

Continuity and Differentiability

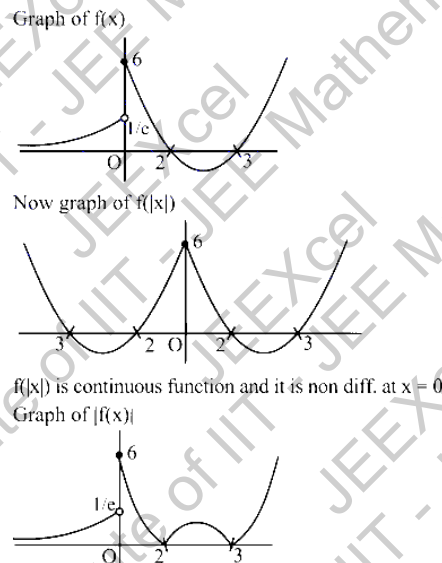
Question 1

Let $f(x) = \begin{cases} e^{x-1}, & x < 0 \\ x^2 - 5x + 6, & x \geq 0 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$. If the number of points where g is not continuous and is not differentiable are α and β respectively, then $\alpha + \beta$ is equal to _____

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Answer: 4

Solution:



First look at $|f(x)|$.

For $x < 0$, we have $f(x) = e^{x-1}$, which is always positive, so $|f(x)| = e^{x-1}$. This part is continuous and differentiable for all $x < 0$.

For $x \geq 0$, $f(x) = x^2 - 5x + 6 = (x-2)(x-3)$. This becomes 0 at $x = 2$ and $x = 3$, and it changes sign between 2 and 3. So when we take absolute value, $|f(x)|$ gets a "sharp corner" at $x = 2$ and $x = 3$, hence it is not differentiable at $x = 2, 3$.

At $x = 0$, check continuity of $|f(x)|$: left value tends to $|e^{-1}| = e^{-1}$, but right value is $|f(0)| = |6| = 6$. Since these are not equal, $|f(x)|$ is discontinuous at $x = 0$ (and therefore also not differentiable at $x = 0$).

Now consider $g(x) = f(|x|) + |f(x)|$.

Here $f(|x|)$ always uses the $x \geq 0$ definition because $|x| \geq 0$. So $f(|x|)$ is continuous for all x , but it is not differentiable at $x = 0$ because of $|x|$.

So, discontinuity in g can only come from $|f(x)|$, which happens at $x = 0$ only. Also, g will be non-differentiable where either term is non-differentiable, i.e., at $x = 0, 2, 3$.

$|f(x)|$ is discontinuous at $x = 0$

$|f(x)|$ is non-diff. at $x = 0, 2, 3$

$g(x) = f(|x|) + |f(x)|$

$g(x)$ will be discontinuous at $x = 0$

$g(x)$ will be non diff at $x = 0, 2, 3$

$\alpha = 1, \beta = 3$

$\alpha + \beta = 1 + 3 = 4$

Question2

The number of points, at which the function

$f(x) = \max \{6x, 2 + 3x^2\} + |x - 1| \cos \left|x^2 - \frac{1}{4}\right|, x \in (-\pi, \pi)$, is not differentiable, is

_____.

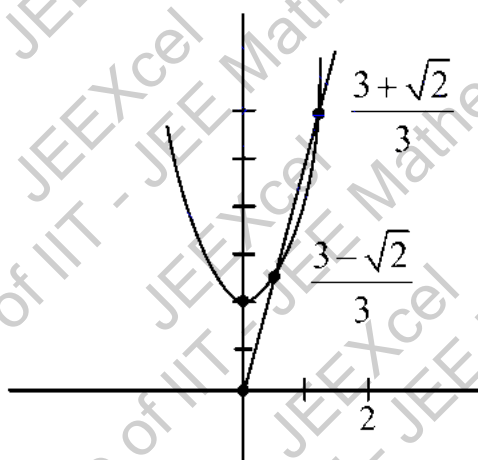
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Answer: 3

Solution:

For the part $\max \{6x, 2 + 3x^2\}$, the function can change its “active” expression only at those points where the two inside expressions become equal.

$$2 + 3x^2 = 6x \Rightarrow 3x^2 - 6x + 2 = 0$$



$$x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{2}}{3}$$

At these two values of x , the max function switches between $6x$ and $2 + 3x^2$. Also, the derivatives of the two expressions are different there (since $\frac{d}{dx}(6x) = 6$ and $\frac{d}{dx}(2 + 3x^2) = 6x$), so $\max \{6x, 2 + 3x^2\}$ is not differentiable at both points.

Now look at the second part $|x - 1| \cos \left|x^2 - \frac{1}{4}\right|$. The factor $\cos \left|x^2 - \frac{1}{4}\right|$ is continuous for all x , and at $x = 1$ it becomes $\cos \left|1 - \frac{1}{4}\right| = \cos \left(\frac{3}{4}\right) \neq 0$.

Because $\cos \left|x^2 - \frac{1}{4}\right|$ is nonzero at $x = 1$, multiplying it by $|x - 1|$ does not remove the “corner” of $|x - 1|$. So $|x - 1| \cos \left|x^2 - \frac{1}{4}\right|$ is not differentiable at $x = 1$.

Total number of points where $f(x)$ is nonDifferentiable is 3

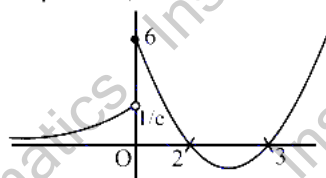
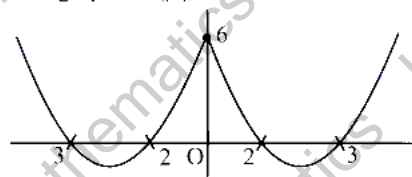
Question 3

Let $f(x) = \begin{cases} e^{x-1}, & x < 0 \\ x^2 - 5x + 6, & x \geq 0 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$. If the number of points where g is not continuous and is not differentiable are α and β respectively, then $\alpha + \beta$ is equal to _____

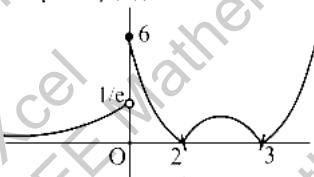
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Answer: 4

Solution:

Graph of $f(x)$ Now graph of $f(|x|)$ 

$f(|x|)$ is continuous function and it is non diff. at $x = 0$

Graph of $|f(x)|$ 

First look at $|f(x)|$.

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$g(x) = f(|x|) + |f(x)|$

$g(x)$ will be discontinuous at $x = 0$

$g(x)$ will be non diff at $x = 0, 2, 3$

$\alpha = 1, \beta = 3$

$\alpha + \beta = 1 + 3 = 4$

Question 4

Let $f(x) = \begin{cases} x^3 + 8; & x < 0, \\ x^2 - 4; & x \geq 0, \end{cases}$ and $g(x) = \begin{cases} (x-8)^{1/3}; & x < 0, \\ (x+4)^{1/2}; & x \geq 0. \end{cases}$

Then the number of points, where the function $g \circ f$ is discontinuous, is ____.

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Answer: 3

Solution:

We need to find the points where the composite function $g \circ f$ is discontinuous.

That means we first find

$$(g \circ f)(x) = g(f(x)).$$

So we must see, for each value of x , which branch of g will be used.

1. Given functions

$$f(x) = \begin{cases} x^3 + 8, & x < 0, \\ x^2 - 4, & x \geq 0 \end{cases}$$

and

$$g(x) = \begin{cases} (x-8)^{1/3}, & x < 0, \\ (x+4)^{1/2}, & x \geq 0. \end{cases}$$

Now,

$$(g \circ f)(x) = g(f(x)).$$

The definition of g depends on whether its input is negative or non-negative. So we must check the sign of $f(x)$.

2. Find where $f(x) < 0$ and where $f(x) \geq 0$

Case 1: $x < 0$

Then

$$f(x) = x^3 + 8.$$

We check when this is negative:

$$x^3 + 8 < 0 \implies x^3 < -8 \implies x < -2.$$

So for $x < 0$:

- if $x < -2$, then $f(x) < 0$,
- if $-2 \leq x < 0$, then $f(x) \geq 0$.

