}$

Answer: C

Solution:

$$(7x^4 \cot y - e^x \operatorname{cosec} y) \frac{dx}{dy} = x^5$$

$$x^5 \frac{dy}{dx} - 7x^4 \cot y = -e^x \operatorname{cosec} y$$

$$\frac{dy}{dx} - \frac{7}{x} \cot y = -\frac{e^x}{x^5} \operatorname{cosec} y$$

$$\sin y \frac{dy}{dx} - \frac{7}{x} \cos y = -\frac{e^x}{x^5}$$

$$\text{Let } -\cos y = t$$

$$\sin y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dt}{dx} + \frac{7}{x} t = -\frac{e^x}{x^5}$$

$$\therefore \text{I.F.} = e^{\int \frac{7}{x} dx} = x^7$$

$$t \cdot x^7 = \int \frac{-e^x}{x^5} \cdot x^7 dx$$

$$-\cos y \cdot x^7 = -\int e^x x^2 dx$$

$$\cos y x^7 = e^x (x^2 - 2x + 2) + c$$

$$\therefore x = 1 \text{ then } y = \frac{\pi}{2} \Rightarrow c = -e$$

$$\therefore \cos y \cdot x^7 = e^x (x^2 - 2x + 2) - e$$

$$\text{When } x = 2 \text{ then } \cos y = \frac{2e^2 - e}{128}$$

Question 61

Let $y = y(x)$ be the solution curve of the differential equation

$x(x^2 + e^x)dy + (e^x(x - 2)y - x^3)dx = 0, x > 0$, passing through the point $(1, 0)$. Then $y(2)$ is equal to :

JEE Main 2025 (Online) 7th April Morning Shift

Options:

A. $\frac{2}{2+e^2}$

B. $\frac{4}{4-e^2}$

C. $\frac{4}{4+e^2}$

D. $\frac{2}{2-e^2}$

Answer: C

Solution:

$$x(x^2 + e^x)dy + (e^x(x-2)y - x^3)dx = 0$$

$$\frac{dy}{dx} + \frac{e^x(x-2)}{x(x^2 + e^x)}y = \frac{x^3}{x(x^2 + e^x)}$$

$$\text{I.F.} = e^{\int \frac{e^x(x-2)}{x(x^2 + e^x)} dx}$$

$$= e^{\int \frac{e^x + 2x}{e^x + x^2} dx - \int \frac{2}{x} dx}$$

$$= e^{\ln e^{-x} + x^2 - 2 \ln x}$$

$$= \frac{e^x + x^2}{x^2}$$

$$\therefore y \left(\frac{e^x + x^2}{x^2} \right) = \int dx + c$$

$$\Rightarrow y \left(\frac{e^x + x^2}{x^2} \right) = x + c$$

Also, $y(1) = 0$

$$\Rightarrow c = -1$$

$$\therefore y \left(\frac{e^x + x^2}{x^2} \right) = x - 1$$

Hence, $y(2) = \frac{4}{e^2 + 4}$

Question 62

Let $y = y(x)$ be the solution of the differential equation

$$(x^2 + 1)y' - 2xy = (x^4 + 2x^2 + 1) \cos x,$$

$$y(0) = 1. \text{ Then } \int_{-3}^3 y(x) dx \text{ is :}$$

JEE Main 2025 (Online) 7th April Evening Shift

Options:

A.

36

B.

24

C.

18

D.

30

Answer: B

Solution:

$$(x^2 + 1) \frac{dy}{dx} - 2xy = (x^4 + 2x^2 + 1) \cos x$$

$$\frac{dy}{dx} - \left(\frac{2x}{x^2 + 1} \right) y = \frac{(x^2 + 1)^2 \cos x}{cx^2 + 1} = (x^2 + 1) \cos x$$

(Linear D.E)

$$P = \frac{-2x}{x^2 + 1}, Q = (x^2 + 1) \cos x$$

$$\text{I.F} = e^{\int P dx} = e^{\int \frac{-2x}{x^2 + 1} dx} = \frac{1}{x^2 + 1}$$

$$y \cdot \frac{1}{x^2 + 1} = \int (x^2 + 1) \cos x \cdot \frac{1}{x^2 + 1} dx$$

$$\frac{y}{x^2 + 1} = \sin x + c \Rightarrow y \cos = 1 \Rightarrow c = 1$$

$$y = (x^2 + 1)(\sin x + 1)$$

$$\int_{-3}^3 y dx = \int_{-3}^3 (x^2 + 1)(\sin x + 1) dx$$

$$dx = \int_{-3}^3 x^2 \sin x + x^2 \sin x + 1 dx$$

$$\Rightarrow \int_{-3}^3 x^2 \sin x dx + \int_{-3}^3 x^2 dx + \int_{-3}^3 \sin x dx + \int_{-3}^3 1 dx$$

$$= 0 + 18 + 0 + 6 = 24$$

Question 63

Let $f(x) = x - 1$ and $g(x) = e^x$ for $x \in \mathbb{R}$. If

$\frac{dy}{dx} = \left(e^{-2\sqrt{x}} g(f(f(x))) - \frac{y}{\sqrt{x}} \right)$, $y(0) = 0$, then $y(1)$ is

JEE Main 2025 (Online) 8th April Evening Shift

Options:

A.

$$\frac{1-e^3}{e^4}$$

B.

$$\frac{e-1}{e^4}$$

C.

$$\frac{1-e^2}{e^4}$$

D.

$$\frac{2e-1}{e^3}$$

Answer: B

Solution:

$$f(x) = x - 1$$

$$f(f(x)) = f(x) - 1 = x - 1 - 1 = x - 2$$

$$g(f(f(x))) = e^{x-2}$$

$$\therefore \frac{dy}{dx} = e^{-2\sqrt{x}} \times e^{x-2} - \frac{1}{\sqrt{x}} y$$

$$\frac{dy}{dx} + \frac{1}{\sqrt{x}} y = e^{x-2\sqrt{x}-2} \text{ which is L.D.E}$$

$$\text{I.F.} = e^{\int \frac{dy}{\sqrt{x}}} = e^{2\sqrt{x}}$$

Its solution is

$$y \times e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \times e^{x-2\sqrt{x}-2} dx + c$$

$$y \times e^{2\sqrt{x}} = \int e^{x-2} dx + c$$

$$y \times e^{2\sqrt{x}} = e^{x-2} + c$$

$$\text{Given } x=0, y=0 \Rightarrow 0 = e^{-2} + c \quad ; c = -e^{-2}$$

$$\therefore y \times e^{2\sqrt{x}} = e^{x-2} - e^{-2}$$

$$\text{when } x=1, y \times e^2 = e^{-1} - e^{-2}$$

$$y = \frac{e^{-1}-e^{-2}}{e^2} = \frac{\frac{1}{e}-\frac{1}{e^2}}{e^2} = \frac{e^2-e}{e^5} = \frac{e-1}{e^4}$$

Option (1) is correct

Question64

Let $x = x(t)$ and $y = y(t)$ be solutions of the differential equations

$\frac{dx}{dt} + ax = 0$ and $\frac{dy}{dt} + by = 0$ respectively, $a, b \in \mathbb{R}$. Given that

$x(0) = 2$; $y(0) = 1$ and $3y(1) = 2x(1)$, the value of t , for which $x(t) = y(t)$, is :
[27-Jan-2024 Shift 1]

Options:

A. $\log_2 \frac{2}{3}$

B. $\log_4 3$

C. $\log_3 4$

D. $\log_4 \frac{2}{3}$

Answer: D

Solution:

$$\frac{dx}{dt} + ax = 0$$

$$\frac{dx}{x} = -a dt$$

$$\int \frac{dx}{x} = -a \int dt$$

$$\ln |x| = -at + c$$

$$at = 0, x = 2$$

$$\ln 2 = 0 + c$$

$$\ln x = -at + \ln 2$$

$$\frac{x}{2} = e^{-at}$$

$$x = 2e^{-at} \dots\dots\dots (I)$$

$$\frac{dy}{dt} + by = 0$$

$$\frac{dy}{y} = -b dt$$

$$\ln |y| = -bt + \lambda$$

$$t = 0, y = 1$$

$$0 = 0 + \lambda$$

$$y = e^{-bt} \dots\dots\dots (II)$$

According to question

$$3y(1) = 2x(1)$$

$$3e^{-b} = 2(2e^{-a})$$

$$e^{a-b} = \frac{4}{3}$$

$$\text{For } x(t) = y(t)$$

$$\Rightarrow 2e^{-at} = e^{-bt}$$

$$2 = e^{(a-b)t}$$

$$2 = \left(\frac{4}{3}\right)^t$$

$$\log_{\frac{4}{3}} 2 = t$$

Question 65

If the solution of the differential equation

$$(2x + 3y - 2) dx + (4x + 6y - 7) dy = 0, y(0) = 3, \text{ is}$$

$\alpha x + \beta y + 3 \log_e 2x + 3y - \gamma = 6$, then $\alpha + 2\beta + 3\gamma$ is equal to _____

[27-Jan-2024 Shift 1]

Answer: 29

Solution:

$$2x + 3y - 2 = t$$

$$4x + 6y - 4 = 2t$$

$$2 + 3 \frac{dy}{dx} = \frac{dt}{dx}$$

$$4x + 6y - 7 = 2t - 3$$

$$\frac{dy}{dx} = \frac{-(2x + 3y - 2)}{4x + 6y - 7}$$

$$\frac{dt}{dx} = \frac{-3t + 4t - 6}{2t - 3} = \frac{t - 6}{2t - 3}$$

$$\int \frac{2t - 3}{t - 6} dt = \int dx$$

$$\int \left(\frac{2t - 12}{t - 6} + \frac{9}{t - 6} \right) \cdot dt = x$$

$$2t + 9 \ln(t - 6) = x + c$$

$$2(2x + 3y - 2) + 9 \ln(2x + 3y - 8) = x + c$$

$$x = 0, y = 3$$

$$c = 14$$

$$4x + 6y - 4 + 9 \ln(2x + 3y - 8) = x + 14$$

$$x + 2y + 3 \ln(2x + 3y - 8) = 6$$

$$\alpha = 1, \beta = 2, \gamma = 8$$

$$\alpha + 2\beta + 3\gamma = 1 + 4 + 24 = 29$$

Question 66

If $y = y(x)$ is the solution curve of the differential equation

$$(x^2 - 4)dy - (y^2 - 3y)dx = 0,$$

$x > 2$, $y(4) = \frac{3}{2}$ and the slope of the curve is never zero, then the value of $y(10)$ equals :

[27-Jan-2024 Shift 2]

Options:

A. $\frac{3}{1+(8)^{1/4}}$

B. $\frac{3}{1+2\sqrt{2}}$

C. $\frac{3}{1-2\sqrt{2}}$

D. $\frac{3}{1-(8)^{1/4}}$

Answer: A

Solution:

$$(x^2 - 4)dy - (y^2 - 3y)dx = 0$$

$$\Rightarrow \int \frac{dy}{y^2 - 3y} = \int \frac{dx}{x^2 - 4}$$

$$\Rightarrow \frac{1}{3} \int \frac{y - (y - 3)}{y(y - 3)} dy = \int \frac{dx}{x^2 - 4}$$

$$\Rightarrow \frac{1}{3} (\ln |y - 3| - \ln |y|) = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + C$$

$$\Rightarrow \frac{1}{3} \ln \left| \frac{y - 3}{y} \right| = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + C$$

$$\text{At } x = 4, y = \frac{3}{2}$$

$$\therefore C = \frac{1}{4} \ln 3$$

$$\therefore \frac{1}{3} \ln \left| \frac{y-3}{y} \right| = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + \frac{1}{4} \ln(3)$$

$$\text{At } x = 10$$

$$\frac{1}{3} \ln \left| \frac{y-3}{y} \right| = \frac{1}{4} \ln \left| \frac{2}{3} \right| + \frac{1}{4} \ln(3)$$

$$\ln \left| \frac{y-3}{y} \right| = \ln 2^{3/4}, \forall x > 2, \frac{dy}{dx} < 0$$

$$\text{as } y(4) = \frac{3}{2} \Rightarrow y \in (0, 3)$$

$$-y + 3 = 8^{1/4} \cdot y$$

$$y = \frac{3}{1 + 8^{1/4}}$$

Question 67

If the solution curve, of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passing through the point (2, 1) is

$$\tan^{-1} \left(\frac{y-1}{x-1} \right) - \frac{1}{\beta} \log_e \left(\alpha + \left(\frac{y-1}{x-1} \right)^2 \right) = \left| \log_e x - 1 \right|$$

then $5\beta + \alpha$ is equal to ____
[27-Jan-2024 Shift 2]

Answer: 11

Solution:

$$\frac{dy}{dx} = \frac{x+y-2}{x-y}$$

$$x = X+h, y = Y+k$$

$$\frac{dY}{dX} = \frac{X+Y}{X-Y}$$

$$\left. \begin{array}{l} h+k-2=0 \\ h-k=0 \end{array} \right\} h=k=1$$

$$Y = vX$$

$$v + \frac{dv}{dX} = \frac{1+v}{1-v} \Rightarrow X - \frac{dv}{dX} = \frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2} dv = \frac{dX}{X}$$

$$\tan^{-1} v - \frac{1}{2} \ln(1+v^2) = \ln|X| + C$$

As curve is passing through (2, 1)

$$\tan^{-1} \left(\frac{y-1}{x-1} \right) - \frac{1}{2} \ln \left(1 + \left(\frac{y-1}{x-1} \right)^2 \right) = \ln|x-1|$$

$$\therefore \alpha = 1 \text{ and } \beta = 2$$

$$\Rightarrow 5\beta + \alpha = 11$$

Question 68

A function $y = f(x)$ satisfies $f(x) \sin 2x + \sin x - (1 + \cos^2 x)f'(x) = 0$ with condition $f(0) = 0$. Then $f\left(\frac{\pi}{2}\right)$ is equal to

[29-Jan-2024 Shift 1]

Options:

- A. 1
- B. 0
- C. -1
- D. 2

Answer: A

Solution:

$$\frac{dy}{dx} - \left(\frac{\sin 2x}{1 + \cos^2 x} \right) y = \sin x$$

$$\text{I.F.} = 1 + \cos^2 x$$

$$y \cdot (1 + \cos^2 x) = \int (\sin x) dx$$

$$= -\cos x + C$$

$$x=0, C=1$$

$$y\left(\frac{\pi}{2}\right) = 1$$

Question 69

If the solution curve $y = y(x)$ of the differential equation

$(1 + y^2)(1 + \log_e x) dx + x dy = 0$, $x > 0$ passes through the point $(1, 1)$ and

$$y(e) = \frac{\alpha - \tan\left(\frac{3}{2}\right)}{\beta + \tan\left(\frac{3}{2}\right)}, \text{ then } \alpha + 2\beta \text{ is}$$

[29-Jan-2024 Shift 1]

Answer: 3

Solution:

$$\int \left(\frac{1}{x} + \frac{\ln x}{x} \right) dx + \int \frac{dy}{1+y^2} = 0$$

$$\ln x + \frac{(\ln x)^2}{2} + \tan^{-1} y = C$$

$$\text{Put } x = y = 1$$

$$\therefore C = \frac{\pi}{4}$$

$$\Rightarrow \ln x + \frac{(\ln x)^2}{2} + \tan^{-1} y = \frac{\pi}{4}$$

Put $x = e$

$$\Rightarrow y = \tan\left(\frac{\pi}{4} - \frac{3}{2}\right) = \frac{1 - \tan \frac{3}{2}}{1 + \tan \frac{3}{2}}$$

$$\therefore \alpha = 1, \beta = 1$$

$$\Rightarrow \alpha + 2\beta = 3$$

Question 70

If $\sin\left(\frac{y}{x}\right) = \log_e |x| + \frac{\alpha}{2}$ is the solution of the differential equation

$x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$ and $y = \frac{\pi}{3}$, then α^2 is equal to

[29-Jan-2024 Shift 2]

Options:

- A. 3
- B. 12
- C. 4
- D. 9

Answer: A

Solution:

Differential equation :-

$$x \cos \frac{y}{x} \frac{dy}{dx} = y \cos \frac{y}{x} + x$$

$$\cos \frac{y}{x} \left[x \frac{dy}{dx} - y \right] = x$$

Divide both sides by x^2

$$\cos \frac{y}{x} \left(\frac{x \frac{dy}{dx} - y}{x^2} \right) = \frac{1}{x}$$

$$\text{Let } \frac{y}{x} = t$$

$$\cos t \left(\frac{dt}{dx} \right) = \frac{1}{x}$$

$$\cos t = \frac{1}{x} dx$$

Integrating both sides

$$\sin t = \ln |x| + c$$

$$\sin \frac{y}{x} = \ln |x| + c$$

$$\text{Using } y(1) = \frac{\pi}{3}, \text{ we get } c = \frac{\sqrt{3}}{2}$$

$$\text{So, } \alpha = \sqrt{3} \Rightarrow \alpha^2 = 3$$

Question 71

Let $y = y(x)$ be the solution of the differential equation

$$(1 - x^2) dy = [xy + (x^3 + 2)\sqrt{3(1 - x^2)}] dx, -1 < x < 1, y(0) = 0. \text{ If } y\left(\frac{1}{2}\right) = \frac{m}{n}, m$$

and n are coprime numbers, then $m + n$ is equal to

[30-Jan-2024 Shift 1]

Answer: 97

Solution:

$$\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{(x^3+2)\sqrt{3(1-x^2)}}{1-x^2}$$

$$\text{IF} = e^{-\int \frac{x}{1-x^2} dx} = e^{+\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2}$$

$$y\sqrt{1-x^2} = \sqrt{3} \int (x^3+2) dx$$

$$y\sqrt{1-x^2} = \sqrt{3} \left(\frac{x^4}{4} + 2x \right) + c$$

$$\Rightarrow y(0) = 0 \quad \therefore c = 0$$

$$y\left(\frac{1}{2}\right) = \frac{65}{32} = \frac{m}{n}$$

$$m+n=97$$

Question 72

Let $y = y(x)$ be the solution of the differential equation $\sec x \, dy + \{2(1-x)\tan x + x(2-x)\} \, dx = 0$ such that $y(0) = 2$. Then $y(2)$ is equal to :

[30-Jan-2024 Shift 1]

Options:

- A. 2
- B. $2\{1 - \sin(2)\}$
- C. $2\{\sin(2) + 1\}$
- D. 1

Answer: A

Solution:

$$\frac{dy}{dx} = 2(x-1)\sin x + (x^2 - 2x)\cos x$$

Now both side integrate

$$y(x) = \int 2(x-1)\sin x \, dx + [(x^2 - 2x)(\sin x) - \int (2x-2)\sin x \, dx]$$

$$y(x) = (x^2 - 2x)\sin x + \lambda$$

$$y(0) = 0 + \lambda \Rightarrow 2 = \lambda$$

$$y(x) = (x^2 - 2x)\sin x + 2$$

$$y(2) = 2$$

Question 73

The solution curve of the differential equation

$$y \frac{dx}{dy} = x(\log_e x - \log_e y + 1), x > 0, y > 0 \text{ passing}$$

through the point (e, 1) is

[31-Jan-2024 Shift 1]

Options:

A. $\left| \log_e \frac{y}{x} \right| = x$

B. $\left| \log_e \frac{y}{x} \right| = y^2$

C. $\left| \log_e \frac{x}{y} \right| = y$

D. $2 \left| \log_e \frac{x}{y} \right| = y + 1$

Answer: C

Solution:

$$\frac{dx}{dy} = \frac{x}{y} \left(\ln \left(\frac{x}{y} \right) + 1 \right)$$

$$\text{Let } \frac{x}{y} = t \Rightarrow x = ty$$

$$\frac{dx}{dy} = t + y \frac{dt}{dy}$$

$$t + y \frac{dt}{dy} = t(\ln(t) + 1)$$

$$y \frac{dt}{dy} = t \ln(t) \Rightarrow \frac{dt}{t \ln(t)} = \frac{dy}{y}$$

$$\Rightarrow \int \frac{dt}{t \cdot \ln(t)} = \int \frac{dy}{y}$$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{dy}{y}$$

$$\text{let } \ln t = p$$

$$\frac{1}{t} dt = dp$$

$$\Rightarrow \ln p = \ln y + c$$

$$\ln(\ln t) = \ln y + c$$

$$\ln \left(\ln \left(\frac{x}{y} \right) \right) = \ln y + c$$

$$\text{at } x = e, y = 1$$

$$\ln \left(\ln \left(\frac{e}{1} \right) \right) = \ln(1) + c \Rightarrow c = 0$$

$$\ln \left| \ln \left(\frac{x}{y} \right) \right| = \ln y$$

$$\left| \ln \left(\frac{x}{y} \right) \right| = e^{\ln y}$$

$$\left| \ln \left(\frac{x}{y} \right) \right| = y$$

Question 74

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = \frac{(\tan x) + y}{\sin x (\sec x - \sin x \tan x)}, \quad x \in \left(0, \frac{\pi}{2} \right) \text{ satisfying the condition } y \left(\frac{\pi}{4} \right) = 2.$$

Then, $y\left(\frac{\pi}{3}\right)$ is

[31-Jan-2024 Shift 1]

Options:

A. $\sqrt{3}(2 + \log_e \sqrt{3})$

B. $\frac{\sqrt{3}}{2}(2 + \log_e 3)$

C. $\sqrt{3}(1 + 2\log_e 3)$

D. $\sqrt{3}(2 + \log_e 3)$

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{\sin x + y \cos x}{\sin x \cdot \cos x \left(\frac{1}{\cos x} - \sin x \cdot \frac{\sin x}{\cos x} \right)}$$

$$= \frac{\sin x + y \cos x}{\sin x (1 - \sin^2 x)}$$

$$\frac{dy}{dx} = \sec^2 x + y \cdot 2(\operatorname{cosec} 2x)$$

$$\frac{dy}{dx} - 2 \operatorname{cosec}(2x) \cdot y = \sec^2 x$$

$$\frac{dy}{dx} + p \cdot y = Q$$

$$\text{I.F.} = e^{\int p dx} = e^{\int -2 \operatorname{cosec}(2x) dx}$$

$$\text{Let } 2x = t$$

$$2 \frac{dx}{dt} = 1$$

$$dx = \frac{dt}{2}$$

$$= e^{-\int \operatorname{cosec}(t) dt}$$

$$= e^{-\ln \left| \tan \frac{t}{2} \right|}$$

$$= e^{-\ln |\tan x|} = \frac{1}{|\tan x|}$$

$$y(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\Rightarrow y \frac{1}{|\tan x|} = \int \sec^2 x \cdot \frac{1}{|\tan x|} + c$$

$$y \cdot \frac{1}{|\tan x|} = \int \frac{dt}{|t|} + c \quad \text{for } \tan x = t$$

$$y \cdot \frac{1}{|\tan x|} = \ln |t| + c$$

$$y = |\tan x| (\ln |\tan x| + c)$$

$$\text{Put } x = \frac{\pi}{4}, y = 2$$

$$2 = \ln 1 + c \Rightarrow c = 2$$

$$y = |\tan x| (\ln |\tan x| + 2)$$

$$y\left(\frac{\pi}{3}\right) = \sqrt{3}(\ln \sqrt{3} + 2)$$

Question 75

The temperature $T(t)$ of a body at time $t = 0$ is 160°F and it decreases continuously as per the differential equation $\frac{dT}{dt} = -K(T - 80)$, where K is positive constant. If $T(15) = 120^\circ \text{F}$, then $T(45)$ is equal to

[31-Jan-2024 Shift 2]

Options:

A. 85°F

B. 95°F

C. 90°F

D. 80°F

Answer: C

Solution:

$$\frac{dT}{dt} = -k(T - 80)$$

$$\int_{160}^T \frac{dT}{(T - 80)} = \int_0^t -K dt$$

$$[\ln |T - 80|]_{160}^T = -kt$$

$$\ln |T - 80| - \ln 80 = -kt$$

$$\ln \left| \frac{T - 80}{80} \right| = -kt$$

$$T = 80 + 80e^{-kt}$$

$$120 = 80 + 80e^{-k \cdot 15}$$

$$\frac{40}{80} = e^{-k \cdot 15} = \frac{1}{2}$$

$$\therefore T(45) = 80 + 80e^{-k \cdot 45}$$

$$= 80 + 80(e^{-k \cdot 15})^3$$

$$= 80 + 80 \times \frac{1}{8}$$

$$= 90$$

Question 76

Let $y = y(x)$ be the solution of the differential equation

$$\sec^2 x \, dx + (e^{2y} \tan^2 x + \tan x) \, dy = 0$$

$$0 < x < \frac{\pi}{2}, \quad y\left(\frac{\pi}{4}\right) = 0. \quad \text{If } y\left(\frac{\pi}{6}\right) = \alpha$$

Then $e^{8\alpha}$ is equal to

[31-Jan-2024 Shift 2]

Answer: 9

Solution:

$$\sec^2 x \frac{dx}{dy} + e^{2y} \tan^2 x + \tan x = 0$$

$$\left(\text{Put } \tan x = t \Rightarrow \sec^2 x \frac{dx}{dy} = \frac{dt}{dy} \right)$$

$$\frac{dt}{dy} + e^{2y} \times t^2 + t = 0$$

$$\frac{dt}{dy} + t = -t^2 \cdot e^{2y}$$

$$\frac{1}{t^2} \frac{dt}{dy} + \frac{1}{t} = -e^{2y}$$

$$\left(\text{Put } \frac{1}{t} = u \Rightarrow \frac{-1}{t^2} \frac{dt}{dy} = \frac{du}{dy} \right)$$

$$\frac{-du}{dy} + u = -e^{2y}$$

$$\frac{du}{dy} - u = e^{2y}$$

$$\text{I.F.} = e^{-\int dy} = e^{-y}$$

$$ue^{-y} = \int e^{-y} \times e^{2y} dy$$

$$\frac{1}{\tan x} \times e^{-y} = e^y + c$$

$$x = \frac{\pi}{4}, y = 0, c = 0$$

$$x = \frac{\pi}{6}, y = \alpha$$

$$\sqrt{3}e^{-\alpha} = e^{\alpha} + 0$$

$$e^{2\alpha} = \sqrt{3}$$

$$e^{8\alpha} = 9$$

Question 77

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1, y(0) = 1.$$

Then, $\left(\frac{1}{\sqrt{2}} + y\left(\frac{1}{\sqrt{2}} \right) \right)^2$ equals :

[1-Feb-2024 Shift 1]

Options:

A. $\frac{4}{4+\sqrt{e}}$

B. $\frac{3}{3-\sqrt{e}}$

C. $\frac{2}{1+\sqrt{e}}$

D. $\frac{1}{2-\sqrt{e}}$

Answer: D

Solution:

$$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$$

$$x+y=t$$

$$\frac{dt}{dx} - 1 = 2xt^3 - xt - 1$$

$$\frac{dt}{2t^3 - t} = x dx$$

$$\frac{tdt}{2t^4 - t^2} = x dx$$

$$\text{Let } t^2 = z$$

$$\int \frac{dz}{2(2z^2 - z)} = \int x dx$$

$$\int \frac{dz}{4z\left(z - \frac{1}{2}\right)} = \int x dx$$

$$\ln \left| \frac{z - \frac{1}{2}}{z} \right| = x^2 + k$$

$$z = \frac{1}{2 - \sqrt{e}}$$

Question 78

If $x = x(t)$ is the solution of the differential equation $(t+1) dx = (2x + (t+1)^4) dt$, $x(0) = 2$, then, $x(1)$ equals _____

[1-Feb-2024 Shift 1]

Answer: 14

Solution:

$$(t+1) dx = (2x + (t+1)^4) dt$$

$$\frac{dx}{dt} = \frac{2x + (t+1)^4}{t+1}$$

$$\frac{dx}{dt} - \frac{2x}{t+1} = (t+1)^3$$

$$I \cdot F = e^{-\int \frac{2}{t+1} dt} = e^{-2 \ln(t+1)} = \frac{1}{(t+1)^2}$$

$$\frac{x}{(t+1)^2} = \int \frac{1}{(t+1)^2} (t+1)^3 dt + c$$

$$\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

$$x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$$

$$\text{put, } t = 1$$

$$x = 2^3 + 6 = 14$$

Question 79

Let α be a non-zero real number. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f(0) = 2$ and $\lim_{x \rightarrow -\infty} f(x) = 1$. If $f'(x) = \alpha f(x) + 3$, for all $x \in \mathbb{R}$, then $f(-\log_e 2)$ is equal to
[1-Feb-2024 Shift 2]

Options:

- A. 3
- B. 5
- C. 9

D. 7

Answer: A

Solution:

$$f(0) = 2, \lim_{x \rightarrow -\infty} f(x) = 1$$

$$f'(x) - \alpha \cdot f(x) = 3$$

$$I \cdot F = e^{-\alpha x}$$

$$y(e^{-\alpha x}) = \int 3 \cdot e^{-\alpha x} dx$$

$$f(x) \cdot (e^{-\alpha x}) = \frac{3e^{-\alpha x}}{-\alpha} + c$$

$$x = 0 \Rightarrow 2 = \frac{-3}{\alpha} + c \Rightarrow \frac{3}{\alpha} = c - 2$$

$$f(x) = \frac{-3}{\alpha} + c \cdot e^{\alpha x}$$

$$x \rightarrow -\infty \Rightarrow 1 = \frac{-3}{\alpha} + c(0)$$

$$\alpha = -3 \therefore c = 1$$

$$f(-\ln 2) = \frac{-3}{\alpha} + c \cdot e^{\alpha x}$$

$$= 1 + e^{3 \ln 2} = 9$$

(But α should be greater than 0 for finite value of c)