Case 2: $x \geq 0$

Then

$$f(x) = x^2 - 4.$$

We check when this is negative:

$$x^2 - 4 < 0 \implies x^2 < 4 \implies -2 < x < 2$$

But here $x \geq 0$, so this becomes

$$0 \leq x < 2.$$

Thus for $x \geq 0$:

- if $0 \leq x < 2$, then $f(x) < 0$,
- if $x \geq 2$, then $f(x) \geq 0$.

3. Write $(g \circ f)(x)$ piecewise

Now use the correct branch of g .

(i) When $x < -2$

Here $f(x) < 0$, so use

$$g(y) = (y-8)^{1/3}.$$

Thus

$$(g \circ f)(x) = ((x^3 + 8) - 8)^{1/3} = (x^3)^{1/3} = x.$$

So for $x < -2$,

$$(g \circ f)(x) = x.$$

(ii) When $-2 \leq x < 0$

Here $f(x) \geq 0$, so use

$$g(y) = \sqrt{y+4}.$$

Thus

$$(g \circ f)(x) = \sqrt{(x^2 - 4) + 4} = \sqrt{x^2 + 12}.$$

So for $-2 \leq x < 0$,

$$(g \circ f)(x) = \sqrt{x^3 + 12}.$$

(iii) When $0 \leq x < 2$

Here $f(x) < 0$, so use

$$g(y) = (y - 8)^{1/3}.$$

Thus

$$(g \circ f)(x) = ((x^2 - 4) - 8)^{1/3} = (x^2 - 12)^{1/3}.$$

So for $0 \leq x < 2$,

$$(g \circ f)(x) = (x^2 - 12)^{1/3}.$$

(iv) When $x \geq 2$

Here $f(x) \geq 0$, so use

$$g(y) = \sqrt{y + 4}.$$

Thus

$$(g \circ f)(x) = \sqrt{(x^2 - 4) + 4} = \sqrt{x^2} = x$$

since $x \geq 2$.

So for $x \geq 2$,

$$(g \circ f)(x) = x.$$

4. Check continuity at possible boundary points

Each expression is continuous in its own interval. So we only check the joining points:

$$x = -2, \quad x = 0, \quad x = 2.$$

At $x = -2$

Left-hand limit:

$$\text{for } x < -2, (g \circ f)(x) = x, \text{ so}$$

$$\lim_{x \rightarrow -2^-} (g \circ f)(x) = -2.$$

Right-hand limit:

$$\text{for } -2 \leq x < 0, (g \circ f)(x) = \sqrt{x^3 + 12}, \text{ so}$$

$$\lim_{x \rightarrow -2^+} (g \circ f)(x) = \sqrt{(-2)^3 + 12} = \sqrt{4} = 2.$$

Since

$$-2 \neq 2,$$

the function is discontinuous at $x = -2$.

At $x = 0$

Left-hand limit:

$$\lim_{x \rightarrow 0^-} (g \circ f)(x) = \sqrt{0^3 + 12} = \sqrt{12}.$$

Right-hand limit:

$$\lim_{x \rightarrow 0^+} (g \circ f)(x) = (0^2 - 12)^{1/3} = (-12)^{1/3}.$$

These are not equal, so discontinuous at $x = 0$.

At $x = 2$

Left-hand limit:

$$\lim_{x \rightarrow 2^-} (g \circ f)(x) = (2^2 - 12)^{1/3} = (-8)^{1/3} = -2.$$

Right-hand limit:

$$\lim_{x \rightarrow 2^+} (g \circ f)(x) = 2.$$

These are not equal, so discontinuous at $x = 2$.

5. Conclusion

The function $g \circ f$ is discontinuous at

$$x = -2, \quad x = 0, \quad x = 2.$$

Hence, the number of points of discontinuity is

3.

Question 5

Let $f : \mathbb{R} \rightarrow (0, \infty)$ be a twice differentiable function such that $f(3) = 18$, $f'(3) = 0$ and $f''(3) = 4$.

Then $\lim_{x \rightarrow 1} \left(\log_e \left(\frac{f(2+x)}{f(3)} \right)^{\frac{18}{(x-1)^2}} \right)$ is equal to :

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Options:

A.

9

B.

18

C.

1

D.

2

Answer: D

Solution:

$$\ln \left(\lim_{x \rightarrow 1} \left(\frac{f(x+2)}{f(3)} \right)^{\frac{18}{(x-1)^2}} \right) (1^\infty)$$

$$\ln \left(e^{\lim_{x \rightarrow 1} \frac{18}{(x-1)^2} \left(\frac{f(x+2)}{f(3)} - 1 \right)} \right)$$

$$\ln \left(e^{\lim_{x \rightarrow 1} \frac{f'(x+2)}{2(X-1)}} \right)$$

$$\ln \left(e^{\lim_{x \rightarrow 1} \frac{f''(x+2)}{2}} \right)$$

$$\lim_{x \rightarrow 1} \frac{f''(3)}{2} = \frac{4}{2} = 2$$

Question 6

Let $[\cdot]$ denote the greatest integer function, and let $f(x) = \min \{ \sqrt{2}x, x^2 \}$.

Let $S = \{x \in (-2, 2) : \text{the function } g(x) = |x| [x^2] \text{ is discontinuous at } x\}$.

Then $\sum_{x \in S} f(x)$ equals

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Options:

A.

$$2 - \sqrt{2}$$

B.

$$2\sqrt{6} - 3\sqrt{2}$$

C.

$$1 - \sqrt{2}$$

D.

$$\sqrt{6} - 2\sqrt{2}$$

Answer: C

Solution:

$$g(x) = |x| \lfloor x^2 \rfloor$$

Points of discontinuity of $g(x)$ in $(-2, 2)$ are $(\pm 1, \pm\sqrt{2}, \pm\sqrt{3})$

$$\therefore S = \{-1, 1, -\sqrt{2}, \sqrt{2}, -\sqrt{3}, \sqrt{3}\}$$

$$\therefore f(x) = \min \{ \sqrt{2x}, x^2 \}$$

$$\therefore \sum_{x \in S} f(x) = -\sqrt{2} + 1 - 2 + 2 - \sqrt{6} + \sqrt{6} = 1 - \sqrt{2}$$

Question 7

If $\lim_{x \rightarrow 0} \frac{e^{(a-1)x} + 2 \cos bx + (c-2)e^{-x}}{x \cos x - \log_e(1+x)} = 2$, then $a^2 + b^2 + c^2$ is equal to :

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Options:

A.

$$3$$

B.

$$5$$

C.

$$9$$

D.

$$7$$

Answer: D

Solution:

$$\lim_{x \rightarrow 0} \frac{\left[1 + \frac{(a-1)}{1!}x + \frac{(a-1)^2}{2!}x^2 + \dots \infty \right] + 2 \left[1 - \frac{(bx)^2}{2!} + \dots \infty \right] + (c-2) \left[1 - \frac{x^1}{1!} + \frac{x^2}{2!} - \dots \infty \right]}{x \left(1 - \frac{x^2}{2!} + \dots \infty \right) - \left(x - \frac{x^2}{2} + \dots \infty \right)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(c+1) + (a-c+1)x + \left[\frac{(a-1)^2}{2} - b^2 + \frac{(c-2)}{2} \right] x^2 + \dots \infty}{\left(\frac{x^2}{2} - \frac{x^3}{2} + \dots \infty \right)} = 2$$

$$c+1=0 \Rightarrow c=-1$$

$$a-c+1=0 \Rightarrow a+1+1=0 \Rightarrow a=-2$$

$$\left(\frac{(a-1)^2}{2} - b^2 + \frac{(c-2)}{2} \right)$$

$$\left(\frac{1}{2} \right)$$

$$\Rightarrow 9 - 2b^2 - 3 = 2 \Rightarrow 2b^2 = 4 \Rightarrow b^2 = 2$$

$$\therefore a^2 + b^2 + c^2 = 4 + 2 + 1 = 7$$

Question 8

Let $f(x) = \begin{cases} \frac{ax^2+2ax+3}{4x^2+4x-3} & , x \neq -\frac{3}{2}, \frac{1}{2} \\ b & , x = -\frac{3}{2}, \frac{1}{2} \end{cases}$ be continuous at $x = -\frac{3}{2}$. If $f \circ f(x) = \frac{7}{5}$, then x is equal to:

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Options:

A.

1.4

B.

2

C.

1

D.