Question 80

If $\frac{dx}{dy} = \frac{1+x-y^2}{y}$, $x(1) = 1$, then $5x(2)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 5

Solution:

$$\frac{dx}{dy} - \frac{x}{y} = \frac{1-y^2}{y}$$

$$\text{Integrating factor} = e^{\int -\frac{1}{y} dy} = \frac{1}{y}$$

$$x \cdot \frac{1}{y} = \int \frac{1-y^2}{y^2} dy$$

$$\frac{x}{y} = \frac{-1}{y} - y + c$$

$$x = -1 - y^2 + cy$$

$$x(1) = 1$$

$$1 = -1 - 1 + c \Rightarrow c = 3$$

$$x = -1 - y^2 + 3y$$

$$5x(2) = 5(-1 - 4 + 6)$$

$$= 5$$

Question81

Let $y = y(x)$ be the solution of the differential equation

$$x^3 dy + (xy - 1) dx = 0, x > 0, y\left(\frac{1}{2}\right) = 3 - e. \text{ Then } y(1) \text{ is equal to}$$

[24-Jan-2023 Shift 1]

Options:

A. 1

B. e

C. $2 - e$

D. 3

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{1 - xy}{x^3} = \frac{1}{x^3} - \frac{y}{x^2}$$

$$\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$$

$$\text{If } \int \frac{1}{x^2} dx = e^{-\frac{1}{x}}$$

$$y \cdot e^{-\frac{1}{x}} = \int e^{-\frac{1}{x}} \cdot \frac{1}{x^3} dx \left(\text{put } -\frac{1}{x} = t \right)$$

$$y \cdot e^{-\frac{1}{x}} = -\int e^t \cdot t dt$$

$$y = \frac{1}{x} + 1 + Ce^{\frac{1}{x}}$$

Where C is constant

$$\text{Put } x = \frac{1}{2}$$

$$3 - e = 2 + 1 + Ce^2$$

$$C = -\frac{1}{e}$$

$$y(1) = 1$$

Question 82

Let $y = y(x)$ be the solution of the differential equation $(x^2 - 3y^2) dx + 3xy dy = 0$, $y(1) = 1$. Then $6y^2(e)$ is equal to [24-Jan-2023 Shift 2]

Options:

A. $3e^2$

B. e^2

C. $2e^2$

D. $\frac{3e^2}{2}$

Answer: C

Solution:

$$(x^2 - 3y^2) dx + 3xy dy = 0$$

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{3xy} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{3} \frac{x}{y}$$

$$\text{Put } y = vx$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$(1) \Rightarrow v + x \frac{dv}{dx} = v - \frac{1}{3} \frac{1}{v}$$

$$\Rightarrow v dv = \frac{-1}{3x}$$

Integrating both side

$$\begin{aligned}\frac{v^2}{2} &= \frac{-1}{3} \ln x + c \\ \Rightarrow \frac{y^2}{2x^2} &= \frac{-1}{3} \ln x + c \\ y(1) &= 1 \\ \Rightarrow \frac{1}{2} &= c \\ \Rightarrow \frac{y^2}{2x^2} &= \frac{-1}{3} \ln x + \frac{1}{2} \\ \Rightarrow y^2 &= -\frac{2}{3} x^2 \ln x + x^2 \\ y^2(e) &= -\frac{2}{3} e^2 + e^2 = \frac{e^2}{3} \\ \Rightarrow 6y^2(e) &= 2e^2\end{aligned}$$

Question 83

Let $y = y(x)$ be the solution curve of the differential equation

$\frac{dy}{dx} = \frac{y}{x}(1 + xy^2(1 + \log_e x))$, $x > 0$, $y(1) = 3$. Then $\frac{y^2(x)}{9}$ is equal to :

[25-Jan-2023 Shift 1]

Options:

A. $\frac{x^2}{5 - 2x^3(2 + \log_e x^3)}$

B. $\frac{x^2}{2x^3(2 + \log_e x^3) - 3}$

C. $\frac{x^2}{3x^3(1 + \log_e x^2) - 2}$

D. $\frac{x^2}{7 - 3x^3(2 + \log_e x^2)}$

Answer: A

Solution:

$$\frac{dy}{dx} - \frac{y}{x} = y^3(1 + \log_e x)$$

$$\frac{1}{y^3} \frac{dy}{dx} - \frac{1}{xy^2} = 1 + \log_e x$$

$$\text{Let } -\frac{1}{y^2} = t \Rightarrow \frac{2}{y^3} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dt}{dx} + \frac{2t}{x} = 2(1 + \log_e x)$$

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = x^2$$

$$\frac{-x^2}{y^2} = \frac{2}{3} \left((1 + \log_e x)x^3 - \frac{x^3}{3} \right) + C$$

$$y(1) = 3$$

$$\frac{y^2}{9} = \frac{x^2}{5 - 2x^3(2 + \log_e x^3)}$$

$$x dy = y dx + xy^3(1 + \log_e x) dx$$

$$\frac{x dy - y dx}{y^3} = x(1 + \log_e x) dx$$

$$- \frac{x}{y} d\left(\frac{x}{y}\right) = x^2(1 + \log_e x) dx$$

$$- \left(\frac{x}{y}\right)^2 = 2 \int x^2(1 + \log_e x) dx$$

Question 84

Let $y = y(t)$ be a solution of the differential equation

$$\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$$

Where, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Then $\lim_{t \rightarrow \infty} y(t)$

[25-Jan-2023 Shift 2]

Options:

A. is 0

B. does not exist

C. is 1

D. is -1

Answer: A

Solution:

$$\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$$

$$\text{I.F.} = e^{\int \alpha dt} = e^{\alpha t}$$

$$\text{Solution} \Rightarrow y \cdot e^{\alpha t} = \int \gamma e^{-\beta t} \cdot e^{\alpha t} dt$$

$$\Rightarrow y e^{\alpha t} = \gamma \frac{e^{(\alpha - \beta)t}}{(\alpha - \beta)} + c$$

$$\Rightarrow y = \frac{\gamma}{e^{\beta t}(\alpha - \beta)} + \frac{c}{e^{\alpha t}}$$

$$\text{So, } \lim_{t \rightarrow \infty} y(t) = \frac{\gamma}{\infty} + \frac{c}{\infty} = 0$$

Question85

Let $y = f(x)$ be the solution of the differential equation $y(x+1)dx - x^2 dy = 0$, $y(1) = e$. Then $\lim_{x \rightarrow 0^+} f(x)$ is equal to

[29-Jan-2023 Shift 1]

Options:

A. 0

B. $\frac{1}{e}$

C. e^2

D. $\frac{1}{e^2}$

Answer: A

Solution:

$$\frac{x+1}{x^2} dx = \frac{dy}{y}$$

$$\ln x - \frac{1}{x} = \ln y + c$$

$$(1, e) \\ c = -2$$

$$\ln x - \frac{1}{x} = \ln y - 2$$

$$y = e^{\ln x - \frac{1}{x} + 2}$$

$$\lim_{x \rightarrow 0^+} e^{\ln x - 1 - \frac{1}{x} + 2} \\ = e^{-\infty} \\ = 0$$

Question86

Let $y = y(x)$ be the solution of the differential equation $x \log_e x \frac{dy}{dx} + y = x^2 \log_e x$, ($x > 1$). If $y(2) = 2$, then $y(e)$ is equal to

[29-Jan-2023 Shift 2]

Options:

A. $\frac{4+e^2}{4}$

B. $\frac{1+e^2}{4}$

C. $\frac{2+e^2}{2}$

D. $\frac{1+e^2}{2}$

Answer: A

Solution:

$$x \log_e x \frac{dy}{dx} + y = x^2 \log_e x, (x > 1)$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \ln x} = x$$

Linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{x \ln x} dx} = |\ln x|$$

$\therefore$ Solution of differential equation

$$y |\ln x| = \int x |\ln x| dx$$

$$= |\ln x| \left(\frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right)$$

$$\Rightarrow y |\ln x| = |\ln x| \left(\frac{x^2}{2} \right) - \frac{x^2}{4} + c$$

For constant

$$y(2) = 2 \Rightarrow c = 1$$

$$\text{So, } y(x) = \frac{x^2}{2} - \frac{x^2}{4 |\ln x|} + \frac{1}{|\ln x|}$$

$$\text{Hence, } y(e) = \frac{e^2}{2} - \frac{e^2}{4} + 1 = 1 + \frac{e^2}{4}$$

Question 87

Let the solution curve $y = y(x)$ of the differential equation

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1+x^6)^{\frac{3}{2}}} y = 2x$$

$\exp \frac{x^3 - \tan^{-1} x^3}{\sqrt{(1+x)^6}}$ pass through the origin.

Then $y(1)$ is equal to :

[30-Jan-2023 Shift 1]

Options:

A. $\exp \left(\frac{4-\pi}{4\sqrt{2}} \right)$

B. $\exp\left(\frac{\pi-4}{4\sqrt{2}}\right)$

C. $\exp\left(\frac{1-\pi}{4\sqrt{2}}\right)$

D. $\exp\left(\frac{4+\pi}{4\sqrt{2}}\right)$

Answer: A

Solution:

$$\frac{dy}{dx} + \left(\frac{-3x^5 \tan^{-1} x^3}{(1+x^6)^{3/2}} \right) y = 2e^{\left\{ \frac{x - \tan x}{\sqrt{1+x^6}} \right\}}$$

$$\int \frac{-3x^5 \tan^{-1} x^3}{(1+x^6)^{3/2}} dx$$

$$\text{I.F.} = e^{\frac{\tan^{-1} x^3 - x^3}{\sqrt{1+x^6}}}$$

Solution of differential equation

$$y \cdot e^{\frac{\tan^{-1} x^3 - x^3}{\sqrt{1+x^6}}} = \int 2x e^{\left(\frac{x^3 - \tan^{-1} x^3}{\sqrt{1+x^6}} \right)} \cdot e^{\left(\frac{\tan^{-1}(x^3) - x^3}{\sqrt{1+x^6}} \right)} dx$$

$$= \int 2x dx + c$$

$$y \cdot e^{\frac{\tan^{-1} x^3 - x^3}{\sqrt{1+x^6}}} = x^2 + c$$

Also it passes through origin

$$c = 0$$

$$y(1) \cdot e^{\frac{\tan^{-1}(1) - 1}{\sqrt{2}}} = 1$$

$$y(1) \cdot e^{\frac{\frac{\pi}{4} - 1}{\sqrt{2}}} = 1$$

$$y(1) \cdot e^{\frac{\pi-4}{4\sqrt{2}}} = 1$$

$$y(1) = \frac{1}{e^{\frac{\pi-4}{4\sqrt{2}}}} = e^{\frac{4-\pi}{4\sqrt{2}}}$$

Question 88

The solution of the differential equation

$$\frac{dy}{dx} = - \left(\frac{x^2 + 3y^2}{3x^2 + y^2} \right), y(1) = 0 \text{ is}$$

[30-Jan-2023 Shift 2]

Options:

A. $\log_e \left| x + y \right| - \frac{xy}{(x+y)^2} = 0$

B. $\log_e \left| x + y \right| + \frac{xy}{(x+y)^2} = 0$

C. $\log_e \left| x + y \right| + \frac{2xy}{(x+y)^2} = 0$

D. $\log_e \left| x + y \right| - \frac{2xy}{(x+y)^2} = 0$

Answer: C

Solution:

Put $y = vx$

$$v + x \frac{dv}{dx} = - \left(\frac{1+3v^2}{3+v^2} \right)$$

$$x \frac{dv}{dx} = - \frac{(v+1)^3}{3+v^2}$$

$$\frac{(3+v^2)dv}{(v+1)^3} + \frac{dx}{x} = 0$$

$$\int \frac{4dv}{(v+1)^3} + \int \frac{dv}{v+1} - \int \frac{2dv}{(v+1)^2} + \int \frac{dx}{x} = 0$$

$$\frac{-2}{(v+1)^2} + \ln(v+1) + \frac{2}{v+1} + \ln x = c$$

$$\frac{-2x^2}{(x+y)^2} + \ln \left(\frac{x+y}{x} \right) + \frac{2x}{x+y} + \ln x = c$$

$$\frac{2xy}{(x+y)^2} + \ln(x+y) = c$$

$$\therefore c = 0, \text{ as } x = 1, y = 0$$

$$\therefore \frac{2xy}{(x+y)^2} + \ln(x+y) = 0$$

Question 89

Let $y = y(x)$ be the solution of the differential equation

$(3y^2 - 5x^2)y \, dx + 2x(x^2 - y^2) \, dy = 0$ such that $y(1) = 1$. then $|(y(2))^3 - 12y(2)|$ is equal to:

[31-Jan-2023 Shift 2]

Options:

A. $32\sqrt{2}$

B. 64

C. $16\sqrt{2}$

D. 32

Answer: A

Solution:

$$(3y^2 - 5x^2)y \cdot dx + 2x(x^2 - y^2)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(5x^2 - 3y^2)}{2x(x^2 - y^2)}$$

Put $y = mx$

$$\Rightarrow m + x \cdot \frac{dm}{dx} = \frac{m(5 - 3m^2)}{2(1 - m^2)}$$

$$x \cdot \frac{dm}{dx} = \frac{(5 - 3m^2)m - 2m(1 - m^2)}{2(1 - m^2)}$$

$$\Rightarrow \frac{dx}{x} = \frac{2(m^2 - 1)}{m(m^2 - 3)} dm$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{2}{m} - \frac{3}{m} + \frac{4m}{m^2 - 3} \right) dm$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{2}{m} + \int \frac{2}{3} \left(\frac{2m}{m^2 - 3} \right) dm$$

$$\Rightarrow \ln |x| = \frac{2}{3} \ln |m| + \frac{2}{3} \ln |m^2 - 3| + C$$

$$\text{Or, } \ln |x| = \frac{2}{3} \ln \left| \frac{y}{x} \right| + \frac{2}{3} \ln \left| \left(\frac{y}{x} \right)^2 - 3 \right| + C$$

Put $(x=1, y=1)$: we get $C = -\frac{2}{3} \ln(2)$

$$\Rightarrow \ln |x| = \frac{2}{3} \ln \left| \frac{y}{x} \right| + \frac{2}{3} \ln \left| \left(\frac{y}{x} \right)^2 - 3 \right| - \frac{2}{3} \ln(2)$$

$$\Rightarrow \left(\frac{y}{x} \right) \left[\left(\frac{y}{x} \right)^2 - 3 \right] = 2 \cdot (x^{3/2})$$

Put $x=2$ to get $y(2)$

$$\Rightarrow y(y^2 - 12) = 4 \times 2 \times 2 \times 2\sqrt{2}$$

$$\Rightarrow y^3 - 12y = 32\sqrt{2}$$

$$\Rightarrow |y^3(2) - 12y(2)| = 32\sqrt{2}$$

Question 90

If $y = y(x)$ is the solution curve of the differential equation

$$\frac{dy}{dx} + y \tan x = x \sec x, \quad 0 \leq x \leq \frac{\pi}{3}, \quad y(0) = 1, \text{ then } y\left(\frac{\pi}{6}\right) \text{ is equal to}$$

[1-Feb-2023 Shift 1]

Options:

A. $\frac{\pi}{12} - \frac{\sqrt{3}}{2} \log_e \left(\frac{2}{e\sqrt{3}} \right)$

B. $\frac{\pi}{12} + \frac{\sqrt{3}}{2} \log_e \left(\frac{2\sqrt{3}}{e} \right)$

C. $\frac{\pi}{12} - \frac{\sqrt{3}}{2} \log_e \left(\frac{2\sqrt{3}}{e} \right)$

D. $\frac{\pi}{12} + \frac{\sqrt{3}}{2} \log_e \left(\frac{2}{e\sqrt{3}} \right)$

Answer: A

Solution:

Here I.F. = $\sec x$

Then solution of D.E :

$$y(\sec x) = x \tan x - \ln(\sec x) + c$$

$$\text{Given } y(0) = 1 \Rightarrow c = 1$$

$$\therefore y(\sec x) = x \tan x - \ln(\sec x) + 1$$

$$\text{At } x = \frac{\pi}{6}, y = \frac{\pi}{12} + \frac{\sqrt{3}}{2} \ln \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

Question 91

Let $\alpha x = \exp(x^\beta y^\gamma)$ be the solution of the differential equation

$2x^2 y \, dy - (1 - xy^2) \, dx = 0$, $x > 0$, $y(2) = \sqrt{\log_e 2}$. Then $\alpha + \beta - \gamma$ equals :

[1-Feb-2023 Shift 2]

Options:

A. 1

B. -1

C. 0

D. 3

Answer: A

Solution:

$$\begin{aligned} \alpha x &= e^{x^8 \cdot y^2} \\ 2x^2 y \frac{dy}{dx} &= 1 - x \cdot y^2 \quad y^2 = t \\ x^2 \frac{dt}{dx} &= 1 - xt \\ \frac{dt}{dx} + \frac{t}{x} &= \frac{1}{x^2} \quad \text{I.F.} = e^{\int \frac{1}{x} dx} = x \\ t(x) &= \int \frac{1}{x^2} \cdot x dx \\ y^2 \cdot x &= \ln x + C \\ \therefore 2 \cdot \ln 2 &= \ln 2 + C \\ \therefore C &= \ln 2 \\ \text{Hence, } xy^2 &= \ln 2x \\ \therefore 2x &= e^{x \cdot y^2} \\ \text{Hence } \alpha &= 2, \beta = 1, \gamma = 2 \end{aligned}$$

Question 92

Let $y = y(x)$ be a solution of the differential

$$(x \cos x) dy + (x y \sin x + y \cos x - 1) dx = 0, \quad 0 < x < \frac{\pi}{2}. \quad \text{If } \frac{\pi}{3} y \left(\frac{\pi}{3} \right) = \sqrt{3}, \text{ then}$$

$$\left| \frac{\pi}{6} y'' \left(\frac{\pi}{6} \right) + 2y' \left(\frac{\pi}{6} \right) \right| \text{ is equal to } \underline{\hspace{2cm}}.$$

[6-Apr-2023 shift 1]

Answer: 2

Solution:

$$\begin{aligned} (x \cos x) dy + (x y \sin x + y \cos x - 1) dx &= 0, \quad 0 < x < \frac{\pi}{2} \\ \frac{dy}{dx} + \left(\frac{x \sin x + \cos x}{x \cos x} \right) y &= \frac{1}{x \cos x} \\ \text{I.F.} &= x \sec x \\ y \cdot x \sec x &= \int \frac{x \sec x}{x \cos x} dx = \tan x + c \\ \text{Since } y \left(\frac{\pi}{3} \right) &= \frac{3\sqrt{3}}{\pi} \quad \text{Hence } c = \sqrt{3} \\ \text{Hence } \left| \frac{\pi}{6} y'' \left(\frac{\pi}{6} \right) + y' \left(\frac{\pi}{6} \right) \right| &= |-2| = 2 \end{aligned}$$

Question93

If the solution curve $f(x, y) = 0$ of the differential equation

$(1 + \log_e x) \frac{dx}{dy} - x \log_e x = e^y$, $x > 0$, passes through the points $(1, 0)$ and $(\alpha, 2)$,

then α^α is equal to :

[6-Apr-2023 shift 2]

Options:

A. $e^{\sqrt{2}c^2}$

B. e^{c^2}

C. $e^{2e^{\sqrt{2}}}$

D. e^{2c^2}

Answer: D

Solution:

$$(1 + \ell \ln x) \frac{dx}{dy} - x \ell \ln x = e^y$$

$$\text{Let } x \ell \ln x = t$$

$$(1 + \ln x) \frac{dx}{dy} = \frac{dt}{dy}$$

$$\frac{dt}{dy} - t = e^y \quad P = -1, Q = e^y$$

$$I \cdot F = e^{\int -dy} = e^{-y}$$

Solution -

$$(t)(e^{-y}) = \int (e^{-y})(e^y) dy$$

$$t(e^{-y}) = y + c$$

$$(x \ell \ln x) e^{-y} = y + c \quad \text{pass } (1, 0) \Rightarrow c = 0$$

$$\text{pass } (\alpha, 2)$$

$$\alpha^\alpha = e^{2c^2}$$

Ans. Option 4

Question94

Let the solution curve $x = x(y)$, $0 < y < \frac{\pi}{2}$, of the differential equation

$(\log_e(\cos y))^2 \cos y dx - (1 + 3x \log_e(\cos y)) \sin y dy = 0$ satisfy

$x\left(\frac{\pi}{3}\right) = \frac{1}{2 \log_e 2}$. If $x\left(\frac{\pi}{6}\right) = \frac{1}{\log_e m - \log_e n}$, where m and n are co-prime, then mn is

equal to
[8-Apr-2023 shift 2]

Answer: 12

Solution:

$$\cos y \ln^2 \cos y \, dx = (1 + 3x \ln \cos y) \sin y \, dy$$

$$\frac{dx}{dy} = \tan y \left(\frac{3x}{\ln \cos y} + \frac{1}{\ln^2 \cos y} \right)$$

$$\frac{dx}{dy} - \left(\frac{3 \tan y}{\ln \cos y} \right) x = \frac{\tan y}{\ln^2 \cos y}$$

$$- \frac{3}{\ln \cos y \cos y} dy$$

$$\ln \cos y = t$$

$$\frac{1}{\cos y} - \sin y \, dy = dt$$

$$\text{If } = e^{\frac{3}{t} dt} = e^{3 \ln t} = t^3 = \ln^3 \cos y$$

$$\text{solution is } x \cdot \ln^3 \cos y = \left\{ \frac{\sin y}{\cos y} \cdot \ln \cos y \, dy + C \right.$$

$$x \ln^3 \cos y = \frac{-\ln^2 \cos y}{2} + C$$

$$x \left(\frac{\pi}{3} \right) = \frac{1}{2 \ln 2} \text{ so } \frac{1}{2 \ln^2} \times \ln^3 \left(\frac{1}{2} \right) = - \frac{\ln^3 \left(\frac{1}{2} \right)}{2} + C$$

$$C = 0 \quad x \ln^3 \frac{\sqrt{3}}{2} = - \frac{1}{2} \ln^2 \frac{\sqrt{3}}{2} + 0$$

$$y = \frac{\pi}{6}$$

$$x = - \frac{1}{2 \ln \left(\frac{\sqrt{3}}{2} \right)}$$

$$x = \frac{1}{\ln \frac{4}{3}} = \frac{1}{\ln 4 - \ln 3}$$

$$mn = 12$$

Question 95

Let f be a differentiable function such that $x^2 f(x) - x = 4 \int_0^x t f(t) \, dt$, $f(1) = \frac{2}{3}$.

Then $18f(3)$ is equal to :
[10-Apr-2023 shift 1]

Options:

A. 180

B. 150

C. 210

D. 160

Answer: D

Solution:

$$x^2 f(x) - x = 4 \int_0^x t f(t) dt$$

Differentiate w.r.t. x

$$x^2 f'(x) + 2xf(x) - 1 = 4xf(x)$$

Let $y = f(x)$

$$\Rightarrow x^2 \frac{dy}{dx} - 2xy - 1 = 0$$

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{1}{x^2}$$

$$\text{I.F.} = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$$

Its solution is

$$\frac{y}{x^2} = \int \frac{1}{x^4} dx + C$$

$$\frac{y}{x^2} = -\frac{1}{3x^3} + C$$

$$\because f(1) = \frac{2}{3} \Rightarrow y(1) = \frac{2}{3}$$

$$\Rightarrow \frac{2}{3} = -\frac{1}{3} + C$$

$$\Rightarrow C = 1$$

$$\therefore y = -\frac{1}{3x} + x^2$$

$$f(x) = -\frac{1}{3x} + x^2$$

$$f(3) = -\frac{1}{9} + 9 = \frac{80}{9} \Rightarrow 18f(3) = 160$$

Question 96

The slope of tangent at any point (x, y) on a curve $y = y(x)$ is $\frac{x^2 + y^2}{2xy}$, $x > 0$. If

$y(2) = 0$, then a value of $y(8)$ is:

[10-Apr-2023 shift 1]

Options:

A. $4\sqrt{3}$

B. $-4\sqrt{2}$

C. $-2\sqrt{3}$

D. $2\sqrt{3}$

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

$$y = vx$$

$$y(2) = 0$$

$$y(8) = ?$$

$$\frac{dv}{dx} = v + x \frac{dv}{dx}$$

$$v + \frac{xdv}{dx} = \frac{x^2 + v^2x^2}{2vx^2}$$

$$x \cdot \frac{dv}{dx} = \left(\frac{v^2 + 1}{2v} - v \right)$$

$$\frac{2vdv}{(1 - v^2)} = \frac{dx}{x}$$

$$-\ln(1 - v^2) = \ln x + C$$

$$\ln x + \ln(1 - v^2) = C$$

$$\ln \left[x \left(1 - \frac{y^2}{x^2} \right) \right] = C$$

$$\ln \left[\left(\frac{x^2 - y^2}{x} \right) \right] = C$$

$$x^2 - y^2 = cx$$

$$y(2) = 0 \text{ at } x = 2, y = 0$$

$$4 = 2C \Rightarrow C = 2$$

$$x^2 - y^2 = 2x$$

$$\text{Hence, at } x = 8$$

$$64 - y^2 = 16$$

$$y = \sqrt{48} = 4\sqrt{3}$$

$$y(8) = 4\sqrt{3}$$

Option (1)

Question97

Let the tangent at any point P on a curve passing through the points (1, 1)

and $\left(\frac{1}{10}, 100 \right)$, intersect positive x axis and y-axis at the points A and B

respectively. If $PA : PB = 1 : k$ and $y = y(x)$ is the solution of the differential

equation $e^{\frac{dy}{dx}} = kx + \frac{k}{2}$, $y(0) = k$, then $4y$ then $4y(1) - 5 \log e^3$ is equal to

[10-Apr-2023 shift 2]

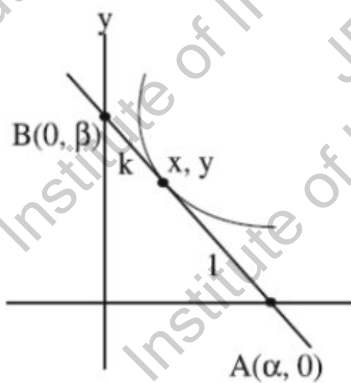
Answer: 5

Solution:

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y = 0$$

$$X = \frac{-y dx}{dy} + x$$



$$\frac{k\alpha + 0}{k+1} = x, \alpha = \frac{k+1}{k}x$$

$$\frac{k+1}{k}x = -y \frac{dx}{dy} + x$$

$$x + \frac{x}{k} = -y \frac{dx}{dy} + x$$

$$x \frac{dy}{dx} + ky = 0$$

$$\frac{dy}{dx} + \frac{k}{x}y = 0$$

$$y \cdot x^k = C$$

$$C = 1$$

$$100 \cdot \left(\frac{1}{10}\right)^k = 1$$

$$K = 2$$

$$\frac{dy}{dx} = \ln(2x+1)$$

$$y = \frac{(2x+1)}{2}(\ln(2x+1) - 1) + c$$

$$2 = \frac{1}{2}(0-1) + C$$

$$C = 2 + \frac{1}{2} = \frac{5}{2}$$

$$y(1) = \frac{3}{2}(\ell \ln 3 - 1) + \frac{5}{2}$$

$$= \frac{3}{2} \ln 3 + 1$$

$$4y(1) = 6 \ln 3 + 4$$

$$4y(1) - 5 \ln 3 = 4 + \ln 3$$

Question 98

Let $y = y(x)$ be a solution curve of the differential equation.

$(1 - x^2 y^2) dx = y dx + x dy$. If the line $x = 1$ intersects the curve $y = y(x)$ at $y = 2$ and the line $x = 2$ intersects the curve $y = y(x)$ at $y = \alpha$, then a value of α is :

[11-Apr-2023 shift 1]

Options:

A. $\frac{1 + 3e^2}{2(3e^2 - 1)}$

B. $\frac{1 - 3e^2}{2(3e^2 + 1)}$

C. $\frac{3e^2}{2(3e^2 - 1)}$

D. $\frac{3e^2}{2(3e^2 + 1)}$

Answer: A

Solution:

$$(1 - x^2 y^2) dx = y dx + x dy, y(1) = 2$$

$$y(2) = \alpha$$

$$dx = \frac{d(xy)}{1 - (xy)^2}$$

$$\int dx = \int \frac{d(xy)}{1 - (xy)^2}$$

$$x = \frac{1}{2} \ln \left| \frac{1 + xy}{1 - xy} \right| + C$$

$$\text{Put } x = 1 \text{ and } y = 2:$$

$$1 = \frac{1}{2} \ln \left| \frac{1 + 2}{1 - 2} \right| + C$$

$$C = 1 - \frac{1}{2} \ln 3$$

Now put $x = 2$:

$$2 = \frac{1}{2} \ln \left| \frac{1 + 2\alpha}{1 - 2\alpha} \right| + 1 - \frac{1}{2} \ln 3$$

$$1 + \frac{1}{2} \ln 3 = \frac{1}{2} \ln \left| \frac{1 + 2\alpha}{1 - 2\alpha} \right|$$

$$2 + \ln 3 = \ln \left| \frac{1 + 2\alpha}{1 - 2\alpha} \right|$$

$$\left| \frac{1+2\alpha}{1-2\alpha} \right| = 3e^2$$

$$\frac{1+2\alpha}{1-2\alpha} = 3e^2, -3e^2$$

$$\frac{1+2\alpha}{1-2\alpha} = 3e^2 \Rightarrow \alpha = \frac{3e^2 - 1}{2(3e^2 + 1)}$$

$$\text{And } \frac{1+2\alpha}{1-2\alpha} = -3e^2 \Rightarrow \alpha = \frac{3e^2 + 1}{2(3e^2 - 1)}$$

Question99

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{5}{x(x^5+1)}y = \frac{(x^5+1)^2}{x^7}, x > 0. \text{ If } y(1) = 2, \text{ then } y(2) \text{ is equal to}$$

[11-Apr-2023 shift 2]

Options:

A. $\frac{693}{128}$

B. $\frac{637}{128}$

C. $\frac{697}{128}$

D. $\frac{679}{128}$

Answer: A

Solution:

$$\text{I.F} = e^{\int \frac{5 dx}{x(x^5+1)}} = e^{\int \frac{5x^{-6} - 6x}{x^5+1} dx}$$

$$\text{Put, } 1+x^{-5} = t \Rightarrow -5x^{-6} dx = dt$$

$$\Rightarrow e^{\int \frac{-dt}{1-t}} = \frac{1}{t} = \frac{x^5}{1+x^5}$$

$$y \cdot \frac{x^5}{1+x^5} = \int \frac{x^5}{(1+x^5)} \times \frac{(1+x^5)^2}{x^7} dx$$

$$= \int x^3 dx + \int x^{-2} dx$$

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + c$$

$$\text{Given than: } x=1 \Rightarrow y=2$$

$$2 \cdot \frac{1}{2} = \frac{1}{4} - 1 + c$$

$$c = \frac{7}{4}$$

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + \frac{7}{4}$$

Now put, $x = 2$

$$y \cdot \left(\frac{32}{33} \right) = \frac{21}{4}$$

$$y = \frac{693}{128}$$

Question 100

Let $y = y(x)$, $y > 0$, be a solution curve of the differential equation $(1+x^2)dy = y(x-y)dx$. If $y(0) = 1$ and $y(2\sqrt{2}) = \beta$, then [12-Apr-2023 shift 1]

Options:

A. $e^{3\beta^{-1}} = e(5 + \sqrt{2})$

B. $e^{3\beta^{-1}} = e(3 + 2\sqrt{2})$

C. $e^{\beta^{-1}} = e^{-2}(3 + 2\sqrt{2})$

D. $e^{\beta^{-1}} = e^{-2}(5 + \sqrt{2})$

Answer: B

Solution:

$$(1+x^2)dy = y(x-y)dx$$

$$Y(0) = 1 \cdot y(2\sqrt{2}) = \beta$$

$$\frac{dy}{dx} = \frac{yx-y^2}{1+x^2}$$

$$\frac{dy}{dx} + y \left(\frac{-x}{1+x^2} \right) = \left(\frac{-1}{1+x^2} \right) y^2$$

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{y} \left(\frac{-x}{1+x^2} \right) = \frac{-1}{1+x^2}$$

$$\text{Put } \frac{1}{y} = t \text{ then } \frac{-1}{y^2} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + t \frac{x}{1+x^2} = \frac{1}{1+x^2}$$

$$\text{If } = e^{\int \frac{x}{1+x^2} dx} = e^{\frac{1}{2} \ln(1+x^2)} \sqrt{1+x^2}$$

$$t \sqrt{1+x^2} = \int \frac{1}{\sqrt{1+x^2}} dx$$

$$\frac{\sqrt{1+x^2}}{y} = \ln(x + \sqrt{x^2+1}) + c$$

$$y(0) = 1 \Rightarrow c = 1$$

$$\Rightarrow \sqrt{1+x^2} = y \ln(e(x + \sqrt{x^2+1}))$$

$$\beta = \frac{3}{\ln(e(3+2\sqrt{2}))} \Rightarrow \frac{3}{\beta} = \ln(e(3+2\sqrt{2}))$$

$$e^{\frac{3}{\beta}} = e(3+2\sqrt{2})$$

Question101

Let $y = y_1(x)$ and $y = y_2(x)$ be the solution curves of the differential equation $\frac{dy}{dx} = y + 7$ with initial conditions $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then the curves $y = y_1(x)$ and $y = y_2(x)$ intersect at [13-Apr-2023 shift 1]

Options:

- A. no point
- B. infinite number of points
- C. one point
- D. two points

Answer: A

Solution:

$$\frac{dy}{dx} = y + 7 \Rightarrow \frac{dy}{dx} - y = 7$$

$$\text{I.F.} = e^{-x}$$

$$ye^{-x} = \int 7e^{-x} dx$$

$$\Rightarrow ye^{-x} = -7e^{-x} + c$$

$$\Rightarrow y = -7 + ce^x$$

$$-7 + 7e^x = -7 + 8e^x \Rightarrow e^x = 0$$

No solution

Question102

If $y = y(x)$ is the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x}{(x^2-1)}y = \frac{x+2}{(x^2-1)^{\frac{5}{2}}}, x > 1 \text{ such that } y(2) = \frac{2}{9}\log_e(2 + \sqrt{3}) \text{ and}$$

$y(\sqrt{2}) = \alpha \log_e(\sqrt{\alpha} + \beta) + \beta - \sqrt{\gamma}$, $\alpha, \beta, \gamma, \in \mathbb{N}$, then $\alpha\beta\gamma$ is equal to _____ [13-Apr-2023 shift 2]

Answer: 6

Solution:

given differential equation $\frac{dy}{dx} + \frac{4x}{(x^2-1)}y = \frac{x+2}{(x^2-1)^{5/2}}$ is linear D.E.

$$\text{I.F.} = \int \frac{4x}{x^{2-1}} dx = e^{\int \frac{4x}{x^2-1} dx} = e^{\ln(x^2-1)^2} = (x^2-1)^2$$

$$y(x^2-1)^2 = \int \frac{x+2}{(x^2-1)^{5/2}} (x^2-1)^2 dx$$

$$= \int \frac{x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx$$

$$= \sqrt{x^2-1} + 2 \ln[x + \sqrt{x^2-1}] + C$$

$$\text{put } y(2) = \frac{2}{9} \ln(2 + \sqrt{3})$$

$$\frac{2}{9} \ln(2 + \sqrt{3})(9) = \sqrt{3} + 2 \ln[2 + \sqrt{3}] + C$$

$$= C = -\sqrt{3}$$

$$\text{put } x = \sqrt{2}$$

$$y = 1 + 2 \ln[\sqrt{2} + 1] - \sqrt{3}$$

$$\alpha = 2, \beta = 1, \gamma = 3$$

$$\alpha\beta\gamma = 2(1)(3) = 6$$

Question 103

Let $x = x(y)$ be the solution of the differential equation

$$2(y+2)\log_e(y+2) dx + (x+4-2\log_e(y+2)) dy = 0, y > -1 \text{ with } x(e^4-2) = 1.$$

Then $x(e^9-2)$ is equal to
[15-Apr-2023 shift 1]

Options:

A. $\frac{4}{9}$

B. $\frac{32}{9}$

C. $\frac{10}{3}$

D. 3

Answer: B

Solution:

$$2(y+2)\ln(y+2)dx + (x+4-2\ln(y+2))\frac{dy}{dx} = 0$$
$$2\ln(y+2) + (x+4-2\ln(y+2))\frac{1}{y+2} \cdot \frac{dy}{dx} = 0$$

$$\text{let, } \ln(y+2) = t$$

$$\frac{1}{y+2} \cdot \frac{dy}{dx} = \frac{dt}{dx}$$

$$2t + (x+4-2t) \cdot \frac{dt}{dx} = 0$$

$$(x+4-2t) \cdot \frac{dt}{dx} = -2t$$

$$\frac{dx}{dt} = \frac{2t-4-x}{2t}$$

$$\frac{dx}{dt} + \frac{x}{2t} = \frac{2t-4}{2t}$$

$$x \cdot t^{1/2} = \int \frac{2t-4}{2t} \cdot t^{1/2} \cdot dt$$

$$x \cdot t^{1/2} = \int \left(t^{1/2} - \frac{2}{t^{1/2}} \right) \cdot dt$$

$$= \frac{3}{2} t^{3/2} - 2 \cdot \frac{t^{1/2}}{1/2} + C$$

$$x \cdot t^{1/2} = \frac{2t^{3/2}}{3} - 4t^{1/2} + C$$

$$x = \frac{2}{3} \cdot t - 4 + C \cdot t^{-1/2}$$

$$x = \frac{2}{3} \ln(y+2) - 4 + C \cdot (\ln(y+2))^{-1/2}$$

$$\text{Put } y = e^4 - 2, x = 1$$

$$1 = \frac{2}{3} \times 4 - 4 + C \times \frac{1}{2}$$

$$\frac{C}{2} = 5 - \frac{8}{3} = \frac{7}{3}$$

$$\Rightarrow C = \frac{14}{3}$$

$$x = \frac{2}{3} \times 9 - 4 + \frac{14}{3} \times \frac{1}{3}$$

$$= 2 + \frac{14}{9}$$

$$= 3$$

Question 104

The slope of normal at any point (x, y) , $x > 0$, $y > 0$ on the curve $y = y(x)$ is given by $\frac{x^2}{xy - x^2y^2 - 1}$. If the curve passes through the point $(1, 1)$, then e. $y(e)$ is equal to

[24-Jun-2022-Shift-2]

Options:

A. $\frac{1 - \tan(1)}{1 + \tan(1)}$

B. $\tan(1)$

C. 1

D. $\frac{1 + \tan(1)}{1 - \tan(1)}$

Answer: D

Solution:

$$\therefore \frac{dx}{dy} = \frac{x^2}{xy - x^2y^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{x^2y^2 - xy + 1}{x^2}$$

$$\text{Let } xy = v \Rightarrow y + x \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} - y = \frac{(v^2 - v + 1)y}{v}$$

$$\therefore \frac{dv}{dx} = \frac{v^2 + 1}{x}$$

$$\therefore y(1) = 1 \Rightarrow \tan^{-1}(xy) = \ln x + \tan^{-1}(1)$$

Put $x = e$ and $y = y(e)$ we get

$$\tan^{-1}(e \cdot y(e)) = 1 + \tan^{-1}1$$

$$\tan^{-1}(e \cdot y(e)) - \tan^{-1}1 = 1$$

$$\therefore e(y(e)) = \frac{1 + \tan(1)}{1 - \tan(1)}$$

Question 105

Let $y = y(x)$ be the solution of the differential equation

$(x + 1)y' - y = e^{3x}(x + 1)^2$, with $y(0) = \frac{1}{3}$. Then, the point $x = -\frac{4}{3}$ for the curve

$y = y(x)$ is :

[25-Jun-2022-Shift-1]

Options:

A. not a critical point

- B. a point of local minima
- C. a point of local maxima
- D. a point of inflection

Answer: B

Solution:

$$(x+1) \frac{dy}{dx} - y = e^{3x}(x+1)^2$$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

$$\text{Iff } e^{-\int \frac{1}{x+1} dx} = e^{-\log(x+1)} = \frac{1}{x+1}$$

$$\therefore y \left(\frac{1}{x+1} \right) = \int \frac{e^{3x}(x+1)}{x+1} dx$$

$$\frac{y}{x+1} = \int e^{3x} dx$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$$

$$\therefore y(0) = \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{3} + c$$

$$\therefore c = 0$$

$$\text{So } y = \frac{e^{3x}}{3}(x+1)$$

$$y' = e^{3x}(x+1) + \frac{e^{3x}}{3} = e^{3x} \left(x + \frac{4}{3} \right)$$

$$y'' = 3e^{3x} \left(x + \frac{4}{3} \right) + e^{3x} = e^{3x}(3x+5)$$

$$y' = 0 \text{ at } x = -\frac{4}{3} \text{ \& } y'' = e^{-4}(1) > 0 \text{ at } x = -\frac{4}{3}$$

$$\Rightarrow x = -\frac{4}{3} \text{ is point of local minima}$$

Question 106

If the solution curve $y = y(x)$ of the differential equation

$y^2 dx + (x^2 - xy + y^2) dy = 0$, which passes through the point $(1, 1)$ and

intersects the line $y = \sqrt{3}x$ at the point $(\alpha, \sqrt{3}\alpha)$, then value of $\log_e(\sqrt{3}\alpha)$ is

equal to :

[25-Jun-2022-Shift-1]

Options:

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{12}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2 - y^2}$$

Put $y = vx$ we get

$$v + x \frac{dv}{dx} = \frac{v^2}{v - 1 - v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - v^2 + v + v^3}{v - 1 - v^2}$$

$$\Rightarrow \int \frac{v - 1 - v^2}{v(1 + v^2)} dv = \int \frac{dx}{x}$$

$$\tan^{-1}\left(\frac{v}{x}\right) - \ln\left(\frac{v}{x}\right) = \ln x + c$$

As it passes through (1, 1)

$$c = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x}\right) = \ln x + \frac{\pi}{4}$$

Put $y = \sqrt{3}x$ we get

$$\Rightarrow \frac{\pi}{3} - \ln \sqrt{3} = \ln x + \frac{\pi}{4}$$

$$\Rightarrow \ln x = \frac{\pi}{12} - \ln \sqrt{3} = \ln \alpha$$

$$\therefore \ln(\sqrt{3}\alpha) = \ln \sqrt{3} + \ln \alpha$$

$$= \ln \sqrt{3} + \frac{\pi}{12} - \ln \sqrt{3} = \frac{\pi}{12}$$

Question 107

If $y = y(x)$ is the solution of the differential equation $2x^2 \frac{dy}{dx} - 2xy + 3y^2 = 0$ such that $y(e) = \frac{e}{3}$, then $y(1)$ is equal to
[25-Jun-2022-Shift-2]

Options:

A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{3}{2}$

D. 3

Answer: B

Solution:

$$2x^2 \frac{dy}{dx} - 2xy + 3y^2 = 0$$

$$\Rightarrow 2x(x dy - y dx) + 3y^2 dx = 0$$

$$\Rightarrow 2 \left(\frac{x dy - y dx}{y^2} \right) + 3 \frac{dx}{x} = 0$$

$$\Rightarrow -\frac{2x}{y} + 3 \ln x = C$$

$$\therefore y(e) = \frac{e}{3} \Rightarrow -6 + 3 = C \Rightarrow C = -3$$

$$\text{Now, at } x = 1, -\frac{2}{y} + 0 = -3$$

$$y = \frac{2}{3}$$

Question 108

Let the solution curve $y = y(x)$ of the differential equation $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ pass through the origin. Then $y(2)$ is

equal to _____

[26-Jun-2022-Shift-1]

Answer: 12

Solution:

$$(4+x^2)dy - 2x(x^2+3y+4)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{6x}{x^2+4} \right)y + 2x$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{6x}{x^2+4} \right)y = 2x$$

$$\text{I. F.} = e^{-3 \ln(x^2+4)} = \frac{1}{(x^2+4)^3}$$

$$\text{So } \frac{y}{(x^2+4)^3} = \int \frac{2x}{(x^2+4)^3} dx + c$$

$$\Rightarrow y = -\frac{1}{2}(x^2+4) + c(x^2+4)^3$$

$$\text{When } x=0, y=0 \text{ gives } c = \frac{1}{32}$$

$$\text{So, for } x=2, y=12$$

Question 109

Let $S = (0, 2\pi) - \left\{ \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \right\}$. Let $y = y(x)$, $x \in S$, be the solution curve

of the differential equation $\frac{dy}{dx} = \frac{1}{1+\sin 2x}$, $y\left(\frac{\pi}{4}\right) = \frac{1}{2}$. If the sum of abscissas of all the points of intersection of the curve $y = y(x)$ with the curve $y = \sqrt{2} \sin x$ is $\frac{k\pi}{12}$, then k is equal to _____

[26-Jun-2022-Shift-1]

Answer: 42

Solution:

$$\frac{dy}{dx} = \frac{1}{1 + \sin 2x}$$

$$\Rightarrow dy = \frac{\sec^2 x dx}{(1 + \tan x)^2}$$

$$\Rightarrow y = -\frac{1}{1 + \tan x} + c$$

$$\text{When } x = \frac{\pi}{4}, y = \frac{1}{2} \text{ gives } c = 1$$

$$\text{So } y = \frac{\tan x}{1 + \tan x} \Rightarrow y = \frac{\sin x}{\sin x + \cos x}$$

$$\text{Now, } y = \sqrt{2} \sin x \Rightarrow \sin x = 0$$

$$\text{or } \sin x + \cos x = \frac{1}{\sqrt{2}}$$

$$\sin x = 0 \text{ gives } x = \pi \text{ only.}$$

$$\text{and } \sin x + \cos x = \frac{1}{\sqrt{2}} \Rightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$$

$$\text{So } x + \frac{\pi}{4} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \Rightarrow x = \frac{7\pi}{12} \text{ or } \frac{23\pi}{12}$$

$$\text{Sum of all solutions} = \pi + \frac{7\pi}{12} + \frac{23\pi}{12} = \frac{42\pi}{12}$$

$$\text{Hence, } k = 42.$$

Question 110

If the solution of the differential equation

$$\frac{dy}{dx} + e^x(x^2 - 2)y = (x^2 - 2x)(x^2 - 2)e^{2x} \text{ satisfies } y(0) = 0, \text{ then the value of } y(2)$$

is

[26-Jun-2022-Shift-2]

Options:

A. -1

B. 1

C. 0

D. e

Answer: C

Solution:

$$\therefore \frac{dy}{dx} + e^x(x^2 - 2)y = (x^2 - 2x)(x^2 - 2)e^{2x}$$

$$\text{Here, } I.F. = e^{\int e^x(x^2 - 2)dx}$$

$$= e^{(x^2 - 2x)e^x}$$

$\therefore$ Solution of the differential equation is

$$y \cdot e^{(x^2 - 2x)e^x} = \int (x^2 - 2x)(x^2 - 2)e^{2x} \cdot e^{(x^2 - 2x)e^x} dx$$

$$= \int (x^2 - 2x)e^x \cdot (x^2 - 2)e^x \cdot e^{(x^2 - 2x)e^x} dx$$

$$\text{Let } (x^2 - 2x)e^x = t$$

$$\therefore (x^2 - 2)e^x dx = dt$$

$$y \cdot e^{(x^2 - 2x)e^x} = \int t \cdot e^t dt$$

$$y \cdot e^{(x^2 - 2x)e^x} = (x^2 - 2x - 1)e^{(x^2 - 2x)e^x} + c$$

$$\therefore y(0) = 0$$

$$\therefore c = 1$$

$$\therefore y = (x^2 - 2x - 1) + e^{(2x - x^2)e^x}$$

$$\therefore y(2) = -1 + 1 = 0$$

Question 111

Let $\frac{dy}{dx} = \frac{ax - by + a}{bx + cy + a}$, where a, b, c are constants, represent a circle passing through the point (2, 5). Then the shortest distance of the point (11, 6) from this circle is :
[27-Jun-2022-Shift-1]

Options:

A. 10

B. 8

C. 7

D. 5

Answer: B

Solution:

$$\frac{dy}{dx} = \frac{ax - by + a}{bx + cy + a}$$

$$= bxdy + cydy + ady = axdx - bydx + adx$$

$$= cydy + ady - axdx - adx + b(xdy + ydx) = 0$$

$$= c \int y dy + a \int x dx - a \int dx + b \int d(xy) = 0$$

$$= \frac{cy^2}{2} + ay - \frac{ax^2}{2} - ax + bxy = k$$

$$= ax^2 - cy^2 + 2ax - 2ay - 2bxy = k$$

Above equation is circle

$$\Rightarrow a = -c \text{ and } b = 0$$

$$ax^2 + ay^2 + 2ax - 2ay = k$$

$$\Rightarrow x^2 + y^2 + 2x - 2y = \lambda \quad \left[\lambda = \frac{k}{a} \right]$$

Passes through (2, 5)

$$4 + 25 + 4 - 10 = \lambda \Rightarrow \lambda = 23$$

$$\text{Circle} \equiv x^2 + y^2 + 2x - 2y - 23 = 0$$

$$\text{Centre } (-1, 1) r = \sqrt{(-1)^2 + 1^2 + 23} = 5$$

$$\text{Shortest distance of } (11, 6) = \sqrt{12^2 + 5^2} - 5$$

$$= 13 - 5$$

$$= 8$$

Question 112

If $\frac{dy}{dx} + \frac{2^{x-y}(2^y-1)}{2^x-1} = 0$, $x, y > 0$, $y(1) = 1$, then $y(2)$ is equal to :

[27-Jun-2022-Shift-1]

Options:

A. $2 + \log_2 3$

B. $2 + \log_3 2$

C. $2 - \log_3 2$

D. $2 - \log_2 3$

Answer: D

Solution:

$$\frac{dy}{dx} + \frac{2^{x-y}(2^y-1)}{2^x-1} = 0, x, y > 0, y(1) = 1$$

$$\frac{dy}{dx} = - \frac{2^x(2^y-1)}{2^y(2^x-1)}$$

$$\int \frac{2^y}{2^y-1} dy = - \int \frac{2^x}{2^x-1} dx$$

$$= \frac{\log_e(2^y-1)}{\log_e 2} = - \frac{\log_e(2^x-1)}{\log_e 2} + \frac{\log_e c}{\log_e 2}$$

$$= | (2^y-1)(2^x-1) | = c$$

$$\because y(1) = 1$$

$$\therefore c = 1$$

$$= |(2^y - 1)(2^x - 1)| = 1$$

For $x = 2$

$$|(2^y - 1)3| = 1$$

$$2^y - 1 = \frac{1}{3} \Rightarrow 2y = \frac{4}{3}$$

Taking log to base 2.

$$\therefore y = 2 - \log_2 3$$

Question 113

If the solution curve of the differential equation $((\tan^{-1}y) - x)dy = (1 + y^2)dx$ passes through the point $(1, 0)$, then the abscissa of the point on the curve whose ordinate is $\tan(1)$, is
[27-Jun-2022-Shift-2]

Options:

A. $2e$

B. $\frac{2}{e}$

C. 2

D. $\frac{1}{e}$

Answer: B

Solution:

$$((\tan^{-1}y) - x)dy = (1 + y^2)dx$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\text{I. F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$\therefore$ Solution

$$x \cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y} \tan^{-1}y}{1+y^2} dy$$

$$\text{Let } e^{\tan^{-1}y} = t$$

$$\frac{e^{\tan^{-1}y}}{1+y^2} = dt$$

$$= x e^{\tan^{-1}y} = \int \ln t dt = t \ln t - t + c$$

$$\therefore x e^{\tan^{-1}y} = e^{\tan^{-1}y} \tan^{-1}y - e^{\tan^{-1}y} + c \dots (i)$$

$\therefore$ It passes through $(1, 0) \Rightarrow c = 2$

Now put $y = \tan 1$, then

$$ex = e - e + 2$$

$$\Rightarrow x = \frac{2}{e}$$

Question 114

Let $y = y(x)$ be the solution of the differential equation

$$(1 - x^2)dy = (xy + (x^3 + 2)\sqrt{1 - x^2})dx, -1 < x < 1, \text{ and } y(0) = 0. \text{ If}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 - x^2} y(x) dx = k, \text{ then } k^{-1} \text{ is equal to } \underline{\hspace{2cm}}$$

[27-Jun-2022-Shift-2]

Answer: 320

Solution:

$$(1 - x^2)dy = (xy + (x^3 + 2)\sqrt{1 - x^2})dx$$

$$\therefore \frac{dy}{dx} - \frac{x}{1 - x^2}y = \frac{x^3 + 2}{\sqrt{1 - x^2}}$$

$$\therefore \text{I.F.} = e^{\int -\frac{x}{1 - x^2} dx} = \sqrt{1 - x^2}$$

Solution is

$$y \cdot \sqrt{1 - x^2} = \int (x^3 + 2) dx$$

$$y \cdot \sqrt{1 - x^2} = \frac{x^4}{4} + 2x + c$$

$$\therefore y(0) = 0 \Rightarrow c = 0$$

$$\therefore y(x) = \frac{x^4 + 12x}{4\sqrt{1 - x^2}}$$

$$\therefore \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 - x^2} y(x) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{x^4 + 12x}{4} \right) dx = \frac{1}{2} \int_0^{\frac{1}{2}} \frac{x^4}{2} dx$$

$$\therefore k = \frac{1}{320}$$

$$\therefore k^{-1} = 320$$

Question 115

Let the solution curve $y = y(x)$ of the differential equation

$$\left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] x \frac{dy}{dx} = x + \left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] y$$

pass through the points $(1, 0)$ and $(2\alpha, \alpha)$, $\alpha > 0$. Then α is equal to
[28-Jun-2022-Shift-1]

Options:

A. $\frac{1}{2} \exp\left(\frac{\pi}{6} + \sqrt{e} - 1\right)$

B. $\frac{1}{2} \exp\left(\frac{\pi}{6} + e - 1\right)$

C. $\exp\left(\frac{\pi}{6} + \sqrt{e} + 1\right)$

D. $2 \exp\left(\frac{\pi}{3} + \sqrt{e} - 1\right)$

Answer: A

Solution:

$$\left(\frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} + e^{\frac{y}{x}} \right) \frac{dy}{dx} = 1 + \left(\frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} + e^{\frac{y}{x}} \right) \frac{y}{x}$$

Putting $y = tx$

$$\left(\frac{1}{\sqrt{1 - t^2}} + e^t \right) \left(t + x \frac{dt}{dx} \right) = 1 + \left(\frac{1}{\sqrt{1 - t^2}} + e^t \right) t$$

$$\Rightarrow x \left(\frac{1}{\sqrt{1 - t^2}} + e^t \right) \frac{dt}{dx} = 1$$

$$\Rightarrow \sin^{-1} t + e^t = \ln x + C$$

$$\Rightarrow \sin^{-1} \left(\frac{y}{x} \right) + e^{y/x} = \ln x + C$$

at $x = 1, y = 0$

$$\text{So, } 0 + e^0 = 0 + C \Rightarrow C = 1$$

at $(2\alpha, \alpha)$

$$\sin^{-1} \left(\frac{y}{x} \right) + e^{y/x} = \ln x + 1$$

$$\Rightarrow \frac{\pi}{6} + e^{\frac{1}{2}} - 1 = \ln(2\alpha)$$

$$\Rightarrow \alpha = \frac{1}{2} e^{\left(\frac{\pi}{6} + e^{\frac{1}{2}} - 1 \right)}$$

Question 116

Let $y = y(x)$ be the solution of the differential equation $x(1 - x^2) \frac{dy}{dx} + (3x^2y - y - 4x^3) = 0$, $x > 1$, with $y(2) = -2$. Then $y(3)$ is equal to [28-Jun-2022-Shift-1]

Options:

- A. -18
- B. -12
- C. -6
- D. -3

Answer: A

Solution:

$$\frac{dy}{dx} + \frac{y(3x^2 - 1)}{x(1 - x^2)} = \frac{4x^3}{x(1 - x^2)}$$

$$\text{I.F.} = e^{\int \frac{3x^2 - 1}{x - x^3} dx} = e^{-\ln|x^3 - x|} = e^{-\ln(x^3 - x)} = \frac{1}{x^3 - x}$$

Solution of D.E. can be given by

$$y \cdot \frac{1}{x^3 - x} = \int \frac{4x^3}{x(1 - x^2)} \cdot \frac{1}{x(x^2 - 1)} dx$$

$$\Rightarrow \frac{y}{x^3 - x} = \int \frac{-4x}{(x^2 - 1)^2} dx$$

$$\Rightarrow \frac{y}{x^3 - x} = \frac{2}{(x^2 - 1)} + c$$

$$\text{at } x = 2, y = -2$$

$$\frac{-2}{6} = \frac{2}{3} + c \Rightarrow c = -1$$

$$\text{at } x = 3 \Rightarrow \frac{y}{24} = \frac{2}{8} - 1 \Rightarrow y = -18$$

Question 117

Let $x = x(y)$ be the solution of the differential equation

$2ye^{x/y^2} dx + (y^2 - 4xe^{x/y^2}) dy = 0$ such that $x(1) = 0$. Then, $x(e)$ is equal to : [28-Jun-2022-Shift-2]

Options:

- A. $e \log_e(2)$

B. $-\log_e(2)$

C. $e^2 \log_e(2)$

D. $-e^2 \log_e(2)$

Answer: D

Solution:

Given differential equation

$$2ye^{\frac{x}{y^2}} dx + \left(y^2 - 4xe^{\frac{x}{y^2}} \right) dy = 0, x(1) = 0$$

$$\Rightarrow e^{\frac{x}{y^2}} [2y dx - 4x dy] = -y^2 dy$$

$$\Rightarrow e^{\frac{x}{y^2}} \left[\frac{2y^2 dx - 4xy dy}{y^4} \right] = \frac{-1}{y} dy$$

$$\Rightarrow 2e^{\frac{x}{y^2}} d\left(\frac{x}{y^2}\right) = -\frac{1}{y} dy$$

$$\Rightarrow 2e^{\frac{x}{y^2}} = -\ln y + c, \dots (i)$$

Now, using $x(1) = 0, c = 2$

So, for $x(e)$, Put $y = e$ in (i)

$$2e^{\frac{x}{e^2}} = -1 + 2$$

$$\Rightarrow \frac{x}{e^2} = \ln\left(\frac{1}{2}\right) \Rightarrow x(e) = -e^2 \ln 2$$

Question 118

Let the solution curve of the differential equation

$$x \frac{dy}{dx} - y = \sqrt{y^2 + 16x^2}, y(1) = 3 \text{ be } y = y(x). \text{ Then } y(2) \text{ is equal to:}$$

[29-Jun-2022-Shift-1]

Options:

A. 15

B. 11

C. 13

D. 17

Answer: A

Solution:

Given,

$$x \frac{dy}{dx} - y = \sqrt{y^2 + 16x}$$

$$\Rightarrow x \frac{dy}{dx} = y + \sqrt{y^2 + 16x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 16}$$

This is a homogenous different equation.

$$\text{Let } \frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = v + \sqrt{v^2 + 16}$$

$$\Rightarrow x \frac{dv}{dx} = \sqrt{v^2 + 16}$$

$$\Rightarrow \frac{dv}{\sqrt{v^2 + 16}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{v^2 + 16}} = \int \frac{dx}{x}$$

$$\Rightarrow \ln \left| v + \sqrt{v^2 + 16} \right| = \ln x + \ln c$$

$$\Rightarrow v + \sqrt{v^2 + 16} = cx$$

Now putting, $v = \frac{y}{x}$, we get

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 16} = cx$$

$$\Rightarrow \frac{y}{x} + \sqrt{\frac{y^2 + 16x^2}{x^2}} = cx$$

$$\Rightarrow y + \sqrt{y^2 + 16x^2} = cx^2 \dots\dots$$

Given, $y(1) = 3$

$\therefore$ When $x = 1$ then $y = 3$.