0

Answer: C

Solution:

$$\lim_{x \rightarrow -\frac{3}{2}} \frac{ax^2+2ax+3}{4x^2+4x-3} = b$$

$$ax^2+2ax+3=0 \quad \text{at } n = -\frac{3}{2}$$

$$\Rightarrow \frac{9a}{4} - 3a + 3 = 0$$

$$\Rightarrow \frac{3a}{4} = 3$$

$$\Rightarrow a = 4$$

$$\lim_{x \rightarrow -\frac{3}{2}} \frac{4x^2+8ax+3}{4x^2+4x-3}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow -\frac{3}{2}} \frac{(2x+1)(2x+3)}{(2x-1)(2x+3)} \\
 &= \frac{-2}{-4} = \frac{1}{2} = b \\
 \therefore f(x) &= \frac{2x+1}{2x-1} \\
 f(f(x)) &= \frac{7}{5} \\
 \therefore \frac{2f(x)+1}{2f(x)-1} &= \frac{7}{5} \\
 \Rightarrow f(x) &= 3 \\
 \frac{2x+1}{2x-1} &= 3 \\
 \Rightarrow x &= 1
 \end{aligned}$$

Question9

If $f(x) = \begin{cases} \frac{a|x|+x^2-2(\sin|x|)(\cos|x|)}{x} & , x \neq 0 \\ b & , x = 0 \end{cases}$

is continuous at $x = 0$, then $a + b$ is equal to :

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Options:

A.

1

B.

2

C.

0

D.

4

Answer: B

Solution:

Refunctoning the given function.

$$f(x) = \begin{cases} \frac{ax+x^2-\sin 2x}{x} & , x > 0 \\ \frac{-ax+x^2+\sin 2x}{x} & , x < 0 \\ b & , x = 0 \end{cases}$$

$$RHL = \lim_{x \rightarrow 0^+} \left(a + x - 2 \frac{\sin 2x}{2x} \right) = a - 2$$

$$LHL = \lim_{x \rightarrow 0^-} \left(-a + x + 2 \frac{\sin 2x}{2x} \right) = -a + 2$$

For continuity:

$$LHL = RHL \Rightarrow a - 2 = -a + 2 \Rightarrow a = 2$$

$$b = f(0) = RHL = 2 - 2 = 0$$

$$a + b = 2 + 0 = 2$$

Question10

If the function $f(x) = \frac{e^x(e^{\tan x - x} - 1) + \log_e(\sec x + \tan x) - x}{\tan x - x}$ is continuous at $x = 0$, then the value of $f(0)$ is equal to

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Options:

A.

$$\frac{2}{3}$$

B.

$$\frac{1}{2}$$

C.

$$2$$

D.

$$\frac{3}{2}$$

Answer: D

Solution:

Given that $f(x)$ is continuous at $x = 0$ means $\lim_{x \rightarrow 0} f(x) = f(0)$

$$f(0) = \lim_{x \rightarrow 0} \frac{e^x(e^{\tan x - x} - 1) + \log_e(\sec x + \tan x) - x}{\tan x - x}$$

$$f(0) = \lim_{x \rightarrow 0} \frac{e^x(e^{\tan x - x} - 1)}{\tan x - x} + \lim_{x \rightarrow 0} \frac{\log_e(\sec x + \tan x) - x}{\tan x - x}$$

Using standard limit $\frac{e^{f(x)} - 1}{f(x)} = 1$ if $f(x) \rightarrow 0$

$$\text{Let } f(x) = \tan x - x, \text{ then } \lim_{x \rightarrow 0} \frac{e^{\tan x - x} - 1}{\tan x - x} = 1$$

Second limit is $\frac{0}{0}$ form use L'Hopital Rule

$$\begin{aligned} f(0) &= 1 + \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\log(\sec x + \tan x) - x)}{\frac{d}{dx}(\tan x - x)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{\frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) - 1}{\sec^2 x - 1} \\ &= 1 + \lim_{x \rightarrow 0} \frac{\sec x \tan x + \sec^2 x - \sec x - \tan x}{(\sec x + \tan x)(\sec^2 x - 1)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{\tan x(\sec x - 1) + \sec x(\sec x - 1)}{(\sec x + \tan x)(\sec^2 x - 1)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{(\sec x - 1)(\sec x + \tan x)}{(\sec x + \tan x)(\sec^2 x - 1)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{1}{\sec x + 1} \\ \Rightarrow f(0) &= 1 + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

Question 11

Let $\alpha, \beta \in \mathbb{R}$ be such that the function $f(x) = \begin{cases} 2\alpha(x^2 - 2) + 2\beta x & , x < 1 \\ (\alpha + 3)x + (\alpha - \beta) & , x \geq 1 \end{cases}$ be differentiable at all $x \in \mathbb{R}$. Then $34(\alpha + \beta)$ is equal to

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Options:

A.

48

B.

84

C.

36

D.

24

Answer: A

Solution:

$$f(x) = \begin{cases} 2\alpha(x^2 - 2) + 2\beta x, & x < 1 \\ (\alpha + 3)x + (\alpha - \beta), & x \geq 1 \end{cases}$$

Given $f(x)$ is differentiable at all $x \in \mathbb{R}$.

This means $f(x)$ is continuous and differentiable at $x = 1$.

1. Continuity at $x = 1$

$$\text{L.H.L} = \text{R.H.L} = f(1)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$$\lim_{x \rightarrow 1^-} [2\alpha(x^2 - 2) + 2\beta x] = \lim_{x \rightarrow 1^+} [(\alpha + 3)x + (\alpha - \beta)]$$

$$2\alpha(1 - 2) + 2\beta(1) = \alpha + 3 + \alpha - \beta$$

$$-2\alpha + 2\beta = 2\alpha - \beta + 3$$

$$3\beta - 4\alpha = 3 - (1)$$

2. Differentiability at $x = 1$

$$f'(x) = \begin{cases} 2\alpha(2x) + 2\beta, & x < 1 \\ \alpha + 3, & x > 1 \end{cases}$$

$$\text{L.H.D} = \text{R.H.D}$$

$$f'(1^-) = f'(1^+)$$

$$2\alpha(2 \cdot 1) + 2\beta = \alpha + 3$$

$$3\alpha + 2\beta = 3 - (2)$$

Solving equation (1) and (2)

$$\text{eqn (1)} \times 2 - \text{eqn (2)} \times 3$$

$$-8\alpha - 9\alpha = -3 \implies \alpha = \frac{3}{17}$$

Substitute $\alpha = \frac{3}{17}$ in equation (1):

$$3\beta - 4 \times \frac{3}{17} = 3$$

$$3\beta = 3 + \frac{12}{17} = \frac{63}{17}$$

$$\beta = \frac{21}{17}$$

$$\text{Value of } 34(\alpha + \beta) = 34\left(\frac{3}{17} + \frac{21}{17}\right) = 6 + 42 = 48$$

Question 12

Let $[t]$ denote the greatest integer less than or equal to t . If the function

$$f(x) = \begin{cases} b^2 \sin\left(\frac{\pi}{2} \left[\frac{\pi}{2}(\cos x + \sin x) \cos x\right]\right), & x < 0 \\ \frac{\sin x - \frac{1}{2} \sin 2x}{x^3}, & x > 0 \\ a, & x = 0 \end{cases}$$

is continuous at $x = 0$, then $a^2 + b^2$ is equal to :

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Options:

A.

$$\frac{1}{2}$$

B.

$$\frac{5}{8}$$

C.

$$\frac{3}{4}$$

D.

$$\frac{9}{16}$$

Answer: C

Solution:

It is given that $f(x)$ is continuous at $x = 0$

This mean at $x = 0$

$$\text{L.H.L} = \text{R.H.L} = f(0) = a$$

Calculating L.H.L using the definition of $f(x)$ for $x < 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} b^2 \sin\left(\frac{\pi}{2} \left[\frac{\pi}{2}(\cos x + \sin x) \cos x\right]\right) = a$$

$$\Rightarrow \lim_{x \rightarrow 0^-} b^2 \sin\left(\frac{\pi}{2} \left[\frac{\pi}{2}(\cos^2 x + \frac{\sin 2x}{2})\right]\right) = a$$

$$\cos^2(0^-) = 1^- \text{ Less than } 1$$

$$\sin 2(0^-) = 0^- \text{ Less than } 0$$

$$\Rightarrow a = b^2 \sin\left(\frac{\pi}{2} \left[\frac{\pi}{2} \left[\frac{3.14}{2}(1^- + 0^-)\right]\right]\right)$$

$$\Rightarrow a = b^2 \sin\left(\frac{\pi}{2} [1.57(1^-)]\right)$$

$$\Rightarrow a = b^2 \sin\left(\frac{\pi}{2} [1.57^-]\right), [1.57^-] = 1\{[t] \rightarrow \text{G.I.F}\}$$

$$\Rightarrow a = b^2 \sin\left(\frac{\pi}{2} \times 1\right) = b^2$$

Calculating R.H.L using the function definition for $x > 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sin x - \frac{1}{2} \sin 2x}{x^2} = a$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{\sin x - \frac{1}{2} \times 2 \sin x \cos x}{x^2} = a$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{\sin x(1 - \cos x)}{x \cdot x^2} = a$$

$$\text{Using limits result } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Putting these result

$$\text{Using limits result } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Putting these result

$$a = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\text{So } a^2 + b^2 = a^2 + a \{b^2 = a \text{ from } L.H.L\}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Question 13

Let $y = y(x)$ be a differentiable function in the interval $(0, \infty)$ such that $y(1) = 2$, and $\lim_{t \rightarrow x} \left(\frac{t^2 y(x) - x^2 y(t)}{x - t} \right) = 3$ for each $x > 0$. Then $2y(2)$ is equal to :

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Options:

A.

27

B.

18

C.

23

D.

12

Answer: C

Solution:

$$\lim_{t \rightarrow x} \frac{t^2 y(x) - x^2 y(t)}{x - t} = 3$$

Substituting

$$t \rightarrow x + h \Rightarrow h \rightarrow 0$$

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 y(x) - x^2 y(x+h)}{-h} = 3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{x^2 y(x) + h^2 y(x) + 2xhy(x) - x^2 y(x+h)}{-h} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-x^2[y(x+h) - y(x)] + y(x)(h^2 + 2xh)}{-h} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{x^2[y(x+h) - y(x)]}{h} - \lim_{h \rightarrow 0} \frac{y(x)(h^2 + 2xh)}{h} = 3$$

Use

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$x^2 y'(x) - 2xy(x) = 3$$

Equation (1) is differential equation can be written by replacing

$$y'(x) = \frac{dy}{dx} \quad \text{and} \quad y(x) = y$$

$$\Rightarrow x^2 \frac{dy}{dx} - 2xy = 3$$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2} \quad (\text{Linear differential equation})$$

Integrating factor

$$\text{I.F} = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$$

$\therefore$ Solution of linear differential equation

$$y \cdot (\text{I.F}) = \int (\text{I.F}) \cdot \frac{3}{x^2} dx$$

$$\Rightarrow y \cdot \frac{1}{x^2} = \int \frac{1}{x^2} \cdot \frac{3}{x^2} dx = \int \frac{3}{x^4} dx$$

$$\Rightarrow \frac{y}{x^2} = 3 \times \frac{x^{-4+1}}{-4+1} = -\frac{1}{x^3} + C$$

$$\Rightarrow y = -\frac{1}{x} + Cx^2 \text{ (General solution)}$$

Apply initial condition

$$y(1) = 2$$

$$\Rightarrow 2 = -\frac{1}{1} + C(1) \Rightarrow C = 3$$

$$\text{So particular solution is : } y(x) = -\frac{1}{x} + 3x^2$$

$$\therefore y(2) = -\frac{1}{2} + 3(2)^2 = 12 - \frac{1}{2} = \frac{23}{2}$$

$$\Rightarrow 2y(2) = 23$$

Question 14

The value of

$$\lim_{x \rightarrow 0} \frac{\log_e (\sec(ex) \cdot \sec(e^2x) \cdots \sec(e^{10}x))}{e^2 - e^{2 \cos x}}$$

is equal to

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Options:

A.

$$\frac{(e^{10}-1)}{2e^2(e^2-1)}$$

B.

$$\frac{(e^{20}-1)}{2e^2(e^2-1)}$$

C.

$$\frac{(e^{10}-1)}{2(e^2-1)}$$

D.

$$\frac{(e^{20}-1)}{2(e^2-1)}$$

Answer: D

Solution:

$$\text{Let } L = \lim_{x \rightarrow 0} \frac{\ln(\sec(ex) \cdot \sec(e^2x) \cdots \sec(e^{10}x))}{e^2 - e^{2 \cos x}}$$

Simplify : we use $\ln(abc \dots) = \ln(a) + \ln(b) + \ln(c) \dots$ and take e^2 common in denominator

$$L = \lim_{x \rightarrow 0} \frac{\ln \sec(ex) + \ln \sec(e^2x) + \dots + \ln \sec(e^{10}x)}{e^2 [1 - e^{2 \cos x - 2}]} \dots \dots (1)$$

Now understanding dominating term

- If $A \rightarrow 1 \Rightarrow \ln A \approx A - 1$
- as $x \rightarrow 0 \Rightarrow \sec(ex) \rightarrow 1 \Rightarrow \ln \sec(ex) \approx \sec(ex) - 1$
- so generalized $x \rightarrow 0 \Rightarrow \ln \sec(e^kx) \approx \sec(e^kx) - 1$
- If $A \rightarrow 0 \Rightarrow e^A - 1 \approx A$
- so in denominator as $x \rightarrow 0 \Rightarrow 2(\cos x - 1) = 0$
- so, $e^{2(\cos x - 1)} - 1 \approx 2(\cos x - 1)$

Substituting this dominating term in equation (1)

$$L = \lim_{x \rightarrow 0} \frac{(\sec(ex) - 1) + (\sec(e^2x) - 1) + \dots + (\sec(e^{10}x) - 1)}{-e^2 \cdot 2(\cos x - 1)}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\cos(ex)} - 1\right) + \left(\frac{1}{\cos(e^2x)} - 1\right) + \dots + \left(\frac{1}{\cos(e^{10}x)} - 1\right)}{2e^2(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos(ex)}{\cos(ex)} + \frac{1 - \cos(e^2x)}{\cos(e^2x)} + \dots + \frac{1 - \cos(e^{10}x)}{\cos(e^{10}x)}}{2e^2(1 - \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos(ex)}{\cos(ex) \cdot 2e^2(1 - \cos x)} + \lim_{x \rightarrow 0} \frac{1 - \cos(e^2x)}{\cos(e^2x) \cdot 2e^2(1 - \cos x)} + \dots + \frac{1 - \cos(e^{10}x)}{\cos(e^{10}x) \cdot 2e^2(1 - \cos x)} \dots \dots (2)$$

Again using dominating term

at $x \rightarrow 0$, $\cos(e^kx) = 1$, $k = \{1, 2, \dots, 10\}$

$$L = \lim_{x \rightarrow 0} \frac{1 - \cos(ex)}{2e^2(1 - \cos x)} + \lim_{x \rightarrow 0} \frac{1 - \cos(e^2x)}{2e^2(1 - \cos x)} + \dots + \frac{1 - \cos(e^{10}x)}{2e^2(1 - \cos x)}$$

Understanding dominating term again

If $A \rightarrow 0 \Rightarrow 1 - \cos A \approx \frac{A^2}{2}$ So $1 - \cos(e^kx) \approx \frac{(e^kx)^2}{2}$, $k \in \{1, 2, \dots, 10\}$

Substitute this value in equation (2) :

$$L = \frac{\lim_{x \rightarrow 0} \frac{(ex)^2}{2}}{2e^2 \cdot \frac{x^2}{2}} + \frac{\lim_{x \rightarrow 0} \frac{(e^2x)^2}{2}}{2e^2 \cdot \frac{x^2}{2}} + \dots + \frac{\lim_{x \rightarrow 0} \frac{(e^{10}x)^2}{2}}{2e^2 \cdot \frac{x^2}{2}}$$

x^2 and $\frac{1}{2}$ get cancelled so L becomes :

$$L = \frac{e^2}{2e^2} + \frac{(e^2)^2}{2e^2} + \dots + \frac{(e^{10})^2}{2e^2}$$

take $\frac{1}{2e^2}$ common $L = \frac{1}{2e^2} [e^2 + e^4 + e^6 + \dots + e^{20}]$

$e^2 + e^4 + \dots + e^{20}$ is a G.P. whose first term is e^2 common ratio is e^2 so sum of G.P. is $a \frac{(r^n - 1)}{r - 1}$ where a is first term and r is common ratio.

$$L = \frac{1}{2e^2} \times e^2 \left[\frac{(e^2)^{10} - 1}{e^2 - 1} \right] L = \frac{e^{20} - 1}{2(e^2 - 1)}$$

Question 15

Let $f(x) = \lim_{\theta \rightarrow 0} \left(\frac{\cos \pi x - x^{\left(\frac{2}{\theta}\right)} \sin(x-1)}{1 + x^{\left(\frac{2}{\theta}\right)}(x-1)} \right)$, $x \in \mathbb{R}$. Consider the following two statements :

(I) $f(x)$ is discontinuous at $x = 1$.