Putting in equation (1) we get,

$$3 + \sqrt{9 + 16} = c \cdot 1$$

$$\Rightarrow c = 8$$

$\therefore$ Solution of equation,

$$y + \sqrt{y^2 + 16x^2} = 8x^2$$

Now, $y(2)$ means when $x = 2$ then $y = ?$

$$\therefore y + \sqrt{y^2 + 16 \times 4} = 8 \times 4$$

$$\Rightarrow y = 15$$

Question 119

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{\sqrt{2}y}{2\cos^4 x - \cos^2 x} = x e^{\tan^{-1}(\sqrt{2} \cot 2x)}, \quad 0 < x < \frac{\pi}{2} \text{ with } y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{32} \text{ If}$$

$$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{18} e^{-\tan^{-1}(\alpha)}, \text{ then the value of } 3\alpha^2 \text{ is equal to } \underline{\hspace{2cm}}$$

[29-Jun-2022-Shift-1]

Answer: 2

Solution:

$$\frac{dy}{dx} + \frac{\sqrt{2}y}{2\cos^4 x - \cos^2 x} = x e^{\tan^{-1}(\sqrt{2} \cot 2x)}$$

$$\int \frac{dx}{2\cos^4 x - \cos^2 x} = \int \frac{\operatorname{cosec}^4 x \, dx}{\cos^4 x + \sin^4 x} = \int \frac{1}{1 + \cot^4 x}$$

$$= -\int \frac{t^2 + 1}{t^4 + 1} dt = -\int \frac{1}{\left(t - \frac{1}{t}\right)^2 + 2} dt = \frac{-1}{\sqrt{2}} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right)$$

$$\cot x = t$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \cot 2x)$$

$$\therefore \text{IF} = e^{-\tan^{-1}(\sqrt{2} \cot 2x)}$$

$$y e^{-\tan^{-1}(\sqrt{2} \cot 2x)} = \int x dx$$

$$y e^{-\tan^{-1}(\sqrt{2} \cot 2x)} = \frac{x^2}{2} + c$$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{32} + c \Rightarrow c = 0$$

$$y = \frac{x^2}{2} e^{\tan^{-1}(\sqrt{2} \cot 2x)}$$

$$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{18} e^{\tan^{-1}(\sqrt{2} \cot \frac{2\pi}{3})}$$

$$= \frac{\pi^2}{18} e^{-\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)}$$

$$\alpha = \sqrt{\frac{2}{3}} \Rightarrow 3\alpha^2 = 2$$

Question 120

If $y = y(x)$ is the solution of the differential equation

$(1 + e^{2x}) \frac{dy}{dx} + 2(1 + y^2)e^x = 0$ and $y(0) = 0$ then $6(y'(0) + (y(\log_e \sqrt{3}))^2)$ is equal to

[29-Jun-2022-Shift-2]

Options:

A. 2

B. -2

C. -4

D. -1

Answer: C

Solution:

$$\frac{dy}{1+y^2} + \frac{2e^x}{1+e^{2x}} dx = 0$$

on integration

$$\tan^{-1} y + 2 \tan^{-1} e^x = c$$

$$\because y(0) = 0$$

$$\text{so, } C = \frac{\pi}{2} \Rightarrow \tan^{-1} y + 2 \tan^{-1} e^x = \frac{\pi}{2} \text{ from eq.(i), } \left(\frac{dy}{dx} \right)_{x=0} = -1 \quad \arg y(\ln \sqrt{3}) = -\frac{1}{\sqrt{3}}$$

$$6 \left[y'(0) + (y(\ln \sqrt{3}))^2 \right] = 6 \left[-1 + \frac{1}{3} \right] = -4.$$

Question 121

Let $y = y(x)$, $x > 1$, be the solution of the differential equation

$(x-1) \frac{dy}{dx} + 2xy = \frac{1}{x-1}$, with $y(2) = \frac{1+e^4}{2e^4}$. If $y(3) = \frac{e^\alpha + 1}{\beta e^\alpha}$, then the value of $\alpha + \beta$

is equal to _____

[29-Jun-2022-Shift-2]

Answer: 14

Solution:

$$\frac{dy}{dx} + \frac{2x}{x-1} \cdot y = \frac{1}{(x-1)^2}$$

$$y = \frac{1}{(x-1)^2} \left[\frac{e^{2x} + 1}{2e^{2x}} \right]$$

$$y(3) = \frac{e^6 + 1}{8e^6}$$

$$\alpha + \beta = 14$$

Question 122

If $x = x(y)$ is the solution of the differential equation

$y \frac{dx}{dy} = 2x + y^3(y+1)e^y$, $x(1) = 0$; then $x(e)$ is equal to:

[24-Jun-2022-Shift-1]

Options:

A. $e^3(e^e - 1)$

B. $e^e(e^3 - 1)$

C. $e^2(e^e + 1)$

D. $e^e(e^2 - 1)$

Answer: A

Solution:

$$\frac{dx}{dy} - \frac{2x}{y} = y^2(y+1)e^y$$

$$\text{If } x = e^{\int -\frac{2}{y} dy} = e^{-2 \ln y} = \frac{1}{y^2}$$

Solution is given by

$$x \cdot \frac{1}{y^2} = \int y^2(y+1)e^y \cdot \frac{1}{y^2} dy$$

$$\Rightarrow \frac{x}{y^2} = \int (y+1)e^y dy$$

$$\Rightarrow \frac{x}{y^2} = ye^y + c$$

$$\Rightarrow x = y^2(ye^y + c) \text{ at } y = 1, x = 0$$

$$\Rightarrow 0 = 1(1 \cdot e^1 + c) \Rightarrow c = -e \text{ at } y = e$$

$$x = e^2(e \cdot e^e - e)$$

Question 123

The slope of the tangent to a curve $C : y = y(x)$ at any point (x, y) on it is $\frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}}$. If C passes through the points $\left(0, \frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)$ and $\left(\alpha, \frac{1}{2}e^{2\alpha}\right)$, then

e^α is equal to :

[25-Jul-2022-Shift-1]

Options:

A. $\frac{3 + \sqrt{2}}{3 - \sqrt{2}}$

B. $\frac{3}{\sqrt{2}} \left(\frac{3 + \sqrt{2}}{3 - \sqrt{2}} \right)$

C. $\frac{1}{\sqrt{2}} \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$

D. $\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$

Answer: B

Solution:

$$\frac{dy}{dx} = \frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}} = e^{2x} - \frac{6e^{-x}}{2 + 9e^{-2x}}$$

$$\int dy = \int e^{2x} dx - 3 \int \frac{1}{1 + \left(\frac{3e^{-x}}{\sqrt{2}}\right)^2} dx \quad (\text{put } e^{-x} = t)$$

$$= \frac{e^{2x}}{2} + 3 \int \frac{dt}{1 + \left(\frac{3t}{\sqrt{2}}\right)^2}$$

$$= \frac{e^{2x}}{2} + \sqrt{2} \tan^{-1} \frac{3t}{\sqrt{2}} + C$$

$$y = \frac{e^{2x}}{2} + \sqrt{2} \tan^{-1} \left(\frac{3e^{-x}}{\sqrt{2}} \right) + C$$

It is given that the curve passes through

$$\left(0, \frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)$$

$$\frac{1}{2} + \frac{\pi}{2\sqrt{2}} = \frac{1}{2} + \sqrt{2} \tan^{-1}\left(\frac{3}{\sqrt{2}}\right) + C$$

$$\Rightarrow C = \frac{\pi}{2\sqrt{2}} - \sqrt{2} \tan^{-1}\left(\frac{3}{\sqrt{2}}\right)$$

Now if $\left(\alpha, \frac{1}{2}e^{2\alpha}\right)$ satisfies the curve, then

$$\frac{1}{2}e^{2\alpha} = \frac{e^{2\alpha}}{2} + \sqrt{2} \tan^{-1}\left(\frac{3e^{-\alpha}}{\sqrt{2}}\right) + \frac{\pi}{2\sqrt{2}} - \sqrt{2} \tan^{-1}\left(\frac{3}{\sqrt{2}}\right)$$

$$\tan^{-1}\left(\frac{3}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{3e^{-\alpha}}{\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\frac{\frac{3}{\sqrt{2}} - \frac{3e^{-\alpha}}{\sqrt{2}}}{1 + \frac{9}{2}e^{-\alpha}} = 1$$

$$\frac{3}{\sqrt{2}}e^{\alpha} - \frac{3}{\sqrt{2}} = e^{\alpha} + \frac{9}{2}$$

$$e^{\alpha} = \frac{\frac{9}{2} + \frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}} - 1} = \frac{3}{\sqrt{2}} \left(\frac{3 + \sqrt{2}}{3 - \sqrt{2}} \right)$$

Question 124

The general solution of the differential equation

$$(x - y^2)dx + y(5x + y^2)dy = 0 \text{ is:}$$

[25-Jul-2022-Shift-1]

Options:

A. $(y^2 + x)^4 = C \mid (y^2 + 2x)^3 \mid$

B. $(y^2 + 2x)^4 = C \mid (y^2 + x)^3 \mid$

$$C. |(y^2 + x)^3| = C(2y^2 + x)^4$$

$$D. |(y^2 + 2x)^3| = C(2y^2 + x)^4$$

Answer: A

Solution:

$$(x - y^2)dx + y(5x + y^2)dy = 0$$

$$y \frac{dy}{dx} = \frac{y^2 - x}{5x + y^2}$$

$$\text{Let } y^2 = t \quad \frac{1}{2} \cdot \frac{dt}{dx} = \frac{t - x}{5x + t}$$

Now substitute, $t = vx$

$$\frac{dt}{dx} = v + x \frac{dv}{dx}$$

$$\frac{1}{2} \left\{ v + x \frac{dv}{dx} \right\} = \frac{v - 1}{5 + v}$$

$$x \frac{dv}{dx} = \frac{2v - 2}{5 + v} - v = \frac{-3v - v^2 - 2}{5 + v}$$

$$\int \frac{5 + v}{v^2 + 3v + 2} dv = \int -\frac{dx}{x}$$

$$\int \frac{4}{v + 1} dv - \int \frac{3}{v + 2} dv = -\int \frac{dx}{x}$$

$$4 \ln |v + 1| - 3 \ln |v + 2| = -\ln x + \ln C$$

$$\left| \frac{(v + 1)^4}{(v + 2)^3} \right| = \frac{C}{x}$$

$$\left| \frac{\left(\frac{y^2}{x} + 1\right)^4}{\left(\frac{y^2}{x} + 2\right)^3} \right| = \frac{C}{x}$$

$$|(y^2 + x)^4| = C |(y^2 + 2x)^3|$$

Question 125

Let a smooth curve $y = f(x)$ be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the points $(1, 2)$ and $(8, 1)$, then $y\left(\frac{1}{8}\right)$ is equal to

[25-Jul-2022-Shift-2]

Options:

A. $2\log_e 2$

B. 4

C. 1

D. $4\log_e 2$

Answer: B

Solution:

$$\frac{dy}{dx} \propto \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-ky}{x} \Rightarrow \int \frac{dy}{y} = -K \int \frac{dx}{x}$$

$$\ln|y| = -K \ln|x| + C$$

If the above equation satisfy $(1, 2)$ and $(8, 1)$

$$\ln 2 = -K \times 0 + C \Rightarrow C = \ln 2$$

$$\ln 1 = -K \ln 8 + \ln 2 \Rightarrow K = \frac{1}{3}$$

$$\text{So, at } x = \frac{1}{8}$$

$$\ln|y| = -\frac{1}{3} \ln\left(\frac{1}{8}\right) + \ln 2 = 2 \ln 2$$

$$|y| = 4$$

Question 126

Let a smooth curve $y = f(x)$ be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the points $(1, 2)$ and $(8, 1)$, then $\left|y\left(\frac{1}{8}\right)\right|$ is equal to

[25-Jul-2022-Shift-2]

Options:

A. $2\log_e 2$

B. 4

C. 1

D. $4\log_e 2$

Answer: B

Solution:

$$\frac{dy}{dx} \propto \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-ky}{x} \Rightarrow \int \frac{dy}{y} = -K \int \frac{dx}{x}$$

$$\ln |y| = -K \ln |x| + C$$

If the above equation satisfy (1, 2) and (8, 1)

$$\ln 2 = -K \times 0 + C \Rightarrow C = \ln 2$$

$$\ln 1 = -K \ln 8 + \ln 2 \Rightarrow K = \frac{1}{3}$$

So, at $x = \frac{1}{8}$

$$\ln |y| = -\frac{1}{3} \ln \left(\frac{1}{8} \right) + \ln 2 = 2 \ln 2$$

$$|y| = 4$$

Question 127

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = \frac{4y^3 + 2yx^2}{3xy^2 + x^3}, y(1) = 1.$$

If for some $n \in \mathbb{N}$, $y(2) \in [n-1, n)$, then n is equal to
[25-Jul-2022-Shift-2]

Answer: 3

Solution:

$$\frac{dy}{dx} = \frac{y(4y^2 + 2x^2)}{x(3y^2 + x^2)}$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\begin{aligned}
 &\Rightarrow v + x \frac{dv}{dx} = \frac{v(4v^2 + 2)}{(3v^2 + 1)} \\
 &\Rightarrow x \frac{dv}{dx} = v \left(\frac{4v^2 + 2 - 3v^2 - 1}{3v^2 + 1} \right) \\
 &\Rightarrow \int (3v^2 + 1) \frac{dv}{v^3 + v} = \int \frac{dx}{x} \\
 &\Rightarrow \ln |v^3 + v| = \ln x + C \\
 &\Rightarrow \ln \left| \left(\frac{y}{x} \right)^3 + \left(\frac{y}{x} \right) \right| = \ln x + C \\
 &\downarrow y(1) = 1 \\
 &\Rightarrow C = \ln 2 \\
 &\therefore \text{for } y(2) \\
 &\ln \left(\frac{y^3}{8} + \frac{y}{2} \right) = 2 \ln 2 \Rightarrow \frac{y^3}{8} + \frac{y}{2} = 4 \\
 &\Rightarrow [y(2)] = 2 \\
 &\Rightarrow n = 3
 \end{aligned}$$

Question 128

If $\frac{dy}{dx} + 2y \tan x = \sin x$, $0 < x < \frac{\pi}{2}$ and $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of

$y(x)$ is :

[26-Jul-2022-Shift-1]

Options:

A. $\frac{1}{8}$

B. $\frac{3}{4}$

C. $\frac{1}{4}$

D. $\frac{3}{8}$

Answer: A

Solution:

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

which is a first order linear differential equation.

$$\begin{aligned}
 \text{Integrating factor (I. F.)} &= e^{\int 2 \tan x \, dx} \\
 &= e^{2 \ln |\sec x|} = \sec^2 x
 \end{aligned}$$

Solution of differential equation can be written as

$$y \cdot \sec^2 x = \int \sin x \cdot \sec^2 x \, dx = \int \sec x \cdot \tan x \, dx \quad y \sec^2 x = \sec x + C$$

$$y\left(\frac{\pi}{3}\right) = 0, 0 = \sec \frac{\pi}{3} + C \Rightarrow C = -2$$

$$y = \frac{\sec x - 2}{\sec^2 x} = \cos x - 2\cos^2 x$$

$$= \frac{1}{8} - 2 \left(\cos x - \frac{1}{4} \right)^2$$

$$y_{\max} = \frac{1}{8}$$

Question 129

Let a curve $y = y(x)$ pass through the point $(3, 3)$ and the area of the region under this curve, above the x -axis and between the abscissae 3 and x ($x \geq 3$) be

$\left(\frac{y}{x} \right)^3$. If this curve also passes through the point $(\alpha, 6\sqrt{10})$ in the first quadrant, then α is equal to _____.

[26-Jul-2022-Shift-1]

Answer: 6

Solution:

$$\int_3^x f(x) dx = \left(\frac{f(x)}{x} \right)^3$$

$$x^3 \cdot \int_3^x f(x) dx = f^3(x)$$

Differentiate w.r.t. x

$$x^3 f(x) + 3x^2 \cdot \frac{f^3(x)}{x^3} = 3f^2(x) f'(x)$$

$$\Rightarrow 3y^2 \frac{dy}{dx} = x^3 y + \frac{3y^3}{x}$$

$$3xy \frac{dy}{dx} = x^4 + 3y^2$$

Let $y^2 = t$

$$\frac{3}{2} \frac{dt}{dx} = x^3 + \frac{3t}{x}$$

$$\frac{dt}{dx} - \frac{2t}{x} = \frac{2x^3}{3}$$

$$I. F. = e^{\int -\frac{2}{x} dx} = \frac{1}{x^2}$$

Solution of differential equation

$$t. \frac{1}{x^2} = \int \frac{2}{3} x dx$$

$$\frac{y^2}{x^2} = \frac{x^2}{3} + C$$

$$y^2 = \frac{x^4}{3} + Cx^2$$

Curve passes through (3, 3) $\Rightarrow C = -2$

$$y^2 = \frac{x^4}{3} - 2x^2$$

Which passes through $(\alpha, 6\sqrt{10})$

$$\frac{\alpha^4 - 6\alpha^2}{3} = 360$$

$$\alpha^4 - 6\alpha^2 - 1080 = 0$$

$$\alpha = 6$$

Question 130

Let the solution curve $y = f(x)$ of the differential equation

$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$, $x \in (-1, 1)$ pass through the origin. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x) dx$ is

equal to

[26-Jul-2022-Shift-2]

Options:

A. $\frac{\pi}{3} - \frac{1}{4}$

B. $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

C. $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

D. $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Answer: B

Solution:

$$\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

which is first order linear differential equation.

$$\text{Integrating factor (I.F.)} = e^{\int \frac{x}{x^2 - 1} dx}$$

$$= e^{\frac{1}{2} \ln |x^2 - 1|} = \sqrt{|x^2 - 1|}$$

$$= \sqrt{1 - x^2}$$

$$\because x \in (-1, 1)$$

Solution of differential equation

$$y\sqrt{1-x^2} = \int (x^4 + 2x) dx = \frac{x^5}{5} + x^2 + c$$

Curve is passing through origin, $c = 0$

$$y = \frac{x^5 + 5x^2}{5\sqrt{1-x^2}}$$

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^5 + 5x^2}{5\sqrt{1-x^2}} dx = 0 + 2 \int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$$

$$\text{put } x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$I = 2 \int_0^{\frac{\pi}{3}} \frac{\sin^2 \theta \cdot \cos \theta d\theta}{\cos \theta}$$

$$= \int_0^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\frac{\pi}{3}}$$

Question 131

Suppose $y = y(x)$ be the solution curve to the differential equation

$\frac{dy}{dx} - y = 2 - e^{-x}$ such that $\lim_{x \rightarrow \infty} y(x)$ is finite. If a and b are respectively the x -

and y -intercepts of the tangent to the curve at $x = 0$, then the value of $a - 4b$ is equal to _____.

[26-Jul-2022-Shift-2]

Answer: 3

Solution:

$$\text{IF} = e^{-x}$$

$$y \cdot e^{-x} = -2e^{-x} + \frac{e^{-2x}}{2} + C$$

$$\Rightarrow y = -2 + e^{-x} + Ce^x$$

$$\lim_{x \rightarrow \infty} y(x) \text{ is finite so } C = 0$$

$$y = -2 + e^{-x}$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \frac{dy}{dx} \Big|_{x=0} = -1$$

Equation of tangent

$$y + 1 = -1(x - 0)$$

$$\text{or } y + x = -1$$

So $a = -1, b = -1$
 $\Rightarrow a - 4b = 3$

Question 132

Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solutions of the differential equation $\frac{dy}{dx} = x + y$, with $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y = y_2(x)$ is
[27-Jul-2022-Shift-1]

Options:

- A. 0
- B. 1
- C. 2
- D. 3

Answer: A

Solution:

$$\frac{dy}{dx} = x + y$$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - 1 = t \Rightarrow \int \frac{dt}{t+1} = \int dx$$

$$\ln |t+1| = x + C'$$

$$|t+1| = Ce^x$$

$$|x + y + 1| = Ce^x$$

For $y_1(x), y_1(0) = 0 \Rightarrow C = 1$

For $y_2(x), y_2(0) = 1 \Rightarrow C = 2$

$y_1(x)$ is given by $|x + y + 1| = e^x$

$y_2(x)$ is given by $|x + y + 1| = 2e^x$

At point of intersection

$$e^x = 2e^x$$

No solution

So, there is no point of intersection of $y_1(x)$ and $y_2(x)$.

Question 133

Let $y = y(x)$ be the solution curve of the differential equation

$$\sin(2x^2) \log_e(\tan x^2) dy + \left(4xy - 4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right) \right) dx = 0, 0 < x < \sqrt{\frac{\pi}{2}},$$

which passes through the point $\left(\sqrt{\frac{\pi}{6}}, 1 \right)$. Then $\left| y\left(\sqrt{\frac{\pi}{3}} \right) \right|$ is equal to

 .
[27-Jul-2022-Shift-1]

Answer: 1

Solution:

$$\frac{dy}{dx} + y \left(\frac{4x}{\sin(2x^2) \ln(\tan x^2)} \right) = \frac{4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right)}{\sin(2x^2) \ln(\tan x^2)}$$

$$\begin{aligned} I.F. &= e^{\int \frac{4x}{\sin(2x^2) \ln(\tan x^2)} dx} \\ &= e^{\ln |\ln(\tan x^2)|} = \ln(\tan x^2) \end{aligned}$$

$$\therefore \int d(y \cdot \ln(\tan x^2)) = \int \frac{4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right)}{\sin(2x^2)} dx$$

$$\Rightarrow y \ln(\tan x^2) = \ln \left| \frac{\sec x^2 + \tan x^2}{\operatorname{cosec} x^2 - \cot x^2} \right| + C$$

$$\ln \left(\frac{1}{\sqrt{3}} \right) = \ln \left(\frac{\frac{3}{\sqrt{3}}}{2 - \sqrt{3}} \right) + C$$

$$e = \ln \left(\frac{1}{\sqrt{3}} \right) - \ln \left(\frac{\sqrt{3}}{2 - \sqrt{3}} \right)$$

$$\text{For } y \left(\sqrt{\frac{\pi}{3}} \right)$$

$$y \ln(\sqrt{3}) = \ln \left| \frac{2 + \sqrt{3}}{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}} \right| + \ln \left(\frac{1}{\sqrt{3}} \right) - \ln \left(\frac{\sqrt{3}}{2\sqrt{3}} \right)$$

$$= \ln(2 + \sqrt{3}) + \ln \left(\frac{1}{\sqrt{3}} \right) + \ln \left(\frac{1}{\sqrt{3}} \right) - \ln \left(\frac{\sqrt{3}}{2 - \sqrt{3}} \right)$$

$$\Rightarrow y \ln \sqrt{3} = \ln \left(\frac{1}{\sqrt{3}} \right)$$

$$\Rightarrow \frac{y}{2} \ln 3 = -\frac{1}{2} \ln 3$$

$$\Rightarrow y = 1$$

$$\therefore \left| y \left(\sqrt{\frac{\pi}{3}} \right) \right| = 1$$

Question 134

Let the solution curve of the differential equation

$x dy = (\sqrt{x^2 + y^2} + y) dx$, $x > 0$, intersect the line $x = 1$ at $y = 0$ and the line $x = 2$ at $y = \alpha$. Then the value of α is :

[28-Jul-2022-Shift-1]

Options:

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. $-\frac{3}{2}$

D. $\frac{5}{2}$

Answer: B

Solution:

$$\frac{x dy - y dx}{\sqrt{x^2 + y^2}} = dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + y^2}}{x} + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1 + \frac{y^2}{x^2}} + \frac{y}{x}$$

$$\text{Let } \frac{y}{x} = v$$

$$\Rightarrow v + x \frac{dv}{dx} = \sqrt{1 + v^2} + v$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

$$\text{OR } \ln(v + \sqrt{1 + v^2}) = \ln x + C$$

$$\text{at } x = 1, y = 0$$

$$\Rightarrow C = 0$$

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = x$$

$$\text{At } x = 2$$

$$\frac{y}{2} + \sqrt{1 + \frac{y^2}{4}} = 2$$

$$\Rightarrow 1 + \frac{y^2}{4} = 4 + \frac{y^2}{4} - 2y$$

$$\text{OR } y = \frac{3}{2}$$

Question 135

If $y = y(x)$, $x \in (0, \pi/2)$ be the solution curve of the differential equation $(\sin^2 2x) \frac{dy}{dx} + (8\sin^2 2x + 2 \sin 4x)y - 2e^{-4x}(2 \sin 2x + \cos 2x)$,

with $y\left(\frac{\pi}{4}\right) = e^{-\pi}$, then $y\left(\frac{\pi}{6}\right)$ is equal to:

[28-Jul-2022-Shift-1]

Options:

A. $\frac{2}{\sqrt{3}}e^{-2\pi/3}$

B. $\frac{2}{\sqrt{3}}e^{2\pi/3}$

C. $\frac{1}{\sqrt{3}}e^{-2\pi/3}$

D. $\frac{1}{\sqrt{3}}e^{2\pi/3}$

Answer: A

Solution:

$$(\sin^2 2x) \frac{dy}{dx} + (8 \sin^2 2x + 2 \sin 4x)y$$

$$= 2e^{-4x}(2 \sin 2x + \cos 2x)$$

$$\frac{dy}{dx} + (8 + 4 \cot 2x)y = 2e^{-4x} \left(\frac{2 \sin 2x + \cos 2x}{\sin^2 2x} \right)$$

Integrating factor

$$(I.F.) = e^{\int (8 + 4 \cot 2x) dx}$$

$$= e^{8x + 2 \ln \sin 2x}$$

Solution of differential equation

$$y \cdot e^{8x + 2 \ln \sin 2x}$$

$$= \int 2e^{(4x + 2 \ln \sin 2x)} \frac{(2 \sin 2x + \cos 2x)}{\sin^2 2x} dx$$

$$= 2 \int e^{4x} (2 \sin 2x + \cos 2x) dx$$

$$y \cdot e^{8x + 2 \ln \sin 2x} = e^{4x} \sin 2x + c$$

$$y\left(\frac{\pi}{4}\right) = e^{-\pi}$$

$$e^{-\pi} \cdot e^{2\pi} = e^{\pi} + c \Rightarrow c = 0$$

$$y\left(\frac{\pi}{6}\right) = \frac{e^{\frac{2\pi\sqrt{3}}{3}}}{e^{\left(\frac{4\pi}{3} + 2 \ln \frac{\sqrt{3}}{2}\right)}}$$

$$= e^{\frac{-2\pi}{3}} \cdot \frac{2}{\sqrt{3}}$$

Question 136

Let $y = y(x)$ be the solution curve of the differential equation

$$\frac{dy}{dx} + \frac{1}{x^2-1}y = \left(\frac{x-1}{x+1}\right)^{1/2}, \quad x > 1 \text{ passing through the point } \left(2, \sqrt{\frac{1}{3}}\right). \text{ Then}$$

$\sqrt{7}y(8)$ is equal to:

[28-Jul-2022-Shift-2]

Options:

A. $11 + 6\log_e 3$

B. 19

C. $12 - 2\log_e 3$

D. $19 - 6\log_e 3$

Answer: B

Solution:

$$\frac{dy}{dx} + \frac{1}{x^2-1}y = \sqrt{\frac{x-1}{x+1}}, \quad x > 1$$

$$\text{Integrating factor I.F.} = e^{\int \frac{1}{x^2-1} dx} = e^{\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right|} = \sqrt{\frac{x-1}{x+1}}$$