(II) $f(x)$ is continuous at $x = -1$.

Then,

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Options:

A.

Neither (I) nor (II) is True

B.

Only (II) is True

C.

Only (I) is True

D.

Both (I) and (II) are True

Answer: A

Solution:

The function is defined as :

$$f(x) = \lim_{\theta \rightarrow 0} \left(\frac{\cos \pi x - x^{\left(\frac{2}{\theta}\right)} \sin(x-1)}{1 + x^{\left(\frac{2}{\theta}\right)} (x-1)} \right)$$

Let $n = \frac{2}{\theta}$. As $\theta \rightarrow 0$, $n \rightarrow \infty$ (specifically, let's consider θ approaching 0 such that the power grows). The behavior of the function depends on the value of x because of the term x^n .

- **Case 1 :** $|x| < 1$ As $n \rightarrow \infty$, $x^n \rightarrow 0$.

$$f(x) = \frac{\cos \pi x - 0}{1 + 0} = \cos \pi x$$

- **Case 2 :** $|x| > 1$ As $n \rightarrow \infty$, $x^n \rightarrow \infty$. We divide the numerator and denominator by x^n :

$$f(x) = \lim_{n \rightarrow \infty} \frac{\frac{\cos \pi x}{x^n} - \sin(x-1)}{\frac{1}{x^n} + (x-1)} = \frac{0 - \sin(x-1)}{0 + (x-1)} = \frac{-\sin(x-1)}{x-1}$$

- **Case 3 :** $x = 1$ If $x = 1$, then $\cos(\pi) = -1$ and $\sin(0) = 0$.

$$f(1) = \frac{-1 - (1^n \cdot 0)}{1 + (1^n \cdot 0)} = -1$$

- **Case 4 :** $x = -1$ If $x = -1$, then $\cos(-\pi) = -1$ and $\sin(-2) = -\sin(2)$. Since $n = 2/\theta$ is even, $(-1)^n = 1$.

$$f(-1) = \frac{-1 - (1 \cdot \sin(-2))}{1 + (1 \cdot -2)} = \frac{-1 + \sin 2}{-1} = 1 - \sin 2$$

So $f(x)$ become

$$f(x) = \begin{cases} \frac{-\sin(x-1)}{x-1} & \text{if } x \in (-\infty, -1) \cup (1, \infty) \\ \cos \pi x & \text{if } x \in (-1, 1) \\ 0 & \text{if } x = 1 \\ 1 - \sin 2 & \text{if } x = -1 \end{cases}$$

Analyze the Statements

Statement (I) : $f(x)$ is discontinuous at $x = 1$.

- Value at $x = 1$: $f(1) = -1$.
- Left-hand limit ($x \rightarrow 1^-$) : Uses the $|x| < 1$ definition: $\lim_{x \rightarrow 1^-} \cos \pi x = \cos \pi = -1$.
- Right-hand limit ($x \rightarrow 1^+$) : Uses the $|x| > 1$ definition: $\lim_{x \rightarrow 1^+} \frac{-\sin(x-1)}{x-1}$. Using the standard limit $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, this equals -1. Since $LHL = RHL = f(1)$, the function is continuous at $x = 1$.

Conclusion : Statement (I) is False.

Statement (II) : $f(x)$ is continuous at $x = -1$.

- Value at $x = -1$: $f(-1) = 1 - \sin 2$.
- Left-hand limit ($x \rightarrow -1^-$) : Uses $|x| > 1$ definition : $\frac{-\sin(-1-1)}{-1-1} = \frac{-\sin(-2)}{-2} = \frac{\sin 2}{-2}$.
- Right-hand limit ($x \rightarrow -1^+$) : Uses $|x| < 1$ definition : $\cos(-\pi) = -1$.

Since the limits do not match, the function is discontinuous at $x = -1$.

Conclusion : Statement (II) is False.

Question 16

If $\lim_{x \rightarrow 2} \frac{\sin(x^3 - 5x^2 + ax + b)}{(\sqrt{x-1} - 1) \log_e(x-1)} = m$, then $a + b + m$ is equal to :

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Options:

- A.
5
B.

6

C.

8

D.

10

Answer: B**Solution:**

$$\therefore \text{Denominator} = 0 \text{ at } x = 2$$

$$\Rightarrow \text{Numerator} = 0 \text{ at } x = 2$$

$$2^3 - 5(2)^2 + a(2) + b = 0$$

$$\Rightarrow 2a + b = 12 \quad \dots (1)$$

$$\therefore m = \lim_{x \rightarrow 2} \frac{\frac{\sin(x^3 - 5x^2 + ax + b)}{x^3 - 5x^2 + ax + b} \cdot \frac{\log_e(1 + (x-2))}{x-2}}{\left(\frac{x-1-1}{\sqrt{x-1}+1}\right) \cdot \frac{\log_e(1 + (x-2))}{x-2}}$$

$$\Rightarrow m = \lim_{x \rightarrow 2} 2 \left(\frac{x^3 - 5x^2 + ax + b}{(x-2)^2} \right)$$

$$\Rightarrow m = \lim_{x \rightarrow 2} 2 \left(\frac{3x^2 - 10x + a}{2(x-2)} \right)$$

$$\therefore \text{Denominator} = 0 \text{ at } x = 2$$

$$\Rightarrow \text{Numerator} = 0 \text{ at } x = 2$$

$$\Rightarrow 3(2)^2 - 10(2) + a = 0 \rightarrow a = 8$$

$$\text{put } a = 8 \text{ in (1), we get } b = -4$$

$$\therefore m = \lim_{x \rightarrow 2} 2 \left(\frac{6x-10}{2} \right) \rightarrow m = 2$$

$$\therefore a + b + m = 6$$

Question 17

The product of all possible values of α , for which

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos(\alpha x) \cos((\alpha+1)x) \cos((\alpha+2)x)}{\sin^2((\alpha+1)x)} \right) = 2, \text{ is :}$$

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Options:

A.

-2

B.

1

C.

-1

D.

 $\frac{5}{4}$ **Answer: C**

Solution:

$$\lim_{x \rightarrow 0} \frac{1 - \cos \alpha x \cdot \cos((\alpha+1)x) \cos((\alpha+2)x)}{(\alpha+1)^2 x^2}$$

By using L.H. Rule we get

$$\Rightarrow \frac{1}{2}(\alpha+1)^2 [\alpha^2 + (\alpha+1)^2 + (\alpha+2)^2] = 2$$

$$\alpha^2 + (\alpha+1)^2 + (\alpha+2)^2 = 4(\alpha+1)^2$$

$$\alpha^2 + (\alpha+2)^2 = 3(\alpha+1)^2$$

$$\Rightarrow \alpha^2 + 2\alpha - 1 = 0$$

$$\text{Product} = -1$$

Question 18

Let $f(x)$ and $g(x)$ be twice differentiable functions satisfying $f''(x) = g''(x)$ for all $x \in \mathbf{R}$, $f'(1) = 2g'(1) = 4$ and $g(2) = 3f(2) = 9$. Then $f(25) - g(25)$ is equal to :

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Options:

A.

20

B.

40

C.

-20

D.

-40

Answer: B

Solution:

Let us define a new function $h(x)$ taking the difference between $f(x)$ and $g(x)$.

$$h(x) = f(x) - g(x)$$

Differentiating this function twice with respect to x , we get:

$$h''(x) = f''(x) - g''(x)$$

We are given that $f''(x) = g''(x)$ for all $x \in \mathbf{R}$. Therefore, their difference is zero:

$$h''(x) = 0$$

Now, let us integrate both sides with respect to x to find $h'(x)$:

$$h'(x) = \int 0 \, dx = c_1$$

where c_1 is a constant of integration.

We are given the values of the first derivatives at $x = 1$. Let us simplify them:

$$f'(1) = 4$$

$$2g'(1) = 4 \implies g'(1) = 2$$

Using these values, we can find the exact value of $h'(1)$:

$$h'(1) = f'(1) - g'(1) = 4 - 2 = 2$$

Since $h'(x) = c_1$ for all values of x , it follows that:

$$c_1 = 2$$

Substituting this back, we get our first derivative equation:

$$h'(x) = 2$$

Next, we need the original function $h(x)$. We find it by integrating $h'(x)$ with respect to x :

$$h(x) = \int 2 \, dx = 2x + c_2$$

where c_2 is another constant of integration.

We are given the values of the functions at $x = 2$. Let us simplify these values as well:

$$g(2) = 9$$

$$3f(2) = 9 \implies f(2) = 3$$

Using these values, we can find the value of $h(2)$:

$$h(2) = f(2) - g(2) = 3 - 9 = -6$$

Now, substitute $x = 2$ and $h(2) = -6$ into our integrated equation for $h(x)$:

$$h(2) = 2(2) + c_2$$

$$-6 = 4 + c_2$$

$$c_2 = -6 - 4$$

$$c_2 = -10$$

Substituting the value of c_2 back into the equation for $h(x)$, we have the completed function rule:

$$h(x) = 2x - 10$$

The question asks for the value of $f(25) - g(25)$, which is represented directly by $h(25)$. Let us substitute $x = 25$ into our function $h(x)$:

$$h(25) = 2(25) - 10$$

$$h(25) = 50 - 10$$

$$h(25) = 40$$

Therefore, $f(25) - g(25) = 40$.

The correct answer is Option B.

Question 19

Let $f(x) = \lim_{y \rightarrow 0} \frac{(1 - \cos(xy)) \tan(xy)}{y^3}$. Then the number of solutions of the equation $f(x) = \sin x$, $x \in \mathbb{R}$ is :

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Options:

A.

0

B.

2

C.

3

D.

1

Answer: C

Solution:

$$f(x) = \lim_{y \rightarrow 0} \frac{1 - \cos(xy)}{(xy)^2} \cdot \frac{\tan xy}{xy} \cdot \frac{(xy)^3}{y^3}$$

$$f(x) = \frac{x^3}{2}$$

$$f(x) = \sin x$$

$$\frac{x^3}{2} = \sin x$$

3 points of intersection

Question20

The value of $\lim_{x \rightarrow 0} \left(\frac{x^2 \sin^2 x}{x^2 - \sin^2 x} \right)$ is :

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Options:

A.

2

B.

3

C.

4

D.

6

Answer: B

Solution:

$$\lim_{x \rightarrow 0} \frac{x^2 \sin^2 x}{x^2 - \sin^2 x}$$

Applying expansion

$$\lim_{x \rightarrow 0} \frac{x^2 \sin^2 x}{x^2 - \left(x - \frac{x^3}{3!} + \dots\right)^2}$$

$$\lim_{x \rightarrow 0} \frac{x^2 \sin^2 x}{x^2 - x^2 + \frac{2x^4}{3!} \dots} = \frac{3!}{2} = 3$$

Question21

For the function $f(x) = e^{\sin|x|} - |x|$, $x \in \mathbf{R}$, consider the following statements :

Statement I : f is differentiable for all $x \in \mathbf{R}$.

Statement II : f is increasing in $\left(-\pi, -\frac{\pi}{2}\right)$.

In the light of the above statements, choose the correct answer from the options given below :

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Options:

A.

Both Statement I and Statement II are true

B.

Both Statement I and Statement II are false

C.

Statement I is true but Statement II is false

D.

Statement I is false but Statement II is true

Answer: D

Solution:

Here $f(x) = e^{\sin|x|} - |x|$. Since $|x|$ is not differentiable at $x = 0$, and also $\sin|x|$ has a “kink” at $x = 0$, the function f is not differentiable at $x = 0$. So, **Statement I is false**.

Now take $x \in (-\pi, -\frac{\pi}{2})$. Here $x < 0$, so $|x| = -x$ and hence $\sin|x| = \sin(-x) = -\sin x$. Therefore, on this interval we can write $f(x) = e^{-\sin x} + x$ and differentiate to check whether f is increasing.

$$f(x) = e^{-\sin x} + x$$

$$f'(x) = -e^{-\sin x} \cos x + 1$$

$$\text{for } x \in \left(-\pi, -\frac{\pi}{2}\right) \quad \cos x < 0$$

$$\therefore f'(x) > 0$$

$$\therefore f(x) \text{ is increasing}$$

Since $\cos x < 0$ on $(-\pi, -\frac{\pi}{2})$ and $e^{-\sin x} > 0$, we get $-e^{-\sin x} \cos x > 0$, so $f'(x) > 0$ on this interval. Hence, **Statement II is true**.

Question 22

Let the function,

$$f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ a^2 + bx, & x \geq 1 \end{cases}$$

be differentiable for all $x \in \mathbf{R}$, where $a > 1$, $b \in \mathbf{R}$. If the area of the region enclosed by $y = f(x)$ and the line $y = -20$ is $\alpha + \beta\sqrt{3}$, $\alpha, \beta \in \mathbf{Z}$, then the value of $\alpha + \beta$ is _____.

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Answer: 34

Solution:

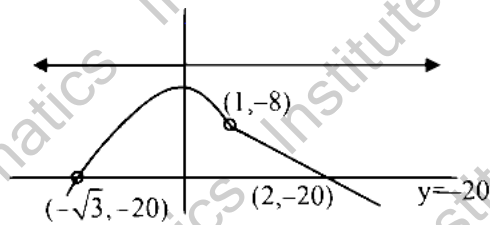
$f(x)$ is continuous and differentiable

$$\text{at } x = 1; \quad \text{LHL} = \text{RHL}, \text{LHD} = \text{RHD}$$

$$-3a - 2 = a^2 + b, -6a = b$$

$$a = 2, 1; b = -12$$

$$f(x) = \begin{cases} -6x^2 - 2, & x < 1 \\ 4 - 12x, & x \geq 1 \end{cases}$$



$$\text{Area} = \int_{-\sqrt{3}}^1 (-6x^2 - 2 + 20) dx + \int_1^2 (4 - 12x + 20) dx$$

$$= 16 + 12\sqrt{3} + 6 = 22 + 12\sqrt{3}$$

$$\therefore \alpha + \beta = 34$$

Question 23

$$\text{Let } f(x) = \begin{cases} 3x, & x < 0 \\ \min\{1 + x + [x], x + 2[x]\}, & 0 \leq x \leq 2 \\ 5, & x > 2 \end{cases}$$

where $[.]$ denotes greatest integer function. If α and β are the number of points, where f is not continuous and is not differentiable, respectively, then $\alpha + \beta$ equals _____.

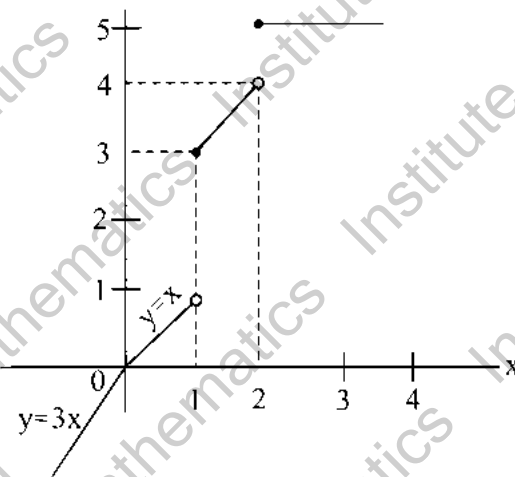
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Answer: 5

Solution:

$$f(x) = \begin{cases} 3x & ; x < 0 \\ \min\{1 + x, x\} & ; 0 \leq x < 1 \\ \min\{2 + x, x + 2\} & ; 1 \leq x < 2 \\ 5 & ; x > 2 \end{cases}$$

$$f(x) = \begin{cases} 3x & ; x < 0 \\ x & ; 0 \leq x < 1 \\ x + 2 & ; 1 \leq x < 2 \\ 5 & ; x > 2 \end{cases}$$



Not continuous at $x \in \{1, 2\} \Rightarrow \alpha = 2$

Not diff. at $x \in \{0, 1, 2\} \Rightarrow \beta = 3$

$$\alpha + \beta = 5$$

Question 24

Let $f(x) = \lim_{n \rightarrow \infty} \sum_{r=0}^n \left(\frac{\tan(x/2^{r+1}) + \tan^3(x/2^{r+1})}{1 - \tan^2(x/2^{r+1})} \right)$. Then $\lim_{x \rightarrow 0} \frac{e^x - e^{f(x)}}{(x - f(x))}$ is equal to _____.