Solution of differential equation

$$y \sqrt{\frac{x-1}{x+1}} = \int \frac{x-1}{x+1} dx = \int \left(1 - \frac{2}{x+1}\right) dx$$

$$y \sqrt{\frac{x-1}{x+1}} = x - 2 \ln |x+1| + C$$

$$\text{Curve passes through } \left(2, \sqrt{\frac{1}{3}}\right)$$

$$\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = 2 - 2 \ln 3 + C$$

$$C = 2 \ln 3 - \frac{5}{3}$$

$$y(8) \times \frac{\sqrt{7}}{3} = 8 - 2 \ln 9 + 2 \ln 3 - \frac{5}{3}$$

$$\sqrt{7} \cdot y(8) = 19 - 6 \ln 3$$

Question 137

**The differential equation of the family of circles passing through the points (0, 2) and (0, -2) is :
[28-Jul-2022-Shift-2]**

Options:

A. $2xy \frac{dy}{dx} + (x^2 - y^2 + 4) = 0$

B. $2xy \frac{dy}{dx} + (x^2 + y^2 - 4) = 0$

C. $2xy \frac{dy}{dx} + (y^2 - x^2 + 4) = 0$

D. $2xy \frac{dy}{dx} - (x^2 - y^2 + 4) = 0$

Answer: A

Solution:

Family of circles passing through the points (0, 2) and (0, -2)

$$x^2 + (y - 2)(y + 2) + \lambda x = 0, \lambda \in \mathbb{R}$$

$$x^2 + y^2 + \lambda x - 4 = 0 \dots (1)$$

Differentiate w.r.t x

$$2x + 2y \frac{dy}{dx} + \lambda = 0 \dots (2)$$

Using (1) and (2), eliminate λ

$$x^2 + y^2 - \left(2x + 2y \frac{dy}{dx}\right)x - 4 = 0$$

$$2xy \frac{dy}{dx} + x^2 - y^2 + 4 = 0$$

Question 138

Let the solution curve $y = y(x)$ of the differential equation

$(1 + e^{2x}) \left(\frac{dy}{dx} + y \right) = 1$ pass through the point $\left(0, \frac{\pi}{2}\right)$. Then, $\lim_{x \rightarrow \infty} e^x y(x)$ is equal to :

[29-Jul-2022-Shift-1]

Options:

A. $\frac{\pi}{4}$

B. $\frac{3\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{2}$

Answer: B

Solution:

$$\text{D.E. } (1 + e^{2x}) \left(\frac{dy}{dx} + y \right) = 1$$

$$\Rightarrow \frac{dy}{dx} + y = \frac{1}{1 + e^{2x}}$$

$$\text{I.F.} = e^{\int 1 \cdot dx} = e^x$$

$\therefore$ Solution

$$e^x y(x) = \int \frac{e^x}{1 + e^{2x}} dx$$

$$\Rightarrow e^x y(x) = \tan^{-1}(e^x) + C$$

$$\therefore \text{It passes through } \left(0, \frac{\pi}{2}\right), C = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} e^x y(x) &= \lim_{x \rightarrow \infty} \tan^{-1}(e^x) + \frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

Question 139

If the solution curve of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passes through the points (2, 1) and (k + 1, 2), k > 0, then
[29-Jul-2022-Shift-2]

Options:

A. $2 \tan^{-1} \left(\frac{1}{k} \right) = \log_e(k^2 + 1)$

B. $\tan^{-1} \left(\frac{1}{k} \right) - \log_e(k^2 + 1)$

C. $2 \tan^{-1} \left(\frac{1}{k+1} \right) = \log_e(k^2 + 2k + 2)$

D. $2 \tan^{-1} \left(\frac{1}{k} \right) = \log_e \left(\frac{k^2 + 1}{k^2} \right)$

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{x+y-2}{x-y} = \frac{(x-1)+(y-1)}{(x-1)-(y-1)}$$

$$\text{Let } x-1 = X, y-1 = Y$$

$$\frac{dY}{dX} = \frac{X+Y}{X-Y}$$

$$\text{Let } Y = tX \Rightarrow \frac{dY}{dX} = t + X \frac{dt}{dX}$$

$$t + X \frac{dt}{dX} = \frac{1+t}{1-t}$$

$$X \frac{dt}{dX} = \frac{1+t}{1-t} - t = \frac{1+t^2}{1-t}$$

$$\int \frac{1-t}{1+t^2} dt = \int \frac{dX}{X}$$

$$\tan^{-1} t - \frac{1}{2} \ln(1+t^2) = \ln|X| + c$$

$$\tan^{-1} \left(\frac{y-1}{x-1} \right) - \frac{1}{2} \ln \left(1 + \left(\frac{y-1}{x-1} \right)^2 \right) = \ln|x-1| + c$$

Curve passes through (2, 1)

$$0 - 0 = 0 + c \Rightarrow c = 0$$

If (k+1, 2) also satisfies the curve

$$\tan^{-1} \left(\frac{1}{k} \right) - \frac{1}{2} \ln \left(\frac{1+k^2}{k^2} \right) = \ln k$$

$$2 \tan^{-1} \left(\frac{1}{k} \right) = \ln(1+k^2)$$

Question 140

Let $y = y(x)$ be the solution curve of the differential equation

$$\frac{dy}{dx} + \left(\frac{2x^2 + 11x + 13}{x^3 + 6x^2 + 11x + 6} \right) y = \frac{(x+3)}{x+1}, x > -1, \text{ which passes through the point } (0, 1).$$

Then $y(1)$ is equal to:

[29-Jul-2022-Shift-2]

Options:

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{7}{2}$

Answer: B

Solution:

$$\frac{dy}{dx} + \left(\frac{2x^2 + 11x + 13}{x^3 + 6x^2 + 11x + 6} \right) y = \frac{(x+3)}{x+1}, x > -1$$

$$\text{Integrating factor I.F.} = e^{\int \frac{2x^2 + 11x + 13}{x^3 + 6x^2 + 11x + 6} dx}$$

$$\text{Let } \frac{2x^2 + 11x + 13}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \quad A=2, B=1, C=-1$$

$$\text{I.F.} = e^{(2 \ln |x+1| + \ln |x+2| - \ln |x+3|)}$$

$$= \frac{(x+1)^2(x+2)}{x+3}$$

Solution of differential equation

$$y \cdot \frac{(x+1)^2(x+2)}{x+3} = \int (x+1)(x+2) dt$$

$$y \cdot \frac{(x+1)^2(x+2)}{x+3} = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$$

Curve passes through (0, 1)

$$1 \times \frac{1 \times 2}{3} = 0 + c \Rightarrow c = \frac{2}{3}$$

$$\text{So, } y(1) = \frac{\frac{1}{3} + \frac{3}{2} + 2 + \frac{2}{3}}{\frac{(2^2 \times 3)}{4}} = \frac{3}{2}$$

Question 141

The population $P = P(t)$ at time 't' of a certain species follows the differential equation $\frac{dP}{dt} = 0.5P - 450$. If $P(0) = 850$, then the time at which population becomes zero is :
24 Feb 2021 Shift 1

Options:

A. $\log_e 18$

B. $\log_e 9$

C. $1/2 \log_e 18$

D. $2 \log_e 18$

Answer: D

Solution:

Given that $\frac{dP}{dt} = 0.5P - 450$

$$\Rightarrow \int_0^t \frac{dp}{P-900} = \int_0^t \frac{dt}{2}$$

$$\Rightarrow [\ln |P(t) - 900|]_0^t = \left[\frac{t}{2} \right]_0^t$$

$$\Rightarrow \ln |P(t) - 900| - \ln |P(0) - 900| = \frac{t}{2}$$

$$\Rightarrow \ln |P(t) - 900| - \ln |50| = \frac{t}{2}$$

For $P(t) = 0$

$$\Rightarrow \ln \left| \frac{900}{50} \right| = \frac{t}{2}$$

$$\Rightarrow t = 2 \ell \ln 18$$

Question 142

Let f be a twice differentiable function defined on $\mathbb{R}$, such that

$f(0) = 1$, $f'(0) = 2$ and $f''(x) \neq 0$ for all $x \in \mathbb{R}$. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} = 0$, for all

$x \in \mathbb{R}$, then the value of $f(1)$ lies in the interval
[24 Feb 2021 Shift 2]

Options:

A. (9, 12)

B. (6, 9)

C. (0, 3)

D. (3, 6)

Answer: B

Solution:

Given, $f(0) = 1$,

$f'(0) = 2$,

$f''(x) \neq 0$

$$\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} = 0$$

$$\Rightarrow f(x)f''(x) - f'(x)f'(x) = 0$$

$$\Rightarrow \frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \int \frac{f''(x)}{f'(x)} dx = \int \frac{f'(x)}{f(x)} dx$$

$$\Rightarrow \log f'(x) = \log f(x) + \log c$$

$$\Rightarrow f'(x) = cf(x)$$

Now, put $x = 0$, we get

$$f'(0) = cf(0)$$

$$\Rightarrow 2 = c \times 1$$

$$\Rightarrow c = 2$$

Putting the value of $c = 2$ in Eq. (i), we get

$$\log f'(x) = \log f(x) + \log 2$$

$$\Rightarrow f'(x) = 2f(x) \Rightarrow \int \frac{f'(x)}{f(x)} dx = \int 2 dx$$

$$\Rightarrow \log f(x) = 2x + D \Rightarrow f(x) = e^{2x+D}$$

$$\Rightarrow f(x) = e^D \cdot e^{2x}$$

$$\Rightarrow f(x) = k \cdot e^{2x} \text{ [Let } k = e^D \text{]}$$

Put $x = 0$, we get

$$f(0) = k \cdot e^0$$

$$\Rightarrow 1 = k \Rightarrow f(x) = k \cdot e^{2x}$$

$$\therefore f(x) = e^{2x}$$

Put $x = 1$, we get

$$f(1) = e^2$$

Clearly, e^2 lies in $(6, 9)$.

Question 143

The difference between degree and order of a differential equation that represents the family of curves given by $y^2 = a \left(x + \frac{\sqrt{a}}{2} \right)$, $a > 0$ is

[26 Feb 2021 Shift 1]

Answer: 2

Solution:

$$\text{Given, } y^2 = a \left[x + \frac{\sqrt{a}}{2} \right], a > 0 \dots (i)$$

Differentiating both sides w.r.t. 'x',

$$2y \frac{dy}{dx} = a[1 + 0] = a \dots (ii)$$

Use Eq. (ii) in Eq. (i) to eliminate the constant 'a'.

$$y^2 = 2y \frac{dy}{dx} \left(x + \sqrt{2y} \sqrt{\frac{dy}{dx}} \right)$$

$$y^2 - 2xy \frac{dy}{dx} = 2\sqrt{2} \cdot y\sqrt{y} \cdot \frac{dy}{dx} \sqrt{\frac{dy}{dx}}$$

Squaring on both sides,

$$y^4 + 4x^2y^2 \left(\frac{dy}{dx} \right)^2 - 4xy^3 \frac{dy}{dx} = 8y^3 \left(\frac{dy}{dx} \right)^3$$

Thus, degree of above differential equation is 3 and its order is 1.

Difference between degree and order = $3 - 1 = 2$

Question 144

Let slope of the tangent line to a curve at any point $P(x, y)$ be given by $\frac{xy^2 + y}{x}$. If the curve intersects the line $x + 2y = 4$ at $x = -2$, then the value of y , for which the point $(3, y)$ lies on the curve, is
[26 Feb 2021 Shift 2]

Options:

A. $\frac{18}{35}$

B. $-\frac{4}{3}$

C. $-\frac{18}{19}$

D. $-\frac{18}{11}$

Answer: C

Solution:

Given, slope of tangent line to curve at (x, y) is $\frac{xy^2 + y}{x}$

i.e., $\frac{dy}{dx} = \frac{xy^2 + y}{x}$

$$\Rightarrow \frac{dy}{dx} = y^2 + \frac{y}{x} \Rightarrow x dy = xy^2 dx + y dx$$

$$\Rightarrow x dy - y dx = xy^2 dx \Rightarrow \frac{x dy - y dx}{y^2} = x dx$$

Integrating both sides, we get

$$\frac{-x}{y} = \frac{x^2}{2} + C \dots (i)$$

The curve intersect line at $x = -2$

Then, $x = -2$, is satisfied by $x + 2y = 4$

Hence, $(-2) + 2y = 4$ Gives, $y = 3$

$\therefore$ Curve passes through $(-2, 3)$.

Use $(-2, 3)$ is Eq. (i), we get

$$\frac{-2}{3} = \frac{(-2)^2}{2} + C \Rightarrow C = -\frac{4}{3}$$

$\therefore$ The curve is

$$\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3} \dots (ii)$$

It also passes through (3, y).

Put (3, 4) in Eq. (ii), we get

$$\Rightarrow \frac{-3}{y} = \frac{(3)^2}{2} - \frac{4}{3} \Rightarrow y = -\frac{18}{19}$$

Question 145

Let $f(x) = \int_0^x e^t f(t) dt + e^x$ be a differentiable function for all $x \in \mathbb{R}$. Then, $f(x)$ equals
[26 Feb 2021 Shift 2]

Options:

A. $2e^{(e^x - 1)} - 1$

B. $e^{e^x} - 1$

C. $2e^{e^x} - 1$

D. $e^{(e^x - 1)}$

Answer: A

Solution:

Given, $f(x) = \int_0^x e^t f(t) dt + e^x \dots (i)$

Since, $f(x)$ is differentiable function, differentiate Eq. (i)

$$f'(x) = e^x f(x) + e^x [\text{Using Newton Leibnitz theorem}]$$

$$f'(x) = e^x (f(x) + 1) \Rightarrow \frac{f'(x)}{f(x) + 1} = e^x$$

Integrating it, $\int \frac{f'(x)}{f(x) + 1} dx = \int e^x dx + C$

Let $f(x) + 1 = u$, then $f'(x) dx = du$

$$\int \frac{du}{u} = e^x + C \Rightarrow \log u = e^x + C$$

$$\log(f(x) + 1) = e^x + C \quad [\because u = f(x) + 1] \dots (ii)$$

Now

$$f(x) = \int_0^x e^t f(t) dt + e^x \Rightarrow f(0) = e^0 = 1$$

Put $x = 0$, in Eq. (ii), we get

$$\log(2) = e^0 + C \Rightarrow C = \log(2) - 1$$

From Eq. (ii), we get

$$\log(f(x) + 1) = e^x + \log 2 - 1$$

$$f(x) + 1 = e^{e^x + \log 2 - 1} = e^{e^x} \cdot e^{\log 2} \cdot e^{-1}$$

$$f(x) + 1 = 2e^{e^x} \cdot e^{-1} = 2e^{e^x - 1}$$

$$\therefore f(x) = 2e^{e^x - 1} - 1$$

Question 146

The rate of growth of bacteria in a culture is proportional to the number of bacteria present and the bacteria count is 1000 at initial time $t = 0$. The number of bacteria is increased by 20% in 2h. If the population of bacteria is 2000 after $\frac{k}{\log_e(6/5)}h$, then $\left(\frac{k}{\log_e 2}\right)^2$ is equal to

[26 Feb 2021 Shift 1]

Options:

- A. 4
- B. 8
- C. 2
- D. 16

Answer: A

Solution:

Let x be the number of bacteria at any time t .

Given that, $\frac{dx}{dt} \propto x$ ($\because$ Rate of growth $= \frac{dx}{dt}$)

$$\Rightarrow \frac{dx}{dt} = \lambda x \Rightarrow \frac{dx}{x} = \lambda dt$$

After integrating it, we get

$$\log x = \lambda t + C \dots (i)$$

$= 0, x = 1000$ which gives

$$\log 1000 = 0 + C \Rightarrow C = \log 1000$$

Given, when $t = 0, x = \log 1000$

$$\log x - \log 1000 = \lambda t \text{ or } \log \left(\frac{x}{1000} \right) = \lambda t \dots (ii)$$

Given that in 2h, number of bacteria increased by 20% i.e. when $t = 2h, x = 1200$ Put, $t = 2$ and $x = 1200$ in Eq.

$$(ii), \log \left(\frac{1200}{1000} \right) = 2\lambda \text{ gives, } \lambda = \frac{1}{2} \log \left(\frac{6}{5} \right)$$

Again, from Eq. (ii),

$$\log \left(\frac{x}{1000} \right) = \frac{1}{2} \log \left(\frac{6}{5} \right)$$

$$\text{or } \frac{x}{1000} = e^{\frac{t}{2} \log \left(\frac{6}{5} \right)} \dots (iii)$$

Given, $x = 2000$ at $t = k / \log_e(6/5)$,

put in Eq. (iii),

$$\frac{2000}{1000} = e^{\frac{k}{2} \log \left(\frac{6}{5} \right) / \log \left(\frac{6}{5} \right)}$$

$$2 = e^{k/2} \text{ or } \log 2 = k/2$$

$$k/\log 2 = 2$$

$$\therefore (k/\log_e 2)^2 = (2)^2 = 4$$

Question 147

If $y = y(x)$ is the solution of the equation

$$e^{\sin y} \cos y \frac{dy}{dx} + e^{\sin y} \cos x = \cos x, y(0) = 0, \text{ then}$$

$$1 + y\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right) + \frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right) \text{ is } \dots\dots\dots$$

[26 Feb 2021 Shift 1]

Answer: 1

Solution:

$$\text{Given } e^{\sin y} \cos y \frac{dy}{dx} + e^{\sin y} \cos x = \cos x \dots (i)$$

$$\text{Let } e^{\sin y} = t, \text{ then } e^{\sin y} \cdot \cos y \cdot \frac{dy}{dx} = \frac{dt}{dx},$$

Putting in Eq. (i),
 $\cos x \dots (ii)$ (Linear form)

$$\text{Then, IF} = e^{\int \cos x dx} = e^{\sin x}$$

Solution of differential Eq. (ii) is,

$$t \cdot IF = \int \cos x \cdot IF dx + C$$

$$t \cdot e^{\sin x} = \int \cos x \cdot e^{\sin x} dx + C$$

$$= e^u \text{ i.e. let } \sin x = u \text{ then } \cos x dx = du$$

$$\Rightarrow t \cdot e^{\sin x} = \int e^u du + C = e^u + C$$

$$\text{Put } u = \sin x \text{ and } t = e^{\sin y}$$

$$\Rightarrow e^{\sin y} \cdot e^{\sin x} = e^{\sin x} + C$$

Given, $y(0) = 0$, this gives $C = 0$

$$\Rightarrow e^{\sin y} \cdot e^{\sin x} = e^{\sin x}$$

$$\Rightarrow e^{\sin y + \sin x} = e^{\sin x}$$

$$\Rightarrow \sin y + \sin x = \sin x$$

$$\Rightarrow \sin y = 0$$

$$\Rightarrow y = 0$$

$$\therefore y(\pi/6) = y(\pi/3) = y(\pi/4) = 0$$

$$\text{Hence, } 1 + y\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right) + \frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right)$$

$$= 1 + 0 + 0 + 0 = 1$$

Question 148

If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $\frac{x^2 - 4x + y + 8}{x - 2}$, then this curve also passes through the point
[25 Feb 2021 Shift 1]

Options:

A. (5, 4)

B. (4, 5)

C. (4, 4)

D. (5, 5)

Answer: D

Solution:

$$\text{Given, slope} = \frac{x^2 - 4x + y + 8}{x - 2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 4x + y + 8}{x - 2} = \frac{(x - 2)^2 + (y + 4)}{(x - 2)}$$

$$= (x - 2) + \frac{y + 4}{x - 2}$$

$$\text{Let } (x - 2) = t \Rightarrow dx = dt$$

$$\text{and } (y + 4) = u \Rightarrow dy = du$$

$$\therefore \frac{dy}{dx} = \frac{du}{dt}$$

$$\text{Now, } \frac{du}{dx} = (x - 2) + \frac{(y + 4)}{(x - 2)}$$

$$\Rightarrow \frac{du}{dt} = t + \frac{u}{t} \Rightarrow \frac{du}{dt} - \frac{u}{t} = t$$

$$\text{Here, integrating factor (IF)} = 1/t$$

$$\Rightarrow u \cdot \left(\frac{1}{t}\right) = \int t \left(\frac{1}{t}\right) dt \Rightarrow u/t = t + c$$

$$\Rightarrow \frac{(y + 4)}{(x - 2)} = (x - 2) + c$$

$\therefore$ It passes through origin [i.e. (0, 0)], then

$$\therefore \frac{4}{-2} = -2 + c$$

$$\Rightarrow -2 + 2 = c \Rightarrow c = 0$$

$$\text{Hence, } \frac{(y + 4)}{(x - 2)} = (x - 2) + 0$$

$$\Rightarrow y + 4 = (x - 2)^2$$

Clearly, this curve passes through (5, 5) as it satisfies the equation.

Question 149

If the curve $y = y(x)$ represented by the solution of the differential equation $(2xy^2 - y)dx + xdy = 0$, passes through the intersection of the lines

$2x - 3y = 1$ and $3x + 2y = 8$, then $|y(1)|$ is equal to
[25 Feb 2021 Shift 2]

Answer: 1

Solution:

Given, $(2xy^2 - y)dx + xdy = 0$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -2y^2$$

$$\Rightarrow \frac{-1}{y^2} \frac{dy}{dx} + \frac{1}{xy} = 2 \dots (i) \text{ [divide by } y^2 \text{]}$$

Let $\frac{1}{y} = v$, then $-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$, putting in Eq. (i)

$$\frac{dv}{dx} + v \left(\frac{1}{x} \right) = 2 \text{ (this is a linear form)}$$

Now, integrating factor (IF) $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$$\therefore (IF)v = \int 2 \cdot (IF) dx = \int 2x dx = 2 \frac{x^2}{2} + C$$

$$\therefore (IA)v = x^2 + C$$

Put $v = \frac{1}{y}$, this gives

$$x^2 + c = \frac{x}{y}$$

Now, first find point of intersection of lines

$2x - 3y = 1$ and $3x = -2y + 8$ by elimination method, we get $x = 2, y = 1$

$\therefore$ The curve $x^2 + c = \frac{x}{y}$ passes through $(2, 1)$.

Put $x = 2, y = 1$, we get $c = -2$

$$\frac{x}{y} = x^2 - 2$$

$$\text{or } y = \frac{x}{x^2 - 2}$$

Put $x = 1$, we get $y(1) = \frac{1}{1 - 2} = -1$

$$\therefore |y(1)| = 1$$

Question 150

If a curve $y = f(x)$ passes through the point $(1, 2)$ and satisfies $x \frac{dy}{dx} + y = bx^4$, then for what value of b , $\int_1^2 f(x) dx = \frac{62}{5}$?

[24 Feb 2021 Shift 2]

Options:

A. 5

B. 10

C. $\frac{62}{5}$

D. $\frac{31}{5}$

Answer: B

Solution:

Given, curve $y = f(x)$ passes through $(1, 2)$ and satisfies

$$x \frac{dy}{dx} + y = bx^4$$

$$\Rightarrow x \frac{dy}{dx} + y = bx^4$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = bx^3$$

$$\text{IF} = e^{\int \frac{1}{x} \cdot dx} = x$$

$$yx = \int bx^4 dx = \frac{bx^5}{5} + C$$

$$\therefore y = \frac{bx^4}{5} + \frac{C}{x} = f(x) \dots (i)$$

$\therefore$ This curve passes through $(1, 2)$.

$$\therefore 2 \times 1 = \frac{b \times (1)^5}{5} + C$$

$$\Rightarrow 2 = \frac{b}{5} + C \dots (ii)$$

$$\text{Also, } \int_1^2 f(x) dx = \frac{62}{5}$$

$$\Rightarrow \int_1^2 \left(\frac{bx^4}{5} + \frac{C}{x} \right) dx = \frac{62}{5}$$

$$\Rightarrow \left[b \times \frac{x^5}{25} + C \log x \right]_1^2 = \frac{62}{5}$$

$$\Rightarrow \left[\left(\frac{b \times 32}{25} + C \log 2 \right) - \left(\frac{b}{25} + C \log 1 \right) \right] = \frac{62}{5}$$

$$\Rightarrow \frac{b \times 32}{25} + C \log 2 - \frac{b}{25} = \frac{62}{5} + 0 \log 2$$

[from Eq. (i)] On comparing, we get

$$\frac{b}{25} \times 31 = \frac{62}{5} \text{ and } C = 0$$

$$b = \frac{62 \times 25}{31 \times 5}$$

$$b = 10$$

Hence, the required value of $b = 10$.

Question 151

The differential equation satisfied by the system of parabolas $y^2 = 4a(x + a)$ is
[18 Mar 2021 Shift 1]

Options:

A. $y \left(\frac{dy}{dx} \right)^2 - 2x \left(\frac{dy}{dx} \right) - y = 0$

B. $y \left(\frac{dy}{dx} \right)^2 - 2x \left(\frac{dy}{dx} \right) + y = 0$

C. $y \left(\frac{dy}{dx} \right)^2 + 2x \left(\frac{dy}{dx} \right) - y = 0$

D. $y \left(\frac{dy}{dx} \right) + 2x \left(\frac{dy}{dx} \right) - y = 0$

Answer: C

Solution:

Given, equation of curve is $y^2 = 4a(x + a)$

$$\Rightarrow y^2 = 4ax + 4a^2 \dots (i)$$

Differentiating Eq. (i) w.r.t. x , we get

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow a = \left(\frac{y}{2} \right) \cdot \frac{dy}{dx} \dots (ii)$$

$\therefore$ Required differential equation is

$$y^2 = 4 \times \frac{y}{2} \times \frac{dy}{dx} + 4 \left(\frac{y}{2} \cdot \frac{dy}{dx} \right)^2 \quad [\text{From Eqs. (i) and (ii)}]$$

$$\Rightarrow y^2 \left(\frac{dy}{dx} \right)^2 + 2xy \left(\frac{dy}{dx} \right) - y^2 = 0$$

$$\Rightarrow y \left[y \left(\frac{dy}{dx} \right)^2 + 2x \left(\frac{dy}{dx} \right) - y \right] = 0$$

As, $y \neq 0$

$$\therefore y \left(\frac{dy}{dx} \right)^2 + 2x \left(\frac{dy}{dx} \right) - y = 0$$

Question 152

Let $y = y(x)$ be the solution of the differential equation

$x dy - y dx = \sqrt{x^2 - y^2} dx$, $x \geq 1$, with $y(1) = 0$. If the area bounded by the

line $x = 1$, $x = e^\pi$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to

[18 Mar 2021 Shift 2]

Answer: 4

Solution:

$$x dy - y dx = \sqrt{x^2 - y^2} dx$$

$$\Rightarrow \frac{x dy - y dx}{x^2} = \frac{1}{x} \sqrt{1 - \frac{y^2}{x^2}} dx \Rightarrow \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \frac{1}{x} dx$$

$$\text{On integrating, } \int \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \cdot d\left(\frac{y}{x}\right) = \int \frac{1}{x} dx$$

$$\Rightarrow \sin^{-1}(y/x) = \log |x| + C$$

$$\text{Now, at } x = 1, y = 0$$

$$\therefore C = 0$$

$$\text{Hence, } y = x \sin(\log x)$$

$$\therefore A = \int_1^e x \sin(\log x) dx$$

$$\text{Put } x = e^t \Rightarrow dx = e^t dt$$

$$\therefore A = \int_0^{\pi} e^{2t} \sin(t) dt$$

$$\left(\text{using } \int e^{ax} \cdot \sin bx = \frac{e^{2x}}{a^2 + b^2} (a \sin bx - b \cos bx) \right)$$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5} (2 \sin t - \cos t) \right)_0^{\pi} = \frac{1 + e^{2\pi}}{5}$$

$$\therefore \alpha = \frac{1}{5}, \beta = \frac{1}{5}$$

$$\therefore 10(\alpha + \beta) = 4$$

Question 153

Which of the following is true for $y(x)$ that satisfies the differential equation

$$\frac{dy}{dx} = xy - 1 + x - y; y(0) = 0$$

[17 Mar 2021 Shift 1]

Options:

A. $y(1) = e^{-\frac{1}{2}} - 1$

B. $y(1) = e^{\frac{1}{2}} - e^{-\frac{1}{2}}$

C. $y(1) = 1$

$$D. y(1) = e^{\frac{1}{2}} - 1$$

Answer: A

Solution:

$$\frac{dy}{dx} = xy - 1 + x - y$$

$$\frac{dy}{dx} = x(y+1) - 1(y+1) \Rightarrow \frac{dy}{dx} = (x-1)(y+1)$$

$$\int \frac{dy}{y+1} = \int (x-1) dx$$

$$\log_e(y+1) = \frac{x^2}{2} - x + C \quad [\because \log_e 1 = 0]$$

$$\log_e 1 = 0 + C \Rightarrow C = 0$$

$$y(1) \Rightarrow \log_e(y+1) = \frac{1}{2} - 1$$

$$y+1 = e^{-1/2}$$

$$y(1) = -1 + e^{-1/2}$$

Question 154

Let the curve $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 2(x+1).$$

If the numerical value of area bounded by the curve $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to.....