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Answer: 1

Solution:

$$f(x) = \lim_{n \rightarrow \infty} \sum_{r=0}^n \left(\tan \frac{x}{2^r} - \tan \frac{x}{2^{r+1}} \right) = \tan x$$
$$\lim_{x \rightarrow 0} \left(\frac{e^x - e^{\tan x}}{x - \tan x} \right) = \lim_{x \rightarrow 0} e^{\tan x} \frac{(e^{x - \tan x} - 1)}{(x - \tan x)}$$
$$= 1$$

Question 25

Let $[t]$ be the greatest integer less than or equal to t . Then the least value of $p \in \mathbb{N}$ for which

$$\lim_{x \rightarrow 0^+} \left(x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{p}{x} \right] \right) - x^2 \left(\left[\frac{1}{x^2} \right] + \left[\frac{2^2}{x^2} \right] + \dots + \left[\frac{p^2}{x^2} \right] \right) \right) \geq 1$$
 is equal to _____.

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Answer: 24

Solution:

To find the least natural number p for which the following inequality holds:

$$\lim_{x \rightarrow 0^+} \left(x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{p}{x} \right] \right) - x^2 \left(\left[\frac{1}{x^2} \right] + \left[\frac{2^2}{x^2} \right] + \dots + \left[\frac{p^2}{x^2} \right] \right) \right) \geq 1$$

we simplify the expression inside the limit.

As $x \rightarrow 0^+$, $\left[\frac{n}{x} \right]$ approximates to $\frac{n}{x}$. Thus, the problem becomes finding:

$$(1 + 2 + \dots + p) - (1^2 + 2^2 + \dots + p^2) \geq 1$$

The sum of the first p natural numbers is given by:

$$\frac{p(p+1)}{2}$$

And the sum of the squares of the first 9 natural numbers is:

$$1^2 + 2^2 + \dots + 9^2 = \frac{9 \cdot 10 \cdot 19}{6}$$

Thus, the inequality becomes:

$$\frac{p(p+1)}{2} - \frac{9 \cdot 10 \cdot 19}{6} \geq 1$$

Solving this, we rewrite:

$$p(p+1) \geq 572$$

The least natural number p satisfying this condition is 24.

Question 26

If $\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = p$, then $96 \log_e p$ is equal to _____

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Answer: 32

Solution:

To solve the given limit problem, we start by analyzing the expression:

$$\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = p$$

This limit exhibits the indeterminate form 1^∞ . To handle this form, we use the transformation:

$$p = e^{\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} - 1 \right) \frac{1}{x^2}}$$

Expanding $\tan x$ using its Taylor series near $x = 0$, we have:

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

Substituting this expansion into the limit, we get:

$$\frac{\tan x - x}{x^3} = \frac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots - x \right)}{x^3} = \frac{\frac{x^3}{3} + \frac{2}{15}x^5 + \dots}{x^3}$$

This simplifies to:

$$\frac{x^3}{3x^3} = \frac{1}{3}$$

Thus, the limit becomes:

$$p = e^{\frac{1}{3}}$$

Therefore, the expression for $\log_e p$ is:

$$\log_e p = \frac{1}{3}$$

Finally, computing $96 \log_e p$:

$$96 \log_e p = 96 \cdot \frac{1}{3} = 32$$

Question 27

Let m and n be the number of points at which the function $f(x) = \max \{x, x^3, x^5, \dots, x^{21}\}$, $x \in \mathbb{R}$, is not differentiable and not continuous, respectively. Then $m + n$ is equal to _____.

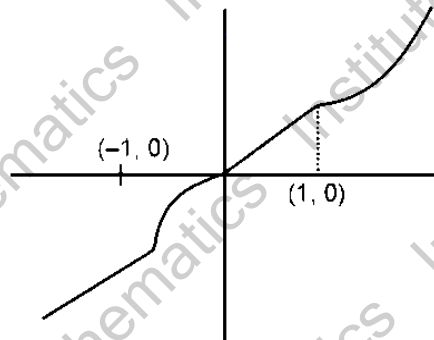
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Answer: 3

Solution:

for $x \geq 1$, $x^{21} \geq x^{19} \geq \dots \geq x$.

$$f(x) = \begin{cases} x & x < -1 \\ x^{21} & -1 \leq x \leq 0 \\ x & 0 < x < 1 \\ x^{21} & x \geq 1 \end{cases}$$



Clearly, $f(x)$ is continuous everywhere.

$$\Rightarrow \alpha = 0$$

$$f'(x) = \begin{cases} 1 & ; x < -1 \\ 21x^{20} & ; -1 \leq x \leq 0 \\ 1 & ; 0 < x < 1 \\ 21 \cdot x^{20} & ; x \geq 1 \end{cases}$$

$$\Rightarrow \beta = 3$$

$$\Rightarrow \alpha + \beta = 3$$

Question28

The number of points of discontinuity of the function $f(x) = \left[\frac{x^2}{2} \right] - [\sqrt{x}]$, $x \in [0, 4]$, where $[\cdot]$ denotes the greatest integer function, is _____.

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Answer: 8

Solution:

To determine the points of discontinuity of the function $f(x) = \left[\frac{x^2}{2} \right] - [\sqrt{x}]$, where $[\cdot]$ denotes the greatest integer function, we need to identify possible values of x where discontinuities might occur within the interval $[0, 4]$.

Discontinuity Analysis

For the term $\left[\frac{x^2}{2} \right]$:

The probable values of x that could cause discontinuities are the roots or specific values where the integer part changes between consecutive integers. The transitions happen when:

$$= 1, 2, 3, 4, 5, 6, 7, 8$$

$$\Rightarrow x = \sqrt{2}, 2, \sqrt{6}, 2\sqrt{2}, \sqrt{10}, 2\sqrt{3}, \sqrt{14}, 4$$

For the term $[\sqrt{x}]$:

The values of x where $[\sqrt{x}]$ changes are straightforward. They occur at:

$$x = 1, 2$$

Discontinuity Check

By evaluating $f(x)$ at all these potential points, we find the function is indeed discontinuous at:

$$x = 1, \sqrt{2}, 2, \sqrt{6}, 2\sqrt{2}, \sqrt{10}, 2\sqrt{3}, \sqrt{14}$$

Thus, the function $f(x)$ has 8 discontinuities on the interval $[0, 4]$.

Question29

For $t > -1$, let α_t and β_t be the roots of the equation

$$((t+2)^{1/7} - 1)x^2 + ((t+2)^{1/6} - 1)x + ((t+2)^{1/21} - 1) = 0. \text{ If } \lim_{t \rightarrow -1^+} \alpha_t = a \text{ and } \lim_{t \rightarrow -1^+} \beta_t = b,$$

then $72(a+b)^2$ is equal to _____.

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Answer: 98

Solution:

$$a + b = \lim_{t \rightarrow -1^+} (\alpha + \beta) = \lim_{t \rightarrow -1^+} -\frac{(t+2)^{\frac{1}{7}} - 1}{(t+2)^{\frac{1}{7}} - 1}$$

$$\text{let } t+2 = y$$

$$a + b = \lim_{y \rightarrow 1^+} \frac{y^{1/6} - 1}{y^{1/7} - 1} = \frac{7}{6}$$

$$72(a+b)^2 = 72 \frac{49}{36} = 98$$

Question30

If the function $f(x) = \frac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$ is continuous at $x = 0$, then $f(0)$ is equal to _____.