[16 Mar 2021 Shift 1]

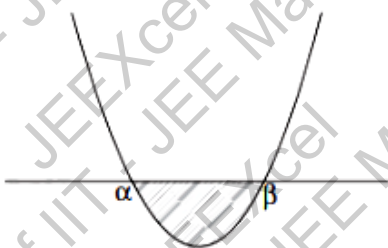
Answer: 2

Solution:

We have, $dy/dx = 2(x+1)$

$$y = \int 2(x+1) dx = 2(x^2/2 + x) + C$$

$$y = x^2 + 2x + C \dots (i)$$



$$\text{Now, } \int_{\alpha}^{\beta} y dx = \frac{4\sqrt{8}}{3}$$

$$\int_{\alpha}^{\beta} (x^2 + 2x + c) dx = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \left[\frac{x^3}{3} + x^2 + cx \right]_{\alpha}^{\beta} = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \left(\frac{\beta^3 - \alpha^3}{3} \right) + (\beta^2 - \alpha^2) + c(\beta - \alpha) = \frac{4\sqrt{8}}{3} \dots (ii)$$

From Eq. (i),

$$\alpha + \beta = -2$$

$$\alpha\beta = c$$

$$\therefore \beta - \alpha = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \sqrt{4 - 4c} = 2\sqrt{1 - c}$$

$$\text{From Eq. (ii), } \frac{1}{3}(\beta - \alpha)(\alpha^2 + \beta^2 + \alpha\beta) + (\beta - \alpha)(\beta + \alpha) + c(\beta - \alpha) = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \frac{1}{3}(2\sqrt{1 - c})(4 - c) + (2\sqrt{1 - c})(-2 + c) = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow (2\sqrt{1 - c})[4 - c + (-6 + 3c)] = 4\sqrt{8}$$

$$\Rightarrow (2\sqrt{1 - c})[2(c - 1)] = 4\sqrt{8}$$

$$\Rightarrow (1 - c)^{3/2} = -\sqrt{8}$$

$$\Rightarrow (1 - c)^3 = 8$$

$$\Rightarrow 1 - c = 2$$

$$\therefore c = -1$$

$$\text{Now, } y = x^2 + 2x - 1$$

$$\therefore y(1) = 1^2 + 2 \cdot 1 - 1 = 2$$

Question 155

Let C_1 be the curve obtained by the solution of differential equation

$2xy \frac{dy}{dx} = y^2 - x^2$, $x > 0$. Let the curve C_2 be the solution of $\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$. If both

the curves pass through $(1, 1)$, then the area enclosed by the curves C_1 and C_2 is equal to

[16 Mar 2021 Shift 2]

Options:

A. $\pi - 1$

B. $\frac{\pi}{2} - 1$

C. $\pi + 1$

D. $\frac{\pi}{4} + 1$

Answer: B

Solution:

Given, $2xy \frac{dy}{dx} = y^2 - x^2, x > 0$

$$\frac{dy}{dx} = \left(\frac{y^2 - x^2}{2xy} \right), x > 0$$

Let $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 - x^2}{2x \cdot vx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v \Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-(v^2 + 1)}{2v} \Rightarrow \int - \left(\frac{2v}{v^2 + 1} \right) dv = \int \frac{dx}{x}$$

$$\Rightarrow -\log |v^2 + 1| = \log x + c$$

$$\Rightarrow \log |x| + \log |v^2 + 1| = c$$

$$\Rightarrow \log |v^2 + 1)x| = c \Rightarrow \left(\frac{y^2}{x^2} + 1 \right) x = c$$

$$\Rightarrow \frac{(y^2 + x^2)}{x} = c \Rightarrow y^2 + x^2 = cx$$

Similarly, for second curve, $x^2 + y^2 = cy$

Both passes through (1, 1),

$$C_1 \Rightarrow 1 + 1 = C_1$$

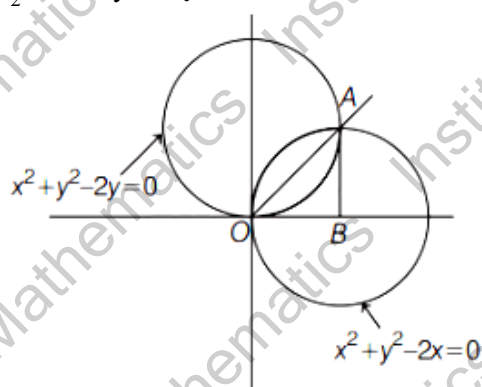
$$\Rightarrow C_1 = 2$$

$$C_1 \Rightarrow x^2 + y^2 = 2x$$

$$C_2 \Rightarrow x^2 + y^2 = C_2 y$$

$$\Rightarrow 1 + 1 = C_2 \Rightarrow C_2 = 2$$

$$C_2 \Rightarrow x^2 + y^2 = 2y$$



$$\therefore \text{Required area} = 2 \left[\frac{\pi \cdot 1^2}{4} - \frac{1}{2} \cdot 1 \cdot 1 \right]$$

$$= 2 \left[\left(\frac{\pi - 2}{4} \right) \right] = \frac{\pi}{2} - 1$$

Question 156

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = (y + 1) \left[(y + 1)e^{x^2/2} - x \right] \quad 0 < x < 2, \quad 1, \text{ with } y(2) = 0. \text{ Then the value of } \frac{dy}{dx}$$

at $x = 1$ is equal to
[18 Mar 2021 Shift 2]

Options:

A. $\frac{-e^{3/2}}{(e^2 + 1)^2}$

B. $-\frac{2e^2}{(1 + e^2)^2}$

C. $\frac{e^{5/2}}{(1 + e^2)^2}$

D. $\frac{5e^{1/2}}{(e^2 + 1)^2}$

Answer: A

Solution:

Given, $\frac{dy}{dx} = (y+1)[(y+1)e^{x^2/2} - x] \dots (i)$

$$\Rightarrow \frac{dy}{dx} = (y+1)^2 e^{x^2/2} - x(y+1)$$

$$\Rightarrow \frac{dy}{dx} + x(y+1) = (y+1)^2 e^{x^2/2}$$

$$\Rightarrow \frac{1}{(y+1)^2} \cdot \frac{dy}{dx} + \frac{1}{y+1} \cdot x = e^{x^2/2} \dots (ii)$$

Let $\frac{1}{y+1} = t \Rightarrow \frac{-1}{(y+1)^2} \cdot \frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow -\frac{dt}{dx} + tx = e^{x^2/2}$$

$$\Rightarrow \frac{dt}{dx} + (-x)t = -e^{x^2/2}$$

which is linear differential equation

$$IF = e^{\int -x dx} = e^{-x^2/2}$$

Now, solution of differential equation is,

$$t \cdot e^{-x^2/2} = -\int e^{-x^2/2} \cdot e^{x^2/2} dx$$

$$\Rightarrow \left(\frac{1}{y+1}\right) e^{-x^2/2} = -x + c, \text{ where } c \text{ is constant of integration} \dots (iii)$$

Given, $y(2) = 0$ i.e. when $x = 2$, then $y = 0$.

From Eq. (iii),

$$e^{-2} = -2 + c \Rightarrow c = 2 + e^{-2}$$

Now, at $x = 1$

$$\frac{1}{y+1} e^{-1/2} = -1 + e^{-2} + 2$$

$$\Rightarrow (y+1) = \frac{e^{-1/2}}{1 + e^{-2}}$$

Now, putting the value of $(y+1)$ in Eq. (i), we get

$$y'(1) = \frac{e^{-1/2}}{1 + e^{-2}} \left(\frac{e^{-1/2}}{1 + e^{-2}} \cdot e^{1/2} - 1 \right)$$

$$\begin{aligned}
 &= \frac{e^{-1/2}}{1+e^{-2}} \left(\frac{1}{1+e^{-2}} - 1 \right) \\
 &= \frac{e^{-1/2}}{1+e^{-2}} \left(\frac{1-1-e^{-2}}{1+e^{-2}} \right) \\
 &= \frac{-e^{-5/2}}{(1+e^{-2})^2} = \frac{-e^{-5/2}}{(e^2+1)^2} = \frac{-e^{3/2}}{(e^2+1)^2} \\
 &\quad \frac{1}{e^4}
 \end{aligned}$$

Question 157

Let $y = y(x)$ be the solution of the differential equation $\cos x(3 \sin x + \cos x + 3) dy = [1 + y \sin x(3 \sin x + \cos x + 3)] dx$

$0 \leq x \leq \frac{\pi}{2}$, $y(0) = 0$. Then, $y\left(\frac{\pi}{3}\right)$ is equal to

[17 Mar 2021 Shift 2]

Options:

A. $2\log_e \left(\frac{2\sqrt{3}+9}{6} \right)$

B. $2\log_e \left(\frac{2\sqrt{3}+10}{11} \right)$

C. $2\log_e \left(\frac{\sqrt{3}+7}{2} \right)$

D. $2\log_e \left(\frac{3\sqrt{3}-8}{4} \right)$

Answer: B

Solution:

Given, $\cos x(3 \sin x + \cos x + 3) dy = [1 + y \sin x(3 \sin x + \cos x + 3)] dx$

$$\Rightarrow \cos x dy = \left(\frac{1}{3 \sin x + \cos x + 3} + y \sin x \right) dx$$

$$\Rightarrow \cos x \cdot \frac{dy}{dx} = y \sin x + \frac{1}{3 \sin x + \cos x + 3}$$

$$\Rightarrow \frac{dy}{dx} = y \frac{\sin x}{\cos x} + \frac{1}{\cos x(3 \sin x + \cos x + 3)}$$

$$\Rightarrow \frac{dy}{dx} - (\tan x)y = \frac{1}{\cos x(3 \sin x + \cos x + 3)} \dots (i)$$

Which is linear differential equation.

$$\begin{aligned}
 \therefore \text{Integrating factor (IF)} &= e^{\int (-\tan x) dx} \\
 &= e^{\log |\cos x|} = |\cos x|
 \end{aligned}$$

$$\because |\cos x| > 0, \forall x \in [0, \pi/2)$$

$$\therefore |\cos x| = \cos x$$

Hence, solution of Eq. (i) is

$$y(\cos x) = \int (\cos x) \cdot \frac{1}{\cos x(3 \sin x + \cos x + 3)} dx$$

$$\Rightarrow y \cos x = \int \frac{1}{3 \sin x + \cos x + 3} dx$$

$$\Rightarrow y \cos x = \int \frac{\sec^2 \frac{x}{2} dx}{2 \tan^2 \frac{x}{2} + 6 \tan \frac{x}{2} + 4} dx = \int \frac{\sec^2 \frac{x}{2} dx}{2 \left(\tan^2 \frac{x}{2} + 3 \tan \frac{x}{2} + 2 \right)} dx$$

$$\text{Putting } \tan \frac{x}{2} = z$$

$$\Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dz$$

$$\therefore y \cos x = \int \frac{dz}{z^2 + 3z + 2} = \int \frac{dz}{(z+1)(z+2)}$$

$$= \int \frac{1}{z+1} dz - \int \frac{1}{z+2} dz = \log(z+1) - \log(z+2) + c$$

$$\Rightarrow y \cos x = \log \left| \frac{z+1}{z+2} \right| + c = \log \left| \frac{\tan \frac{x}{2} + 1}{\tan \frac{x}{2} + 2} \right| + c \dots (i)$$

Since, $y(0) = 0$

$$\therefore C = -\log \left(\frac{1}{2} \right) = \log 2$$

$$\text{From Eq. (i), } y \cos x = \log \left| \frac{\tan \frac{x}{2} + 1}{\tan \frac{x}{2} + 2} \right| + \log 2$$

$$\begin{aligned} \therefore y \left(\frac{\pi}{3} \right) &= 2 \left[\log \left| \frac{\frac{1}{\sqrt{3}} + 1}{\frac{1}{\sqrt{3}} + 2} \right| + \log 2 \right] = 2 \log \left| 2 \left(\frac{\sqrt{3} + 1}{2\sqrt{3} + 1} \right) \right| \\ &= 2 \log \left| 2 \left(\frac{\sqrt{3} + 1}{2\sqrt{3} + 1} \times \frac{2\sqrt{3} - 1}{2\sqrt{3} - 1} \right) \right| = 2 \log \left| \frac{2\sqrt{3} + 10}{11} \right| \end{aligned}$$

Question 158

If the curve $y = y(x)$ is the solution of the differential equation

$$2(x^2 + x^{5/4}) dy - y(x + x^{1/4}) dx = 2x^{9/4} dx, x > 0 \text{ which passes through the point}$$

$\left(1, 1 - \frac{4}{3} \log_e 2 \right)$, then the value of $y(16)$ is equal to

[17 Mar 2021 Shift 2]

Options:

$$\text{A. } 4 \left(\frac{31}{3} + \frac{8}{3} \log_e 3 \right)$$

$$B. \left(\frac{31}{3} + \frac{8}{3} \log_e 3 \right)$$

$$C. 4 \left(\frac{31}{3} - \frac{8}{3} \log_e 3 \right)$$

$$D. \left(\frac{31}{3} - \frac{8}{3} \log_e 3 \right)$$

Answer: C

Solution:

Given, $2(x^2 + x^{5/4})dy - y \cdot (x + x^{1/4})dx = 2x^{9/4}dx$, where $x > 0$

After rearranging, we get

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{x^{9/4}}{x^{5/4}(x^{3/4} + 1)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or function of x.

$\therefore$ Integrating factor (IF) = $e^{\int P dx}$

$$\begin{aligned} &= e^{\int -\frac{1}{2x} dx} = e^{-\frac{1}{2} \int \frac{1}{x} dx} \\ &= e^{-\frac{1}{2} \log x} = e^{\log(x)^{-1/2}} = \frac{1}{x^{1/2}} \end{aligned}$$

Its solution is

$$y \times (IF) = \int Q \times (IF) dx$$

$$\Rightarrow y \times \frac{1}{x^{1/2}} = \int \frac{x^{9/4}}{x^{5/4}(x^{3/4} + 1)} \times x^{-1/2} dx = \int \frac{(x)^{\frac{9}{4} - \frac{5}{4} - \frac{1}{2}}}{(x^{3/4} + 1)} dx$$

$$\Rightarrow y \times \frac{1}{x^{1/2}} = \int \frac{x^{1/2} dx}{(x^{3/4} + 1)} \dots (i)$$

Putting $x = z^4$

$$\Rightarrow dx = 4z^3 \cdot dz$$

RHS of Eq. (i) becomes,

$$\int \frac{z^2 \cdot 4z^3}{(z^3 + 1)} \cdot dz = 4 \int \frac{z^2(z^3 + 1 - 1)}{(z^3 + 1)} dz$$

$$= 4 \left[\int \frac{z^2(z^3 + 1)}{(z^3 + 1)} dz - \int \frac{z^2}{(z^3 + 1)} dz \right]$$

$$= 4 \left[\frac{z^3}{3} - \frac{1}{3} \cdot \int \frac{3z^2}{z^3 + 1} dz \right]$$

$$= 4 \left[\frac{z^3}{3} - \frac{1}{3} \cdot \log |z^3 + 1| \right]$$

$$\left(\because \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C \right)$$

$$= \frac{4z^3}{3} - \frac{4}{3} \log |z^3 + 1| + C$$

$$= \frac{4x^{3/4}}{3} - \frac{4}{3} \log |x^{3/4} + 1| + C \left(\begin{array}{l} \because x = z^4 \\ \therefore x^{3/4} = z^3 \end{array} \right)$$

$\therefore$ Eq. (i) becomes,

$$y \times x^{-1/2} = \frac{4x^{3/4}}{3} - \frac{4}{3} \log|x^{3/4} + 1| + C$$

Since, this passes through $\left(1, 1 - \frac{4}{3} \log_e 2\right)$

Then,

$$\left(1 - \frac{4}{3} \log_e 2\right) \times 1 = \frac{4 \times 1}{3} - \frac{4}{3} \log|1+1| + C$$

$$\Rightarrow C = 1 - \frac{4}{3} \Rightarrow C = \frac{-1}{3}$$

Hence,

$$y = \frac{4}{3} x^{5/4} - \frac{4}{3} \sqrt{x} \log x^{3/4} + 1 - \frac{\sqrt{x}}{3}$$

$\because x > 0$

$$\therefore x^{3/4} > 0 \Rightarrow x^{3/4} + 1 > 0 \text{ i.e., } x^{3/4} + 1 = x^{3/4} + 1$$

$$\therefore y = \frac{4}{3} x^{5/4} - \frac{4}{3} \sqrt{x} \log(x^{3/4} + 1) - \frac{\sqrt{x}}{3}$$

Now, putting $x = 16$, we get

$$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4 \log 9 - \frac{4}{3}$$

$$= \frac{124}{3} - \frac{32}{3} \log 3$$

$$= 4 \left(\frac{31}{3} - \frac{8}{3} \log 3 \right)$$

Question 159

If $y = y(x)$ is the solution of the differential equation,

$$\frac{dy}{dx} + 2y \tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0, \text{ then the maximum value of the function}$$

$y(x)$ over $\mathbb{R}$ is equal to

[16 Mar 2021 Shift 1]

Options:

A. 8

B. $\frac{1}{2}$

C. $-\frac{15}{4}$

D. $\frac{1}{8}$

Answer: D

Solution:

Given, $\frac{dy}{dx} + 2y \tan x = \sin x$

This differential equation is of the form $\frac{dy}{dx} + Py = Q$ where P and Q is function of x.
which is a linear differential equation.

Here, $P = 2 \tan x$ and $Q = \sin x$

The integrating factor of linear differential equation is $e^{\int P dx}$.

Here, $e^{\int 2 \tan x dx} = e^{\int \frac{2 \sin x}{\cos x} dx} = e^{-2 \log(\cos x)} = \sec^2 x$

Now, $\frac{dy}{dx} + 2y \tan x = \sin x$

On multiplying $\sec^2 x$ both the sides,

$$\sec^2 x \frac{dy}{dx} + 2y \sec^2 x \tan x = \sin x \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(y \sec^2 x) = \sin x \sec^2 x$$

$$\Rightarrow y \sec^2 x = \int \sin x \sec^2 x dx$$

$$\Rightarrow y \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx$$

Let $\cos x = t$

$$(-\sin x) dx = dt$$

$$\int \frac{-dt}{t^2} = \frac{1}{t} + c$$

$$\text{So, } y \sec^2 x = \frac{1}{\cos x} + c$$

$$y \sec^2 x = \sec x + c$$

Now, $x = \pi/3, y = 0$

$$0 = 2 + c$$

$$\Rightarrow c = -2$$

$$\text{So, } y \sec^2 x = \sec x - 2$$

$$y = \cos^2 x \left(\frac{1}{\cos x} - 2 \right) = -2 \cos^2 x + \cos x$$

$$\Rightarrow y = -2 \left(\cos^2 x - \frac{\cos x}{2} \right)$$

$$\Rightarrow y = -2 \left[\left(\cos^2 x - \frac{\cos x}{2} + \frac{1}{16} \right) - \frac{1}{16} \right]$$

$$\Rightarrow y = -2 \left[(\cos x - 1/4)^2 - \frac{1}{16} \right]$$

$$\Rightarrow y = \frac{1}{8} - 2(\cos x - 1/4)^2$$

$$\text{So, } y_{\min} = \frac{1}{8}$$

Question 160

If $y = y(x)$ is the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \sin x, 0 \leq x \leq \frac{\pi}{3}, \text{ with } y(0) = 0, \text{ then } y\left(\frac{\pi}{4}\right) \text{ is equal to}$$

[16 Mar 2021 Shift 2]

Options:

A. $\frac{1}{4} \log_e 2$

B. $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$

C. $\log_e 2$

D. $\frac{1}{2}\log_e 2$

Answer: B

Solution:

Given, $\frac{dy}{dx} + (\tan x)y = \sin x, x \in \left[0, \frac{\pi}{3}\right]$

which is a linear differential equation of the form of $\frac{dy}{dx} + Py = Q$

Here, $P = \tan x$

$\therefore IF = e^{\int P dx}$

$\Rightarrow e^{\int \tan x dx} = e^{\log(\sec x)} = \sec x$

Multiplying by $\sec x$ on both sides

$\frac{dy}{dx} + (\tan x)y = \sin x$

$\sec x \frac{dy}{dx} + (\tan x \sec x)y = \sin x \sec x$

$\Rightarrow \frac{d}{dx}(y \sec x) = \tan x \Rightarrow y \sec x = \int \tan x dx$

$\Rightarrow y \sec x = \log(\sec x) + c$

$y = \cos x \log(\sec x) + c \cdot \cos x$

$y(0) = 0$

$\Rightarrow 0 = 1 \cdot 0 + c \cdot 1 \Rightarrow c = 0$

$\therefore y = \cos x \cdot \log(\sec x)$

$\Rightarrow y\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) \cdot \log\left(\sec\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} \log(\sqrt{2})$

$= \frac{1}{2\sqrt{2}} \log 2$

Question 161

In a triangle ABC, if $|\vec{BC}| = 8, |\vec{CA}| = 7, |\vec{AB}| = 10$, then the projection of the vector $\vec{AB}$ on $\vec{AC}$ is equal to [18 Mar 2021 Shift 2]

Options:

A. $\frac{25}{4}$

B. $\frac{85}{14}$

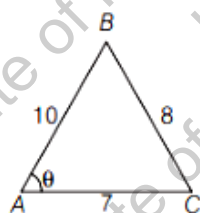
C. $\frac{127}{20}$

D. $\frac{115}{16}$

Answer: B

Solution:

Projection of AB on AC = AB cos θ = $10 \cos \theta$



$$= 10 \cdot \left(\frac{10^2 + 7^2 - 8^2}{2 \times 7 \times 10} \right) = \frac{85}{14} \left(\text{using } \cos \theta = \frac{c^2 + b^2 - a^2}{2bc} \right)$$

Question 162

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 1 + x e^{y-x}, \quad -\sqrt{2} < x < \sqrt{2}, \quad y(0) = 0$$

then, the minimum value of $y(x)$, $x \in (-\sqrt{2}, \sqrt{2})$ is equal to:

[25 Jul 2021 Shift 1]

Options:

A. $(2 - \sqrt{3}) - \log_e 2$

B. $(2 + \sqrt{3}) + \log_e 2$

C. $(1 + \sqrt{3}) - \log_e(\sqrt{3} - 1)$

D. $(1 - \sqrt{3}) - \log_e(\sqrt{3} - 1)$

Answer: D

Solution:

$$\frac{dy - dx}{e^{y-x}} = x dx$$

$$\Rightarrow \frac{dy - dx}{e^{y-x}} = x dx$$

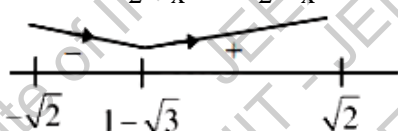
$$\Rightarrow -e^{x-y} = \frac{x^2}{2} + c$$

$$\text{At } x=0, y=0 \Rightarrow c=-1$$

$$\Rightarrow e^{x-y} = \frac{2-x^2}{2}$$

$$\Rightarrow y = x - \ln\left(\frac{2-x^2}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{2x}{2-x^2} = \frac{2+2x-x^2}{2-x^2}$$



So minimum value occurs at $x = 1 - \sqrt{3}$

$$y(1 - \sqrt{3}) = (1 - \sqrt{3}) - \ln\left(\frac{2 - (4 - 2\sqrt{3})}{2}\right)$$

$$= (1 - \sqrt{3}) - \ln(\sqrt{3} - 1)$$

Question 163

Let $y = y(x)$ be solution of the differential equation $\log_e\left(\frac{dy}{dx}\right) = 3x + 4y$, with

$y(0) = 0$ If $y\left(-\frac{2}{3}\log_e 2\right) = \alpha \log_e 2$, then the value of α is equal to:

[27 Jul 2021 Shift 1]

Options:

A. $-\frac{1}{4}$

B. $\frac{1}{4}$

C. 2

D. $-\frac{1}{2}$

Answer: A

Solution:

$$\begin{aligned}\frac{dy}{dx} &= e^{3x} \cdot e^{4y} \Rightarrow \int e^{-4y} dy = \int e^{3x} dx \\ \frac{e^{-4y}}{-4} &= e^{3x} \cdot 3 + C \Rightarrow -\frac{1}{4} - \frac{1}{3} = C \Rightarrow C = -\frac{7}{12} \\ \frac{e^{-4y}}{-4} &= \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow e^{-4y} = \frac{4e^{3x} - 7}{-3} \\ e^{4y} &= \frac{3}{7 - 4e^{3x}} \Rightarrow 4y = \ln\left(\frac{3}{7 - 4e^{3x}}\right) \\ 4y &= \ln\left(\frac{3}{6}\right) \text{ when } x = -\frac{2}{3} \ln 2 \\ y &= \frac{1}{4} \ln\left(\frac{1}{2}\right) = -\frac{1}{4} \ln 2\end{aligned}$$

Question 164

If $y = y(x)$, $y \in \left[0, \frac{\pi}{2}\right)$ is the solution of the differential equation

$\sec y \frac{dy}{dx} - \sin(x+y) - \sin(x-y) = 0$, with $y(0) = 0$ then $5y'\left(\frac{\pi}{2}\right)$ is equal to

[27 Jul 2021 Shift 1]

Answer: 2

Solution:

$$\sec y \frac{dy}{dx} = 2 \sin x \cos y$$

$$\sec^2 y dy = 2 \sin x dx$$

$$\tan y = -2 \cos x + c$$

$$c = 2$$

$$\tan y = -2 \cos x + 2 \Rightarrow \text{at } x = \frac{\pi}{2}$$

$$\tan y = 2$$

$$\sec^2 y \frac{dy}{dx} = 2 \sin x$$

$$5 \frac{dy}{dx} = 2$$

Question 165

Let $y = y(x)$ be the solution of the differential equation

$(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$ If $y(3) = 3$, then $y(4)$ is equal to :

[27 Jul 2021 Shift 2]

Options:

- A. 4
- B. 12
- C. 8
- D. 16

Answer: B

Solution:

$$\begin{aligned}(x - x^3)dy &= (y + yx^2 - 3x^4)dx \\ \Rightarrow xdy - ydx &= (yx^2 - 3x^4)dx + x^3dy \\ \Rightarrow \frac{xdy - ydx}{x^2} &= (ydx + xdy) - 3x^2dx\end{aligned}$$

$$\Rightarrow d\left(\frac{y}{x}\right) = d(xy) - d(x^3)$$

$$\text{Integrate } \Rightarrow \frac{y}{x} = xy - x^3 + c$$

$$\text{given } f(3) = 3$$

$$\Rightarrow \frac{3}{3} = 3 \times 3 - 3^3 + c$$

$$\Rightarrow c = 19$$

$$\therefore \frac{y}{x} = xy - x^3 + 19$$

$$\text{at } x = 4, \frac{y}{4} = 4y - 64 + 19$$

$$15y = 4 \times 45$$

$$\Rightarrow y = 12$$

Question 166

Let $y = y(x)$ be the solution of the differential equation

$dy = e^{ax+y} dx$; $a \in \mathbb{N}$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e \left(\frac{1}{2}\right)$, then the value of a is equal to _____.

[27 Jul 2021 Shift 2]

Answer: 2

Solution:

$$\int e^{-y} dy = \int e^{\alpha x} dx$$

$$\Rightarrow e^{-y} = \frac{e^{\alpha x}}{\alpha} + C \dots\dots(i)$$

$$\text{Put } (x, y) = (\ln 2, \ln 2)$$

$$\frac{-1}{2} = \frac{2^{\alpha}}{\alpha} + C \dots\dots(ii)$$

$$\text{Put } (x, y) = (0, -\ln 2) \text{ in (i)}$$

$$-2 = \frac{1}{\alpha} + C \dots\dots(iii)$$

$$(ii) - (iii)$$

$$\frac{2^{\alpha} - 1}{\alpha} = \frac{3}{2}$$

$$\Rightarrow \alpha = 2 \text{ (as } \alpha \in \mathbb{N} \text{)}$$

Question 167

Let $y = y(x)$ be the solution of the differential equation

$x dy = (y + x^3 \cos x) dx$ with $y(\pi) = 0$, then $y\left(\frac{\pi}{2}\right)$ is equal to:

[25 Jul 2021 Shift 2]

Options:

A. $\frac{\pi^2}{4} + \frac{\pi}{2}$

B. $\frac{\pi^2}{2} + \frac{\pi}{4}$

C. $\frac{\pi^2}{2} - \frac{\pi}{4}$

D. $\frac{\pi^2}{4} - \frac{\pi}{2}$

Answer: A

Solution:

$$x dy = (y + x^3 \cos x) dx$$

$$x dy = y dx + x^3 \cos x dx$$

$$\frac{x dy - y dx}{x^2} = \frac{x^3 \cos x dx}{x^2}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = \int x \cos x dx$$

$$\Rightarrow \frac{y}{x} = x \sin x - \int 1 \cdot \sin x dx$$

$$\frac{y}{x} = x \sin x + \cos x + C$$

$$\Rightarrow 0 = -1 + C \Rightarrow C = 1, x = \pi, y = 0$$

$$\text{so } \frac{y}{x} = x \sin x + \cos x + 1$$

$$y = x^2 \sin x + x \cos x + x \quad x = \frac{\pi}{2}$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + \frac{\pi}{2}$$

Question 168

Let a curve $y = f(x)$ pass through the point $(2, (\log_e 2)^2)$ and have slope $\frac{2y}{x \log_e x}$ for all positive real value of x . Then the value of $f(e)$ is equal to _____.

[25 Jul 2021 Shift 2]

Answer: 1

Solution:

$$y' = \frac{2y}{x \ln x}$$

$$\Rightarrow \frac{dy}{y} = \frac{2dx}{x \ln x}$$

$$\Rightarrow 1 \ln |y| = 2 \ln |\ln x| + C$$

$$\text{put } x = 2, y = (\ln 2)^2$$

$$\Rightarrow C = 0$$

$$\Rightarrow y = (\ln x)^2$$

$$\Rightarrow f(e) = 1$$

Question 169

Let $y = y(x)$ be the solution of the differential equation

$\left((x+2)e^{\left(\frac{y+1}{x+2}\right)} + (y+1) \right) dx = (x+2) dy, y(1) = 1$. If the domain of $y = y(x)$ is an open interval (α, β) , then $|\alpha + \beta|$ is equal to _____.

[22 Jul 2021 Shift 2]

Answer: 4

Solution:

$$y + 1 = Y \Rightarrow dy = dY$$

$$x + 2 = X \Rightarrow dx = dX$$

$$\Rightarrow \left(X e^{\frac{Y}{X}} + Y \right) dX = X dY$$
$$\Rightarrow X dY - Y dX = X e^{Y/X} dX$$

$$\Rightarrow d \left(\frac{Y}{X} \right) e^{-\frac{Y}{X}} = \frac{dX}{X}$$

$$-e^{-Y/X} = 1 \mid X \mid + c$$

$$(3, 2) \rightarrow -e^{-\frac{2}{3}} = 1 \mid 3 \mid + c$$

$$-e^{-\frac{Y}{X}} = 1 \ln \mid X \mid - e^{-\frac{2}{3}} - 1 \ln 3$$

$$e^{-\frac{Y}{X}} = e^{2/3} + 1 \ln 3 - 1 \ln \mid X \mid > 0$$

$$\ln \mid X \mid < (e^{2/3} + 1 \ln 3)$$

$$\text{Let } \lambda = (e^{2/3} + 1 \ln 3)$$

$$\mid x + 2 \mid < e^{\lambda}$$

$$-e^{\lambda} < x + 2 < e^{\lambda}$$

$$-e^{\lambda} - 2 < x < e^{\lambda} - 2$$

$$\alpha + \beta = -4 \Rightarrow \mid \alpha + \beta \mid = 4$$

Although $x = -2$ should be excluded from domain but according to the given problem it will be the most appropriate solution.

Question 170

Let $y = y(x)$ be the solution of the differential equation

$$x \tan \left(\frac{y}{x} \right) dy = \left(y \tan \left(\frac{y}{x} \right) - x \right) dx - 1 \leq x \leq 1, y \left(\frac{1}{2} \right) = \frac{\pi}{6}. \text{ Then the area of the}$$

region bounded by the curves $x = 0$, $x = \frac{1}{\sqrt{2}}$ and $y = y(x)$ in the upper half

plane is:

[20 Jul 2021 Shift 1]

Options:

A. $\frac{1}{8}(\pi - 1)$

B. $\frac{1}{12}(\pi - 3)$

C. $\frac{1}{4}(\pi - 2)$

D. $\frac{1}{6}(\pi - 1)$

Answer: A

Solution:

We have

$$\frac{dy}{dx} = \frac{x \left(\frac{y}{x} \cdot \tan \frac{y}{x} - 1 \right)}{x \tan \frac{y}{x}}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} - \cot \left(\frac{y}{x} \right)$$

$$\text{Put } \frac{y}{x} = v$$

$$\Rightarrow y = vx$$

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Now, we get } v + x \frac{dv}{dx} = v - \cot(v)$$

$$\Rightarrow \int (\tan v) dv = - \int \frac{dx}{x}$$

$$\therefore \ln \left| \sec \left(\frac{y}{x} \right) \right| = - \ln |x| + C$$

$$\text{As } \left(\frac{1}{2} \right) = \left(\frac{y}{x} \right) \Rightarrow C = 0$$

$$\therefore \sec \left(\frac{y}{x} \right) = \frac{1}{x}$$

$$\Rightarrow \cos \left(\frac{y}{x} \right) = x$$

$$\therefore y = x \cos^{-1}(x)$$

So, required bounded area

$$= \int_0^{1/\sqrt{2}} x (\cos^{-1} x) dx = \left(\frac{\pi - 1}{8} \right)$$

$\therefore$ option (1) is correct.

Question 171

Let $y = y(x)$ be the solution of the differential equation

$$e^x \sqrt{1 - y^2} dx + \left(\frac{y}{x} \right) dy = 0, y(1) = -1. \text{ Then the value of } (y(3))^2 \text{ is equal to:}$$

[20 Jul 2021 Shift 1]

Options:

A. $1 - 4e^3$

B. $1 - 4e^6$

C. $1 + 4e^3$

D. $1 + 4e^6$

Answer: B

Solution:

$$e^x \sqrt{1-y^2} dx + \frac{y}{x} dy = 0$$

$$\Rightarrow e^x \sqrt{1-y^2} dx + \frac{-y}{x} dy$$

$$\Rightarrow \int \frac{-y}{\sqrt{1-y^2}} dy = \int e^x x dx$$

$$\Rightarrow \sqrt{1-y^2} = e^x(x-1) + c$$

Given : At $x = 1, y = -1$

$$\Rightarrow 0 = 0 + c \Rightarrow c = 0$$

$$\therefore \sqrt{1-y^2} = e^x(x-1)$$

$$\text{At } x = 3, 1 - y^2 = (e^3)^2 \Rightarrow y^2 = 1 - 4e^6$$

Question 172

Let a curve $y = y(x)$ be given by the solution of the differential equation

$$\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right) dx = \sqrt{e^{2x}-1} dy$$

If it intersects y -axis at $y = -1$, and the intersection point of the curve with x -axis is $(\alpha, 0)$, then e^α is equal to _____.

[20 Jul 2021 Shift 2]

Answer: 2

Solution:

$$\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right) dx = \sqrt{e^{2x}-1} dy$$

Put $\cos^{-1}(e^{-x}) = \theta, \theta \in [0, \pi]$

$$\cos \theta = e^{-x} \Rightarrow 2\cos^2 \theta - 1 = e^{-x}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{e^{-x}+1}{2}} = \sqrt{\frac{e^x+1}{2e^x}}$$

$$\sqrt{\frac{e^x+1}{2e^x}} dx = \sqrt{e^{2x}-1} dy$$

$$\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{e^x} \sqrt{e^x+1}} = \int dy$$

Put $e^x = t, \frac{dt}{dx} = e^x$

$$\frac{1}{\sqrt{2}} \int \frac{dt}{e^x \sqrt{e^x} \sqrt{e^x - 1}} = \int dy$$

$$\int \frac{dt}{t \sqrt{t^2 - 1}} = \sqrt{2} y$$

$$\text{Put } t = \frac{1}{z}, \frac{dt}{dz} = -\frac{1}{z^2}$$

$$\int \frac{-\frac{dz}{z^2}}{\frac{1}{z} \sqrt{\frac{1}{z^2} - 1}} = \frac{\sqrt{2}}{y}$$

$$-\int \frac{dz}{\sqrt{1 - z}} = \sqrt{2} y$$

$$\frac{-2(1 - z)^{1/2}}{-1} = \sqrt{2} y + c$$

$$2\left(1 - \frac{1}{t}\right)^{1/2} = \sqrt{2} y + c$$

$$2(1 - e^{-x})^{1/2} = \sqrt{2} y + c \xrightarrow{(0, -1)} \Rightarrow c = \sqrt{2}$$

$$2(1 - e^{-x})^{1/2} = \sqrt{2}(y + 1), \text{ passes through } (\alpha, 0)$$

$$2(1 - e^{-\alpha})^{1/2} = \sqrt{2}$$

$$\sqrt{1 - e^{-\alpha}} = \frac{1}{\sqrt{2}} \Rightarrow 1 - e^{-\alpha} = \frac{1}{2}$$

$$e^{-\alpha} = \frac{1}{2} \Rightarrow e^{\alpha} = 2$$

Question 173

Let $y = y(x)$ be solution of the following differential

$$\text{equation } e^{y \frac{dy}{dx}} - 2e^y \sin x + \sin x \cos^2 x = 0, y\left(\frac{\pi}{2}\right) = 0 \text{ If } y(0) = \log_e(\alpha + \beta e^{-2}),$$

then $4(\alpha + \beta)$ is equal to _____.

[25 Jul 2021 Shift 1]

Answer: 4

Solution:

$$\text{Let } e^y = t$$

$$\Rightarrow \frac{dt}{dx} - (2 \sin x)t = -\sin x \cos^2 x$$

$$\text{I.F.} = e^{2 \cos x}$$

$$\Rightarrow t \cdot e^{2 \cos x} = \int e^{2 \cos x} \cdot (-\sin x \cos^2 x) dx$$

$$\Rightarrow e^y \cdot e^{2 \cos x} = \int e^{2z} \cdot z^2 dz, z = e^{2 \cos x}$$

$$\text{at } x = \frac{\pi}{2}, y = 0 \Rightarrow C = \frac{3}{4}$$

$$\Rightarrow e^y = \frac{1}{2} \cos^2 x - \frac{1}{2} \cos x + \frac{1}{4} + \frac{3}{4} \cdot e^{-2 \cos x}$$

$$\Rightarrow y = \log \left[\frac{\cos^2 x}{2} - \frac{\cos x}{2} + \frac{1}{4} + \frac{3}{4} e^{-2 \cos x} \right]$$

Put $x = 0$

$$\Rightarrow y = \log \left[\frac{1}{4} + \frac{3}{4} e^{-2} \right] \Rightarrow \alpha = \frac{1}{4}, \beta = \frac{3}{4}$$

Question 174

Let $y = y(x)$ be the solution of the differential equation

$\operatorname{cosec}^2 x \, dy + 2 \, dx = (1 + y \cos 2x) \operatorname{cosec}^2 x \, dx$, with $y\left(\frac{\pi}{4}\right) = 0$. Then, the value of $(y(0) + 1)^2$ is equal to:
[22 Jul 2021 Shift 2]

Options:

A. $e^{1/2}$

B. $e^{-1/2}$

C. e^{-1}

D. e

Answer: C

Solution:

$$\frac{dy}{dx} + 2 \sin^2 x = 1 + y \cos 2x$$

$$\Rightarrow \frac{dy}{dx} + (-\cos 2x)y = \cos 2x$$

$$\text{I.F.} = e^{\int -\cos 2x \, dx} = e^{-\frac{\sin 2x}{2}}$$

Solution of D.E.

$$y \left(e^{-\frac{\sin 2x}{2}} \right) = \int (\cos 2x) \left(e^{-\frac{\sin 2x}{2}} \right) dx + c$$

$$\Rightarrow y \left(e^{-\frac{\sin 2x}{2}} \right) = -e^{-\frac{\sin 2x}{2}} + c$$

Given

$$y\left(\frac{\pi}{4}\right) = 0$$

$$\Rightarrow 0 = -e^{-\frac{1}{2}} + c \Rightarrow c = e^{-\frac{1}{2}}$$

$$\Rightarrow y \left(e^{-\frac{\sin 2x}{2}} \right) = -e^{-\frac{\sin 2x}{2}} + e^{-1/2}$$

at $x = 0$

$$y = -1 + e^{-\frac{1}{2}}$$

$$\Rightarrow y(0) = -1 + e^{-\frac{1}{2}} \Rightarrow (y(0) + 1)^2 = e^{-1}$$

Question 175

Let $y = y(x)$ satisfies the equation $\frac{dy}{dx} - |A| = 0$, for all $x > 0$, where

$$A = \begin{bmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x} \end{bmatrix}. \text{ If } y(\pi) = \pi + 2, \text{ then the value of } y\left(\frac{\pi}{2}\right) \text{ is :}$$

[20 Jul 2021 Shift 2]

Options:

A. $\frac{\pi}{2} + \frac{4}{\pi}$

B. $\frac{\pi}{2} - \frac{1}{\pi}$

C. $\frac{3\pi}{2} - \frac{1}{\pi}$

D. $\frac{\pi}{2} - \frac{4}{\pi}$

Answer: A

Solution:

$$|A| = -\frac{y}{x} + 2 \sin x + 2$$

$$\frac{dy}{dx} = |A|$$

$$\frac{dy}{dx} = -\frac{y}{x} + 2 \sin x + 2$$

$$\frac{dy}{dx} + \frac{y}{x} = 2 \sin x + 2$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = x$$

$$\Rightarrow yx = \int x(2 \sin x + 2) dx$$

$$yx = x^2 - 2x \cos x + 2 \sin x + c \dots\dots(i)$$

$$\text{Now } x = \pi, y = \pi + 2$$

$$\text{Use in (i) } c = 0$$

Now (i) becomes

$$yx = x^2 - 2x \cos x + 2 \sin x$$

$$\text{put } x = \pi/2$$

$$\frac{\pi}{2}y = \left(\frac{\pi}{2}\right)^2 - 2 \cdot \frac{\pi}{2} \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2}$$

$$\frac{\pi}{2}y = \frac{\pi^2}{4} + 2$$

Question 176

Let $\vec{a} = \hat{i} - \alpha\hat{j} + \beta\hat{k}$, $\vec{b} = 3\hat{i} + \beta\hat{j} - \alpha\hat{k}$ and $\vec{c} = -\alpha\hat{i} - 2\hat{j} + \hat{k}$, where α and β are integers.

If $\vec{a} \cdot \vec{b} = -1$ and $\vec{b} \cdot \vec{c} = 10$, then $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is equal to ____.

[27 Jul 2021 Shift 2]

Answer: 9

Solution:

$$\vec{a} = (1, -\alpha, \beta)$$

$$\vec{b} = (3, \beta, -\alpha)$$

$$\vec{c} = (-\alpha, -2, 1); \alpha, \beta \in \mathbb{I}$$

$$\vec{a} \cdot \vec{b} = -1 \Rightarrow 3 - \alpha\beta - \alpha\beta = -1$$

$$\Rightarrow \alpha\beta = 2$$

$$1 \quad 2$$

$$2 \quad 1$$

$$-1 \quad -2$$

$$-2 \quad -1$$

$$\vec{b} \cdot \vec{c} = 10$$

$$\Rightarrow -3\alpha - 2\beta - \alpha = 10$$

$$\Rightarrow 2\alpha + \beta + 5 = 0$$

$$\therefore \alpha = -2; \beta = -1$$

$$\left[\begin{array}{ccc} \vec{a} & \vec{b} & \vec{c} \end{array} \right] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= 1(-1+4) - 2(3-4) - 1(-6+2)$$

$$= 3 + 2 + 4 = 9$$

Question 177

A differential equation representing the family of parabolas with axis parallel to Y-axis and whose length of latus rectum is the distance of the

point $(2, -3)$ from the line $3x + 4y = 5$, is given by
[27 Aug 2021 Shift 2]

Options:

A. $10 \frac{d^2 y}{dx^2} = 11$

B. $11 \frac{d^2 x}{dy^2} + 10$

C. $10 \frac{d^2 x}{dy^2} = 11$

D. $11 \frac{d^2 y}{dx^2} = 10$

Answer: D

Solution:

Let (h, k) be the vertex of parabola. Then, equation of parabola parallel to Y-axis is

$$(x - h)^2 = 4a(y - k) \dots (i)$$

Also,

Length of latusrectum = Distance of point $(2, -3)$ from the line $3x + 4y = 5$

$$\Rightarrow 4a = \frac{|6 - 12 - 5|}{\sqrt{3^2 + 4^2}}$$

$$\Rightarrow 4a = \frac{11}{5}$$

$\therefore$ From Eq. (i),

$$(x - h)^2 = \frac{11}{5}(y - k)$$

Differentiating w.r.t. x , we get

$$2(x - h) = \frac{11}{5} \frac{dy}{dx}$$

Again, differentiating w.r.t. x

$$2 = \frac{11}{5} \frac{d^2 y}{dx^2}$$

$$\Rightarrow 11 \frac{d^2 y}{dx^2} = 10$$

Question 178

Let f be a non-negative function in $[0, 1]$ and twice differentiable in $(0, 1)$. If

$$\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt, \quad 0 \leq x \leq 1 \text{ and } f(0) = 0, \text{ then } \lim_{x \rightarrow 0} \frac{1}{x^2} \int_0^x f(t) dt$$

[31 Aug 2021 Shift 1]

Options:

- A. equals 0
- B. equals 1
- C. does not exist
- D. equals $\frac{1}{2}$

Answer: D

Solution:

$$\int_0^x [1 - (f(t))^2]^{\frac{1}{2}} dt = \int_0^x f(t) dt$$

Differentiating on both sides,

$$\sqrt{1 - [f(x)]^2} = [f(x)]$$

$$\Rightarrow 1 - [f(x)]^2 = [f(x)]^2$$

$$\Rightarrow 1 - [f(x)]^2 = [f(x)]^2$$

$$\Rightarrow f(x) = \sqrt{1 - [f(x)]^2}$$

$$\Rightarrow \int \frac{f(x) dx}{\sqrt{1 - [f(x)]^2}} = \int 1 dx$$

$$\Rightarrow \sin^{-1} f(x) = x + C$$

$$\because f(0) = 0$$

$$C = 0$$

$$f(x) = \sin x$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\int_0^x f(t) dt}{x^2} = \lim_{x \rightarrow 0} \frac{\int_0^x \sin x dt}{x^2} \quad [\text{applying L'Hopital Rule}]$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin x}{2x} \right) = \frac{1}{2}$$

Question 179

If $\frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y}$, $y(0) = 1$, then $y(1)$ is equal to

[31 Aug 2021 Shift 1]

Options:

- A. $\log_2(2 + e)$
- B. $\log_2(1 + e)$
- C. $\log_2(2e)$
- D. $\log_2(1 + e^2)$

Answer: B

Solution:

$$\frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y} = \frac{2^x(2^y - 1)}{2^y}$$

$$\int \frac{2^y}{2^y - 1} dy = \int 2^x dx$$

$$\frac{\ln(2^y - 1)}{\ln 2} = \frac{2^x}{\ln 2} + C$$

$$\text{As, } y(0) = 1$$

$$\Rightarrow 0 = \frac{1}{\log 2} + C$$

$$\text{For } y(1), \ln_2(2^y - 1) = 2^1 - 1$$

$$\Rightarrow 2^y - 1 = e$$

$$y = \log_2(e + 1)$$

Question 180

If $\frac{dy}{dx} = \frac{2^x y + 2^y \cdot 2^x}{2^x + 2^{x+y} \log_e 2}$, $y(0) = 0$, then for $y = 1$, the value of x lies in the interval

[31 Aug 2021 Shift 2]

Options:

A. (1, 2)

B. $\left(\frac{1}{2}, 1\right]$

C. (2, 3)

D. $\left(0, \frac{1}{2}\right]$

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{2^x y + 2^y \cdot 2^x}{2^x + 2^{x+y} \log_e 2}, y(0) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + 2^y}{1 + 2^y \log_e 2}$$

$$\Rightarrow \int \frac{1 + 2^y \cdot \log_e 2 dy}{y + 2^y} = \int dx$$

$$\Rightarrow \log |y + 2^y| = x + C$$

$$\because y(0) = 0$$

$$\Rightarrow C = 0$$

$$\therefore \ln |y + 2^y| = x$$

$$\text{For } y = 1$$

$$x = \ln |1 + 2| = \ln 3$$

$$\Rightarrow x \in (1, 2)$$

Question 181

If $y \frac{dy}{dx} = x \left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)} \right]$, $x > 0$, $\phi > 0$, and $y(1) = -1$, then $\phi\left(\frac{y^2}{4}\right)$ is equal

to

[31 Aug 2021 Shift 2]

Options:

A. $4 \phi(2)$

B. $4 \phi(1)$

C. $2 \phi(1)$

D. $\phi(1)$

Answer: B

Solution:

Given,

$$y \frac{dy}{dx} = x \left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)} \right] \dots (i)$$

$$\text{Let } t = \frac{y}{x}$$

$$\Rightarrow y = xt$$

$$\Rightarrow \frac{dy}{dx} = t + x \frac{dt}{dx}$$

$\therefore$ Eq. (i) becomes

$$t \left(t + x \frac{dt}{dx} \right) = \left(t^2 + \frac{\phi(t^2)}{\phi'(t^2)} \right)$$

$$\Rightarrow xt \frac{dt}{dx} = \frac{\phi(t^2)}{\phi'(t^2)}$$

$$\Rightarrow \frac{t \phi'(t^2)}{\phi(t^2)} dt = \frac{dx}{x}$$

Integrating both sides

$$\int \frac{t\phi'(t^2)}{\phi t^2} dt = \int \frac{dx}{x}$$

$$\text{Let } \phi(t^2) = u$$

$$\Rightarrow t\phi'(t^2)dt = \frac{du}{2}$$

$$\therefore \frac{1}{2} \int \frac{du}{u} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \ln u = \ln x + C$$

$$\Rightarrow \frac{1}{2} \ln \phi(t^2) = \ln x + C$$

$$\Rightarrow \frac{1}{2} \ln \left(\phi \left(\frac{y^2}{x^2} \right) \right) = \ln x + C$$

$$\text{If } x = 1, y = -1, \text{ then } C = \frac{1}{2} \ln(\phi(1))$$

$$\therefore \frac{1}{2} \ln \left(\phi \left(\frac{y^2}{x^2} \right) \right) = \ln x + 2 \ln(\phi(1))$$

$$\text{If } x = 2, \text{ then } \ln \left(\phi \left(\frac{y^2}{4} \right) \right) = \ln 4 + \ln[\phi(1)]$$

$$\text{Or } \phi \left(\frac{y^2}{4} \right) = 4\phi(1)$$

Question 182

Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 2(y + 2 \sin x - 5)x - 2 \cos x \text{ such that, } y(0) = 7.$$

Then $y(\pi)$ is equal to

[27 Aug 2021 Shift 1]

Options:

A. $2e^{\pi^2} + 5$

B. $e^{\pi^2} + 5$

C. $3e^{\pi^2} + 5$

D. $7e^{\pi^2} + 5$

Answer: A

Solution:

$$\text{Given, } \frac{dy}{dx} = 2(y + 2 \sin x - 5)x - 2 \cos x, y(0) = 7$$

$$\Rightarrow \frac{dy}{dx} + 2 \cos x = 2(y + 2 \sin x - 5)x \dots (i)$$

$$\text{Let } y + 2 \sin x - 5 = t$$

$$\Rightarrow \frac{dy}{dx} + 2 \cos x = \frac{dt}{dx}$$

Then, Eq. (i) becomes

$$\frac{dt}{dx} = 2tx$$

$$\Rightarrow \frac{dt}{t} = 2x dx$$

On integrating

$$\ln t = x^2 + C$$

$$\Rightarrow \ln(y + 2 \sin x - 5) = x^2 + C \dots (ii)$$

$$\therefore y(0) = 7$$

$$\Rightarrow \ln(7 + 0 - 5) = 0 + C$$

$$\Rightarrow C = \ln 2$$

$\therefore$ From Eq. (ii),

$$\ln(y + 2 \sin x - 5) = x^2 + \ln 2$$

Now, at $x = \pi$

$$\ln(y(\pi) + 2 \sin \pi - 5) = \pi^2 + \ln 2$$

$$\Rightarrow \ln(y(\pi) - 5) = \pi^2 + \ln 2$$

$$\Rightarrow y(\pi) - 5 = e^{\pi^2 + \ln 2}$$

$$\Rightarrow y(\pi) = 2e^{\pi^2} + 5$$

Question 183

Let $y(x)$ be the solution of the differential equation

$2x^2 dy + (e^y - 2x)dx = 0$, $x > 0$. If $y(e) = 1$, then $y(1)$ is equal to

[26 Aug 2021 Shift 2]

Options:

A. 0

B. 2

C. $\log_e 2$

D. $\log_e(2e)$

Answer: C

Solution:

We have, $2x^2 dy + (e^y - 2x)dx = 0$

$$\frac{dy}{dx} + \frac{e^y - 2x}{2x^2} = 0$$

$$\frac{dy}{dx} + \frac{e^y}{2x^2} - \frac{1}{x} = 0$$

$$e^{-y} \frac{dy}{dx} - \frac{e^{-y}}{x} = -\frac{1}{2x^2} \dots (i)$$

$$e^{-y} = t \dots (ii)$$

$$-e^{-y} dy = dt$$

$$dy = -\left(\frac{dt}{t}\right) \dots (iii)$$

$$\frac{-dt}{dx} - \frac{t}{x} = -\frac{1}{2x^2} \text{ [From Eq. (i)]}$$

$$xdt + tdx = \frac{dx}{2x}$$

$$\int d(xt) = \int \frac{dx}{2x}$$

$$xt = \frac{1}{2} \log(x) + \frac{C}{2}$$

$$2xe^{-y} = \log x + c$$

$$\text{When } x = e, y = 1$$

$$2e \cdot e^{-1} = \log e + c$$

$$c = 1$$

$$\therefore 2xe^{-y} = \log x + 1$$

$$\text{When } x = 1,$$

$$e^{-y} = 0 + 1$$

$$e^y = 2$$

$$\Rightarrow y = \log_e 2$$

Question 184

Let us consider a curve, $y = f(x)$ passing through the point $(-2, 2)$ and the slope of the tangent to the curve at any point $(x, f(x))$ is given by $f(x) + xf'(x) = x^2$ Then
[27 Aug 2021 Shift 1]

Options:

A. $x^2 + 2xf(x) - 12 = 0$

B. $x^3 + 2xf(x) + 12 = 0$

C. $x^3 - 3xf(x) - 4 = 0$

D. $x^2 + 2xf(x) + 4 = 0$

Answer: C

Solution:

$$\text{Given, } f(x) + xf'(x) = x^2$$

$$\Rightarrow f'(x) + \frac{f(x)}{x} = x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = x \left[\because y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \right]$$

This is linear differential equation.

$\therefore$ Integrating factor IF = $e^{\int \frac{1}{x} dx} = x$

Solution, $y \cdot x = \int x \cdot x dx + C$

$$\text{or } xy = \frac{x^3}{3} + C$$

$\therefore$ It passes through $(-2, 2)$.

$$\therefore -2 \cdot 2 = \frac{(-2)^3}{3} + C$$

$$C = -\frac{4}{3}$$

$$\text{Hence, } xf(x) = \frac{x^3}{3} - \frac{4}{3}$$

$$\text{or } x^3 - 3f(x) \cdot x - 4 = 0$$

Question 185

If the solution curve of the differential equation $(2x - 10y^3)dy + ydx = 0$, passes through the points $(0, 1)$ and $(2, \beta)$, then β is a root of the equation [27 Aug 2021 Shift 2]

Options:

A. $y^5 - 2y - 2 = 0$

B. $2y^5 - 2y - 1 = 0$

C. $2y^5 - y^2 - 2 = 0$

D. $y^5 - y^2 - 1 = 0$

Answer: D

Solution:

Given, differential equation

$$(2x - 10y^3)dy + ydx = 0$$

$$\Rightarrow \frac{dx}{y} + \frac{2x}{y} = 10y^2 \dots (i)$$

This is Linear differential equation

$$\text{Integrating factor IF} = e^{\int \frac{2}{y} dy} = y^2$$

Solution of differential Eq. (i),

$$x \cdot y^2 = \int 10y^2 \cdot y^2 dy + C$$

$$\Rightarrow xy^2 = 2y^5 + C \dots (ii)$$

Solution Eq. (ii) passes through $(0, 1)$

$$\Rightarrow 0 \cdot 1^2 = 2 \cdot 1^5 + C$$

$$\Rightarrow C = -2$$

$\therefore$ Solution of Eq. (i) is

$$xy^2 = 2y^5 - 2$$

Now, this equation passes through $(2, \beta)$.

$$\begin{aligned}\therefore 2 \cdot \beta^2 &= 2\beta^5 - 2 \\ \Rightarrow \beta^5 - \beta^2 - 1 &= 0 \\ \Rightarrow \beta &\text{ is root of the equation } y^5 - y^2 - 1 = 0\end{aligned}$$

Question 186

Let $y = y(x)$ be a solution curve of the differential equation

$$(y + 1)\tan^2 x \, dx + \tan x \, dy + y \, dx = 0, \quad x \in \left(0, \frac{\pi}{2}\right). \text{ If } \lim_{x \rightarrow 0^+} xy(x) = 1, \text{ then the}$$

value of $y\left(\frac{\pi}{4}\right)$ is

[26 Aug 2021 Shift 1]

Options:

- A. $-\frac{\pi}{4}$
- B. $\frac{\pi}{4} - 1$
- C. $\frac{\pi}{4} + 1$
- D. $\frac{\pi}{4}$

Answer: D

Solution:

We have, $(y + 1)\tan^2 x \, dx + \tan x \, dy + y \, dx = 0$

$$\Rightarrow [(y + 1)\tan^2 x + y] \, dx + \tan x \, dy = 0$$

$$\Rightarrow \frac{dy}{dx} + (y + 1)\tan x + \frac{y}{\tan x} = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{y\tan^2 x + \tan^2 x + y}{\tan x} = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{y\sec^2 x}{\tan x} + \tan^2 x = 0$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{\sec^2 x}{\tan x}\right)y = -\tan x$$

This is a linear differential equation

$$\therefore \text{IF} = e^{\int \frac{\sec^2 x}{\tan x} \, dx} = e^{\ln(\tan x)} = \tan x$$

So, solution is given by

$$(y \tan x) = \int -\tan^2 x \, dx = \int (1 - \sec^2 x) \, dx = x - \tan x + C$$

$$y = x \cot x - 1 + C \cot x$$

$$\text{Now, } \lim_{x \rightarrow 0^+} x \cdot y = 1$$

$$\Rightarrow \lim_{x \rightarrow 0^+} (x^2 \cot x - x + Cx \cot x) = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(x \cdot \frac{x}{\tan x} - x + \frac{Cx}{\tan x} \right) = 1$$

$$\Rightarrow 0 - 0 + C = 1$$

$$\Rightarrow C = 1$$

$$\therefore y = x \cot x - 1 + \cot x$$

$$\text{Now, } x = \frac{\pi}{4}$$

$$y = \frac{\pi}{4} - 1 + 1 = \frac{\pi}{4}$$

Question 187

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors mutually perpendicular to each other and have same magnitude. If a vector $\mathbf{r}$ satisfies.

$\mathbf{a} \times \{(\mathbf{r} - \mathbf{b}) \times \mathbf{a}\} + \mathbf{b} \times \{(\mathbf{r} - \mathbf{c}) \times \mathbf{b}\} + \mathbf{c} \times \{(\mathbf{r} - \mathbf{a}) \times \mathbf{c}\} = \mathbf{0}$, then $\mathbf{r}$ is equal to
[31 Aug 2021 Shift 2]

Options:

A. $\frac{1}{3} (\mathbf{a} + \mathbf{b} + \mathbf{c})$

B. $\frac{1}{3} (2\mathbf{a} + \mathbf{b} - \mathbf{c})$

C. $\frac{1}{2} (\mathbf{a} + \mathbf{b} + \mathbf{c})$

D. $\frac{1}{2} (\mathbf{a} + \mathbf{b} + 2\mathbf{c})$

Answer: C

Solution:

$$\mathbf{a} \times [(\mathbf{r} - \mathbf{b}) \times \mathbf{a}] + \mathbf{b} \times [(\mathbf{r} - \mathbf{c}) \times \mathbf{b}] + \mathbf{c} \times [(\mathbf{r} - \mathbf{a}) \times \mathbf{c}] = \mathbf{0}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{a} (\mathbf{r} - \mathbf{b}) - (\mathbf{a} \cdot (\mathbf{r} - \mathbf{b})) \mathbf{a} + \mathbf{b} \cdot \mathbf{b} (\mathbf{r} - \mathbf{c}) - (\mathbf{b} \cdot (\mathbf{r} - \mathbf{c})) \mathbf{b} + \mathbf{c} \cdot \mathbf{c} (\mathbf{r} - \mathbf{a}) - (\mathbf{c} \cdot (\mathbf{r} - \mathbf{a})) \mathbf{c} = \mathbf{0}$$

$$\Rightarrow |\mathbf{a}|^2 (\mathbf{r} - \mathbf{b}) - (\mathbf{r} \cdot \mathbf{a}) \mathbf{a} + |\mathbf{b}|^2 (\mathbf{r} - \mathbf{c}) - (\mathbf{r} \cdot \mathbf{b}) \mathbf{b} + |\mathbf{c}|^2 (\mathbf{r} - \mathbf{a}) - (\mathbf{r} \cdot \mathbf{c}) \mathbf{c} = \mathbf{0} \quad [\because \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ are mutually perpendicular; } \therefore \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0]$$

$$\Rightarrow |\mathbf{a}|^2 [3\mathbf{r} - (\mathbf{a} + \mathbf{b} + \mathbf{c})] - [(\mathbf{r} \cdot \mathbf{a}) \mathbf{a} + (\mathbf{r} \cdot \mathbf{b}) \mathbf{b} + (\mathbf{r} \cdot \mathbf{c}) \mathbf{c}] = \mathbf{0} \quad [\because |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|]$$

$$\Rightarrow |\mathbf{a}|^2 [3\mathbf{r} - (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{r}] = \mathbf{0}$$

$$\therefore 3\mathbf{r} - (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{r} = \mathbf{0}$$

$$\Rightarrow \mathbf{r} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{2}$$

Question 188

If $y = y(x)$ is the solution curve of the differential equation

$$x^2 dy + \left(y - \frac{1}{x}\right) dx = 0 ; x > 0$$

and $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to

[1 Sep 2021 Shift 2]

Options:

A. $\frac{3}{2} - \frac{1}{\sqrt{e}}$

B. $3 + \frac{1}{\sqrt{e}}$

C. $3 + e$

D. $3 - e$

Answer: D

Solution:

$$x^2 dy + \left(y - \frac{1}{x}\right) dx = 0$$

$$\Rightarrow x^2 dy + y dx = \frac{dx}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$$

This is linear differential equations

$$\text{IF} = e^{\int -\frac{1}{x^2} dx} = e^{\frac{1}{x}}$$

$$\Rightarrow y \cdot e^{-1/x} = \int e^{-1/x} \cdot \frac{1}{x^3} dx + c$$

$$\text{Let } \frac{-1}{x} = t$$

$$\Rightarrow \frac{1}{x^2} dx = dt$$

$$\Rightarrow y \cdot e^{-1/x} = \int -te^t dt + c = -(te^t - e^t) + c$$

$$\Rightarrow ye^{-1/x} = \frac{1}{x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}} + c$$

Put $x = 1, y = 1$

$$(1) e^{-1} = \frac{e^{-1}}{1} + e^{-1} + c$$

$$\Rightarrow c = -e^{-1}$$

$$\text{Solution } y \cdot e^{-1/x} = \frac{1}{x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}} - e^{-1}$$

$$\Rightarrow y = \frac{1}{x} + 1 - \frac{e^{1/x}}{e}$$

$$\text{Put } x = \frac{1}{2}$$

$$y = 2 + 1 - \frac{e^2}{e}$$

$$\Rightarrow y = 3 - e$$

Question 189

If for $x \geq 0$, $y = y(x)$ is the solution of the differential equation, $(x+1)dy = ((x+1)^2 + y - 3)dx$, $y(2) = 0$ then $y(3)$ is equal to _____.

[NA Jan. 09, 2020 (I)]

Answer: 3

Solution:

$$(x+1)dy = ((x+1)^2 + (y-3))dx = 0$$

$$\Rightarrow \frac{dy}{dx} = (1+x) + \left(\frac{y-3}{1+x} \right)$$

$$\frac{dy}{dx} - \frac{1}{(1+x)}y = (1+x) - \frac{3}{(1+x)}$$

$$\text{I.F.} = e^{-\int \frac{1}{(1+x)} dx} = \frac{1}{(1+x)}$$

$$\therefore \frac{d}{dx} \left(\frac{y}{1+x} \right) = 1 - \frac{3}{(1+x)^2}$$

$$y = (1+x) \left[x + \frac{3}{(1+x)} + C \right]$$

$$\because \text{At } x=2, y=0$$

$$\therefore 0 = 3(2+1+C) \Rightarrow C = -3$$

$$\text{Then, } y = (1+x) \left[x + \frac{3}{1+x} - 3 \right]$$

$$\text{Now, at } x=3, y = (1+3) \left[3 + \frac{3}{1+3} - 3 \right] = 3$$

Question 190

Let $y = y(x)$ be the solution curve of the differential equation, $(y^2 - x) \frac{dy}{dx} = 1$, satisfying $y(0) = 1$. This curve intersects the x -axis at a point whose abscissa is:

[Jan. 7, 2020 (II)]

Options:

A. $2 - e$

B. $-e$

C. 2

D. $2 + e$

Answer: A

Solution:

The given differential equation is $\frac{dx}{dy} + x = y^2$

Comparing with $\frac{dx}{dy} + Px = Q$, where $P = 1$, $Q = y^2$

Now, I.F. = $e^{\int 1 \cdot dy} = e^y$

$$x \cdot e^y = \int (y^2)e^y \cdot dy = y^2 \cdot e^y - \int 2y \cdot e^y \cdot dy$$

$$= y^2 e^y - 2(y \cdot e^y - e^y) + C$$

$$\Rightarrow x \cdot e^y = y^2 e^y - 2y e^y + 2e^y + C$$

$$\Rightarrow x = y^2 - 2y + 2 + C \cdot e^{-y} \dots\dots(i)$$

As $y(0) = 1$, satisfying the given differential eqn,

$\therefore$ put $x = 0$, $y = 1$ in eqn. (i)

$$0 = 1 - 2 + 2 + \frac{C}{e}$$

$$C = -e$$

$$y = 0, x = 0 - 0 + 2 + (-e)(e^{-0})$$

$$x = 2 - e$$

Question191

The differential equation of the family of curves, $x^2 = 4b(y + b)$, $b \in \mathbb{R}$, is:

[Jan. 8, 2020 (II)]

Options:

A. $x(y')^2 = x + 2yy'$

B. $x(y')^2 = 2yy' - x$

C. $xy'' = y'$

D. $x(y')^2 = x - 2yy'$

Answer: A

Solution:

$$\text{Since, } x^2 = 4b(y + b)$$

$$x^2 = 4by + 4b^2$$

$$2x = 4by'$$

$$\Rightarrow b = \frac{x}{2y'}$$

So, differential equation is

$$x^2 = \frac{2x}{y'} \cdot y + \left(\frac{x}{y'}\right)^2$$

$$x(y')^2 = 2yy' + x$$

Question 192

If $f(x) = \tan^{-1}(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f(0) = 0$, then $f(1)$ is equal to:

[Jan. 9, 2020 (I)]

Options:

A. $\frac{\pi+1}{4}$

B. $\frac{1}{4}$

C. $\frac{\pi-1}{4}$

D. $\frac{\pi+2}{4}$

Answer: A

Solution:

$$f'(x) = \tan^{-1}(\sec x + \tan x)$$

$$= \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}$$

Integrate both sides, we get

$$\int f'(x) dx = \int \left(\frac{\pi}{4} + \frac{x}{2}\right) dx$$

$$f(x) = \frac{\pi}{4}x + \frac{x^2}{4} + C$$

$$\therefore f(0) = 0$$

$$C = 0 \Rightarrow f(x) = \frac{\pi}{4}x + \frac{x^2}{4}$$

$$\text{So, } f(1) = \frac{\pi+1}{4}$$

Question 193

If $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$; $y(1) = 1$; then a value of x satisfying $y(x) = e$ is:

[Jan. 9, 2020 (II)]

Options:

A. $\frac{1}{2}\sqrt{3}e$

B. $\frac{e}{\sqrt{2}}$

C. $\sqrt{2}e$

D. $\sqrt{3}e$

Answer: D

Solution:

The given differential equation,

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Then, } v + x \frac{dv}{dx} = \frac{vx^2}{x^2+v^2x^2} = \frac{v}{1+v^2}$$

$$\Rightarrow \frac{1+v^2}{v^3} dv = -\frac{1}{x} dx$$

$$\Rightarrow \int \left(\frac{1}{v^3} + \frac{1}{v} \right) dv = \int -\frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{2} \left(\frac{1}{v^2} \right) + \ln v = -\ln x + c$$

$$\Rightarrow -\frac{x^2}{2y^2} = -\ln y + c \quad \left[v = \frac{y}{x} \right]$$

$$\text{When } x = 1, y = 1, \text{ then } -\frac{1}{2} = c$$

$$\Rightarrow x^2 = y^2(1 + 2 \ln y)$$

$$\text{At } y = e, x^2 = e^2(3)$$

$$\Rightarrow x = \pm \sqrt{3}e$$

$$\text{So, } x = \sqrt{3}e$$

Question 194

Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, $|x| > 1$. If $\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(\sin^{-1}(f(x)))$ and $y(\sqrt{3}) = \frac{\pi}{6}$, then $y(-\sqrt{3})$ is equal to:

[Jan. 8, 2020 (I)]

Options:

A. $\frac{2\pi}{3}$

B. $-\frac{\pi}{6}$

C. $\frac{5\pi}{6}$

D. $\frac{\pi}{3}$

Answer: B

Solution:

$$\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(\sin^{-1}f(x))$$

$$2y = \sin^{-1}f(x) + C = \sin^{-1}(\sin(2\tan^{-1}x)) + C$$

$$\Rightarrow 2\left(\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) + C$$

$$\frac{\pi}{3} = \frac{\pi}{3} + C \quad \therefore C = 0$$

$$\text{for } x = -\sqrt{3}, 2y = \sin^{-1}\left(\sin\left(\frac{-2\pi}{6}\right)\right) + 0$$

$$\Rightarrow 2y = \frac{-\pi}{3} \Rightarrow y = \frac{-\pi}{6}$$

Question 195

Let $y = y(x)$ be a solution of the differential equation,

$$\sqrt{1-x^2} \frac{dy}{dx} + \sqrt{1-y^2} = 0, |x| < 1$$

If $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$, then $y\left(\frac{-1}{\sqrt{2}}\right)$ is equal to:

[Jan. 8, 2020 (I)]

Options:

A. $\frac{\sqrt{3}}{2}$

B. $-\frac{1}{\sqrt{2}}$

C. $\frac{1}{\sqrt{2}}$

D. $-\frac{\sqrt{3}}{2}$

Answer: C

Solution:

The given differential eqn. is

$$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0 \Rightarrow \sin^{-1}y + \sin^{-1}x = c$$

At $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2} \Rightarrow c = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}y = \cos^{-1}x$$

Hence, $y\left(-\frac{1}{\sqrt{2}}\right) = \sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

$$= \sin\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}$$

Question 196

If $y = y(x)$ is the solution of the differential equation, $e^y = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to:

[Jan. 7, 2020 (I)]

Options:

A. $1 + \log_e 2$

B. $2 + \log_e 2$

C. $2e$

D. $\log_e 2$

Answer: A

Solution:

$$\text{Let } e^y = t$$

$$e^y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore dt dx - t = e^x \left[\because e^y \frac{dy}{dx} - e^y = e^x \right]$$

$$\text{I.F.} = e^{\int -1 \cdot dx} = e^{-x}$$

$$t(e^{-x}) = \int e^x \cdot e^{-x} dx \Rightarrow e^{y-x} = x + c$$

Put $x = 0, y = 0$, then we get $c = 1$

$$e^{y-x} = x + 1$$

$$y = x + \log_e(x + 1)$$

$$\text{Put } x = 1 \therefore y = 1 + \log_e 2$$

Question 197

The general solution of the differential equation $\sqrt{1+x^2+y^2+x^2y^2} + xy \frac{dy}{dx} = 0$

is:

(where C is a constant of integration)

: [Sep. 06, 2020 (I)]

Options:

A. $\sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2} \log_e \left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1} \right) + C$

B. $\sqrt{1+y^2} - \sqrt{1+x^2} = \frac{1}{2} \log_e \left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1} \right) + C$

C. $\sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2} \log_e \left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} \right) + C$

D. $\sqrt{1+y^2} - \sqrt{1+x^2} = \frac{1}{2} \log_e \left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} \right) + C$

Answer: A

Solution:

$$\sqrt{1+x^2} \cdot \sqrt{1+y^2} = -xy \frac{dy}{dx}$$

$$\int \frac{\sqrt{1+x^2}}{x} dx = - \int \frac{y}{\sqrt{1+y^2}} dy$$

$$\text{Let } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\begin{aligned}
 &\Rightarrow \int \frac{\sec^3 \theta d\theta}{\tan \theta} = - \int \frac{2y}{2\sqrt{1+y^2}} dy \\
 &\Rightarrow \int \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cdot \cos^2 \theta} d\theta = - \int \frac{1}{\sqrt{1+y^2}} dy \\
 &\Rightarrow \int (\tan \theta \cdot \sec \theta + \operatorname{cosec} \theta) d\theta = - \sqrt{1+y^2} \\
 &\Rightarrow \sec \theta + \log_e |\operatorname{cosec} \theta - \cot \theta| = - \sqrt{1+y^2} + C \\
 &\therefore \sqrt{1+x^2} + \log_e \left| \frac{\sqrt{1+x^2}-1}{x} \right| = - \sqrt{1+y^2} + C \\
 &\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} = \frac{1}{2} \log_e \left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1} \right) + C
 \end{aligned}$$

Question 198

If $y = \left(\frac{2}{\pi}x - 1 \right) \operatorname{cosec} x$ is the solution of the differential equation, $d y d x + p(x)y = \frac{2}{\pi} \operatorname{cosec} x$, $0 < x < \frac{\pi}{2}$, then the function $p(x)$ is equal to:
[Sep. 06, 2020 (II)]

Options:

- A. $\cot x$
- B. $\operatorname{cosec} x$
- C. $\sec x$
- D. $\tan x$

Answer: A

Solution:

$$\begin{aligned}
 \because y &= \left(\frac{2}{\pi}x - 1 \right) \operatorname{cosec} x \\
 \frac{dy}{dx} &= \frac{2}{\pi} \operatorname{cosec} x - \left(\frac{2}{\pi}x - 1 \right) \operatorname{cosec} x \cdot \cot x \\
 &= \operatorname{cosec} x \left[\frac{2}{\pi} - \left(\frac{2}{\pi}x - 1 \right) \cot x \right] \\
 \Rightarrow \frac{dy}{dx} - \frac{2}{\pi} \operatorname{cosec} x &= y \cot x \dots\dots\dots (i) \\
 \text{It is given that,} \\
 \Rightarrow \frac{dy}{dx} - \frac{2}{\pi} \operatorname{cosec} x &= -yp(x) \dots\dots\dots (ii)
 \end{aligned}$$

By comparison of (i) and (ii), we get
 $p(x) = \cot x$

Question 199

If $y = y(x)$ is the solution of the differential equation $\frac{5+e^x}{2+y} \cdot \frac{dy}{dx} + e^x = 0$ satisfying $y(0) = 1$, then a value of $y(\log_e 13)$ is :

[Sep. 05, 2020 (I)]

Options:

- A. 1
- B. -1
- C. 0
- D. 2

Answer: B

Solution:

$$\begin{aligned}\frac{5+e^x}{2+y} \cdot \frac{dy}{dx} &= -e^x \\ \int \frac{dy}{2+y} &= - \int \frac{e^x}{5+e^x} dx \\ \Rightarrow \log_e |2+y| \cdot \log_e |5+e^x| &= \log_e C \\ \Rightarrow |(2+y)(5+e^x)| &= C \cdot y(0) = 1 \\ C &= 18 \\ \therefore (2+y) \cdot (5+e^x) &= 18 \\ \text{When } x = \log_e 13 \text{ then } (2+y) \cdot 18 &= 18 \\ \Rightarrow 2+y &= \pm 1 \\ \therefore y &= -1, -3 \\ \therefore y(\ln 13) &= -1\end{aligned}$$

Question 200

The solution of the differential equation $\frac{dy}{dx} - \frac{y+3x}{\log_e(y+3x)} + 3 = 0$ is (where C is a constant of integration.)

[Sep. 04, 2020 (II)]

Options:

- A. $x - \frac{1}{2}(\log_e(y+3x))^2 = C$
- B. $x - \log_e(y+3x) = C$

C. $y + 3x - \frac{1}{2}(\log_e x)^2 = C$

D. $x - 2\log_e(y + 3x) = C$

Answer: A

Solution:

Let $y + 3x = t$

$\Rightarrow dy dx + 3 = \frac{dt}{dx}$

Putting these value in given differential equation

$\frac{dt}{dx} = \frac{t}{\log_e t}$

$\Rightarrow \int \frac{\log_e t}{t} dt = \int dx$

$\Rightarrow \frac{(\log_e t)^2}{2} = x - C$

$\Rightarrow x - \frac{1}{2}(\ln(y + 3x))^2 = C$

Question 201

Let $f : (0, \infty) \rightarrow (0, \infty)$ be a differentiable function such that $f(1) = e$ and

$\lim_{t \rightarrow x} \frac{t^2 f^2(x) - x^2 f^2(t)}{t - x} = 0$

If $f(x) = 1$, then x is equal to :

[Sep. 04, 2020 (II)]

Options:

A. $\frac{1}{e}$

B. $2e$

C. $\frac{1}{2e}$

D. e

Answer: A

Solution:

$$\lim_{t \rightarrow x} \frac{t^2 f^2(x) - x^2 f^2(t)}{t - x} = 0$$

$$\Rightarrow \lim_{t \rightarrow x} \frac{2tf^2(x) - 2x^2f(t) \cdot f'(t)}{1} = 0$$

Using L'Hospital's rule

$$\Rightarrow f(x) = xf'(x)$$

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{x} dx$$

$$\log_e f(x) = \log_e x + \log_e C$$

$$\Rightarrow f(x) = Cx \quad \because f(1) = e$$

$$\Rightarrow C = e; \text{ so } f(x) = ex$$

$$\text{When } f(x) = 1 = ex \Rightarrow x = \frac{1}{e}$$

Question 202

The solution curve of the differential equation, $(1 + e^{-x})(1 + y^2) \frac{dy}{dx} = y^2$, which passes through the point $(0, 1)$, is :
[Sep. 03, 2020 (I)]

Options:

A. $y^2 + 1 = y \left(\log_e \left(\frac{1 + e^{-x}}{2} \right) + 2 \right)$

B. $y^2 + 1 = y \left(\log_e \left(\frac{1 + e^x}{2} \right) + 2 \right)$

C. $y^2 = 1 + y \log_e \left(\frac{1 + e^x}{2} \right)$

D. $y^2 = 1 + y \log_e \left(\frac{1 + e^{-x}}{2} \right)$

Answer: C

Solution:

$$\int \left(\frac{y^2 + 1}{y^2} \right) dy = \int \frac{e^x dx}{e^x + 1}$$

$$\Rightarrow y - \frac{1}{y} = \log_e |e^x + 1| + c$$

$$\because \text{Passes through } (0, 1)$$

$$\therefore c = -\log_e 2$$

$$\Rightarrow y^2 - 1 = y \log_e \left(\frac{e^x + 1}{2} \right)$$

$$\Rightarrow y^2 = 1 + y \log_e \left(\frac{e^x + 1}{2} \right)$$

Question 203

If $x^3 dy + xy dx = x^2 dy + 2y dx$; $y(2) = e$ and $x > 1$, then $y(4)$ is equal to:
[Sep. 03, 2020 (II)]

Options:

A. $\frac{3}{2} + \sqrt{e}$

B. $\frac{3}{2}\sqrt{e}$

C. $\frac{1}{2} + \sqrt{e}$

D. $\frac{\sqrt{e}}{2}$

Answer: B

Solution:

$$x^3 dy + xy dx = 2y dx + x^2 dy$$

$$\Rightarrow (x^3 - x^2) dy = (2 - x) y dx$$

$$\Rightarrow \frac{dy}{y} = \frac{2 - x}{x^2(x - 1)} dx$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2 - x}{x^2(x - 1)} dx$$

$$\text{Let } \frac{2 - x}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$$

$$\Rightarrow 2 - x = A(x - 1) + B(x - 1) + Cx^2$$

Compare the coefficients of x , x^2 and constant term.

$$C = 1, B = -2 \text{ and } A = -1$$

$$\therefore \int \frac{dy}{y} = \int \left\{ \frac{-1}{x} - \frac{2}{x^2} + \frac{1}{x - 1} \right\} dx$$

$$\Rightarrow \ln y = -\ln x + \frac{2}{x} + \ln |x - 1| + C$$

$$\therefore y(2) = e$$

$$\Rightarrow 1 = -\ln 2 + 1 + 0 + C \quad [\because \log e = 1]$$

$$\Rightarrow C = \ln 2$$

$$\Rightarrow \ln y = -\ln x + \frac{2}{x} + \ln |x - 1| + \ln 2$$

$$\text{At } x = 4,$$

$$\Rightarrow \ln y(4) = -\ln 4 + \frac{1}{2} + \ln 3 + \ln 2$$

$$\Rightarrow \ln y(4) = \ln \left(\frac{3}{2} \right) + \frac{1}{2} = \ln \left(\frac{3}{2} e^{1/2} \right) \quad [\because \log m + \log n = \log(mn)]$$

$$\Rightarrow y(4) = \frac{3}{2} e^{1/2}$$

Question 204

Let $y = y(x)$ be the solution of the differential equation, $2 + \sin xy + 1 \cdot \frac{dy}{dx} = -\cos x$, $y > 0$, $y(0) = 1$. If $y(\pi) = a$ and $\frac{dy}{dx}$ at $x = \pi$ is b , then the ordered pair (a, b) is equal to :
[Sep. 02, 2020 (I)]

Options:

A. $\left(2, \frac{3}{2} \right)$

B. $(1, -1)$

C. $(1, 1)$

D. $(2, 1)$

Answer: C

Solution:

The given differential equation is

$$\frac{2 + \sin x}{y + 1} \frac{dy}{dx} = -\cos x, \quad y > 0$$

$$\Rightarrow \frac{dy}{y + 1} = -\frac{\cos x}{2 + \sin x} dx$$

Integrate both sides,

$$\int \frac{dy}{y + 1} = \int \frac{(-\cos x) dx}{2 + \sin x}$$

$$\ln |y + 1| = -\ln |2 + \sin x| + \ln C$$

$$\Rightarrow \ln |y + 1| + \ln |2 + \sin x| = \ln C$$

$$\Rightarrow \ln |(y + 1)(2 + \sin x)| = \ln C$$

$$\because y(0) = 1 \Rightarrow \ln 4 = \ln C \Rightarrow C = 4$$

$$\therefore (y + 1)(2 + \sin x) = 4$$

$$\Rightarrow y = \frac{4}{2 + \sin x} - 1$$

$$\therefore y = \frac{2 - \sin x}{2 + \sin x} \Rightarrow y(\pi) = \frac{2 - \sin \pi}{2 + \sin \pi} = 1$$

$$\Rightarrow a = 1$$

$$\text{Now, } \frac{dy}{dx} = \frac{(2 + \sin x)(-\cos x) - (2 - \sin x) \cdot \cos x}{(2 + \sin x)^2}$$

$$\left. \frac{dy}{dx} \right|_{x=\pi} = 1 \Rightarrow b = 1$$

Ordered pair $(a, b) = (1, 1)$

Question 205

If a curve $y = f(x)$, passing through the point $(1, 2)$, is the solution of the differential equation, $2x^2 dy = (2xy + y^2) dx$, then $f\left(\frac{1}{2}\right)$ is equal to
[Sep. 02, 2020 (II)]

Options:

A. $\frac{1}{1 + \log_e 2}$

B. $\frac{1}{1 - \log_e 2}$

C. $1 + \log_e 2$

D. $\frac{1}{1 + \log_e 2}$

Answer: A

Solution:

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

It is homogeneous differential equation.

$\therefore$ Put $y = vx$

$$\Rightarrow v + x \frac{dv}{dx} = v + \frac{v^2}{2} \Rightarrow \int 2 \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{-2}{v} = \log_e x + c \Rightarrow \frac{-2x}{y} = \log_e x + c$$

Put $x = 1, y = 2$, we get $c = -1$

$$\Rightarrow \frac{-2x}{y} = \log_e x - 1$$

$$\text{Hence, put } x = \frac{1}{2} \Rightarrow y = \frac{1}{1 + \log_e 2}$$

Question 206

Let $y = y(x)$ be the solution of the differential equation

$$\cos x \frac{dy}{dx} + 2y \sin x = \sin 2x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

If $y(\pi/3) = 0$, then $y(\pi/4)$ is equal to :
[Sep. 05, 2020 (II)]

Options:

A. $2 - \sqrt{2}$

B. $2 + \sqrt{2}$

C. $\sqrt{2} - 2$

D. $\frac{1}{\sqrt{2}} - 1$

Answer: C

Solution:

$$\frac{dy}{dx} + 2y \tan x = 2 \sin x$$

$$I.F. = e^{\int 2 \tan x \, dx} = \sec^2 x$$

The solution of the differential equation is

$$y \times I.F. = \int I.F. \times 2 \sin x \, dx + C$$

$$\Rightarrow y \cdot \sec^2 x = \int 2 \sin x \cdot \sec^2 x \, dx + C$$

$$\Rightarrow y \sec^2 x = 2 \sec x + C \dots\dots(i)$$

$$\text{When } x = \frac{\pi}{3}, y = 0; \text{ then } C = -4$$

$$\therefore \text{From (i), } y \sec^2 x = 2 \sec x - 4$$

$$\Rightarrow y = \frac{2 \sec x - 4}{\sec^2 x} \Rightarrow y\left(\frac{\pi}{4}\right) = \sqrt{2} - 2$$

Question 207

Let $y = y(x)$ be the solution of the differential

equation, $xy' - y = x^2(x \cos x + \sin x)$, $x > 0$. If $y(\pi) = \pi$, then $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right)$ is

equal to :

[Sep. 04, 2020 (I)]

Options:

A. $2 + \frac{\pi}{2}$

B. $2 + \frac{\pi}{2} + \frac{\pi^2}{4}$

C. $2 + \frac{\pi}{2} + \frac{\pi^2}{4}$

D. $1 + \frac{\pi}{2}$

Answer: A

Solution:

$$\frac{dy}{dx} - \frac{y}{x} = x(x \cos x + \sin x)$$

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$$

$$\therefore \int d\left(\frac{y}{x}\right) = \int (x \cos x + \sin x) dx$$

$$\Rightarrow \frac{y}{x} = x \sin x + C \because y(\pi) = \pi \Rightarrow C = 1$$

$$y = x^2 \sin x + x \Rightarrow y\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + \frac{\pi}{2}$$

$$y' = 2x \sin x + x^2 \cos x + 1$$

$$y'' = 2 \sin x - x^2 \sin x \Rightarrow y''\left(\frac{\pi}{2}\right) = 2 - \frac{\pi^2}{4}$$

$$\therefore y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right) = 2 - \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi}{2} = 2 + \frac{\pi}{2}$$