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Answer: 2

Solution:

$$\lim_{x \rightarrow 0} \frac{\frac{\tan(\tan x) - \tan x}{\tan^3 x} + \frac{\sin x - \sin(\sin x)}{\sin^3 x}}{\frac{\tan x - \sin x}{x^3}}$$

$$= \frac{\frac{1}{3} + \frac{1}{2} + \frac{1}{6}}{\frac{1}{2}} = 2$$

Question31

If $\sum_{r=1}^n T_r = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{64}$, then $\lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{T_r} \right)$ is equal to :

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Options:

A. $\frac{2}{3}$

B. $\frac{1}{3}$

C. 1

D. 0

Answer: A

Solution:

$$T_n = S_n - S_{n-1}$$

$$\Rightarrow T_n = \frac{1}{8}(2n-1)(2n+1)(2n+3)$$

$$\Rightarrow \frac{1}{T_n} = \frac{8}{(2n-1)(2n+1)(2n+3)}$$

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{T_r} = \lim_{n \rightarrow \infty} 8 \sum_{r=1}^n \frac{1}{(2r-1)(2r+1)(2r+3)}$$

$$= \lim_{n \rightarrow \infty} \frac{8}{4} \sum \left(\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)} \right)$$

$$= \lim_{n \rightarrow \infty} 2 \left[\left(\frac{1}{1.3} - \frac{1}{3.5} \right) + \left(\frac{1}{3.5} - \frac{1}{5.7} \right) + \dots \right]$$

$$= \frac{2}{3}$$

Question 32

If $\lim_{x \rightarrow \infty} \left(\left(\frac{e}{1-e} \right) \left(\frac{1}{e} - \frac{x}{1+x} \right) \right)^x = \alpha$, then the value of $\frac{\log_e \alpha}{1+\log_e \alpha}$ equals :

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Options:

A. e^{-2}

B. e^2

C. e

D. e^{-1}

Answer: C

Solution:

$$\alpha = \lim_{x \rightarrow \infty} \left[\left(\frac{e}{1-e} \right) \left(\frac{1}{e} - \frac{x}{1+x} \right) \right]^x.$$

Begin by rewriting the expression inside the limit. Notice that

$$\frac{x}{1+x} = 1 - \frac{1}{1+x},$$

so

$$\frac{1}{e} - \frac{x}{1+x} = \frac{1}{e} - \left(1 - \frac{1}{1+x} \right) = \frac{1}{e} - 1 + \frac{1}{1+x}.$$

Express the constant part as

$$1 - \frac{1}{e} = \frac{e-1}{e},$$

to obtain

$$\frac{1}{e} - \frac{x}{1+x} = -\frac{e-1}{e} + \frac{1}{1+x}.$$

Multiplying by the prefactor, we have

$$\left(\frac{e}{1-e} \right) \left(\frac{1}{e} - \frac{x}{1+x} \right) = \frac{e}{1-e} \left[-\frac{e-1}{e} + \frac{1}{1+x} \right].$$

Split the expression:

$$\frac{e}{1-e} \cdot \left(-\frac{e-1}{e} \right) + \frac{e}{1-e} \cdot \frac{1}{1+x} = -\frac{e-1}{1-e} + \frac{e}{(1-e)(1+x)}.$$

Since

$$1 - e = -(e - 1),$$

it follows that

$$-\frac{e-1}{1-e} = -\frac{e-1}{-(e-1)} = 1.$$

Hence, the expression simplifies to

$$1 + \frac{e}{(1-e)(1+x)}.$$

Again, replacing $1 - e$ by $-(e - 1)$,

$$1 + \frac{e}{-(e-1)(1+x)} = 1 - \frac{e}{(e-1)(1+x)}.$$

Thus, the limit becomes

$$\alpha = \lim_{x \rightarrow \infty} \left(1 - \frac{e}{(e-1)(1+x)} \right)^x.$$

For large x , note that

$$1 + x \sim x,$$

so we approximate

$$\alpha = \lim_{x \rightarrow \infty} \left(1 - \frac{e}{(e-1)x} \right)^x.$$

Recall the limit

$$\lim_{x \rightarrow \infty} \left(1 - \frac{a}{x} \right)^x = e^{-a},$$

with

$$a = \frac{e}{e-1}.$$

Therefore,

$$\alpha = e^{-\frac{e}{e-1}}.$$

Taking the natural logarithm,

$$\ln \alpha = -\frac{e}{e-1}.$$

Now, compute

$$\frac{\ln \alpha}{1 + \ln \alpha} = \frac{-\frac{e}{e-1}}{1 - \frac{e}{e-1}}.$$

Express the denominator with a common denominator:

$$1 - \frac{e}{e-1} = \frac{e-1}{e-1} - \frac{e}{e-1} = -\frac{1}{e-1}.$$

Thus,

$$\frac{-\frac{e}{e-1}}{-\frac{1}{e-1}} = \frac{e}{1} = e.$$

The final result is

$$\frac{\ln \alpha}{1 + \ln \alpha} = e.$$

Question 33

If the function

$$f(x) = \begin{cases} \frac{2}{x} \{ \sin(k_1 + 1)x + \sin(k_2 - 1)x \}, & x < 0 \\ 4, & x = 0 \\ \frac{2}{x} \log_e \left(\frac{2+k_1x}{2+k_2x} \right), & x > 0 \end{cases}$$

is continuous at $x = 0$, then $k_1^2 + k_2^2$ is equal to :

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Options:

- A. 5
B. 10
C. 20
D. 8

Answer: B

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2}{x} \{ \sin(k_1 + 1)x + \sin(k_2 - 1)x \} &= 4 \\ \Rightarrow 2(k_1 + 1) + 2(k_2 - 1) &= 4 \\ \Rightarrow k_1 + k_2 &= 2 \\ \Rightarrow \lim_{x \rightarrow 0^+} \frac{2}{x} \ln \left(\frac{2 + k_1 x}{2 + k_2 x} \right) &= 4 \\ \Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{x} \ln \left(1 + \frac{(k_1 - k_2)x}{2 + k_2 x} \right) &= 2 \\ \Rightarrow \frac{k_1 - k_2}{2} &= 2 \\ \Rightarrow k_1 - k_2 &= 4 \\ \therefore k_1 = 3, k_2 = -1 \\ k_1^2 + k_2^2 &= 9 + 1 = 10 \end{aligned}$$

Question34

$\lim_{x \rightarrow \infty} \frac{(2x^2 - 3x + 5)(3x - 1)^{\frac{x}{2}}}{(3x^2 + 5x + 4)\sqrt{(3x + 2)^x}}$ is equal to :

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Options:

- A. $\frac{2e}{3}$
B. $\frac{2}{3\sqrt{e}}$
C. $\frac{2e}{\sqrt{3}}$
D. $\frac{2}{\sqrt{3e}}$

Answer: B

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\left(2 - \frac{3}{x} + \frac{5}{x^2}\right) \left(1 - \frac{1}{3x}\right)^{x/2}}{\left(3 + \frac{5}{x} + \frac{4}{x^2}\right) \left(1 + \frac{2}{3x}\right)^{x/2}} \\ = \lim_{x \rightarrow \infty} \frac{2}{3} \cdot \frac{e^{\frac{x}{2} \left(1 - \frac{1}{3x} - 1\right)}}{e^{\frac{x}{2} \left(1 + \frac{2}{3x} - 1\right)}} \\ = \frac{2}{3} \cdot \frac{e^{-\frac{1}{6}}}{e^{1/3}} = \frac{2}{3} e^{-\frac{1}{2}} \end{aligned}$$

Question35

Let $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ be a function such that $f(x) - 6f\left(\frac{1}{x}\right) = \frac{35}{3x} - \frac{5}{2}$. If the $\lim_{x \rightarrow 0} \left(\frac{1}{\alpha x} + f(x)\right) = \beta$; $\alpha, \beta \in \mathbb{R}$, then $\alpha + 2\beta$ is equal to

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Options:

- A. 6
- B. 5
- C. 3
- D. 4

Answer: D

Solution:

$$F(x) - 6f(1/x) = \frac{35}{3x} - \frac{5}{2} \dots (1)$$

Replace $x \rightarrow \frac{1}{x}$

$$F(1/x) - 6(x) = \frac{35x}{3} - \frac{5}{2} \dots (2)$$

Using (1) & (2)

$$f(x) = -2x - \frac{1}{3x} + \frac{1}{2}$$

$$B = \lim_{x \rightarrow 0} \left(\frac{1}{\alpha x} + f(x) \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\alpha x} - 2x - \frac{1}{3x} + \frac{1}{2} \right)$$

$$\alpha = 3, \quad B = 1/2$$

$$\text{So, } \alpha + 2B = 3 + 1 = 4$$

Question 36

$$\lim_{x \rightarrow 0} \operatorname{cosec} x \left(\sqrt{2 \cos^2 x + 3 \cos x} - \sqrt{\cos^2 x + \sin x + 4} \right) \text{ is:}$$

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Options:

- A. $\frac{1}{\sqrt{15}}$
- B. $\frac{1}{2\sqrt{5}}$
- C. 0
- D. $-\frac{1}{2\sqrt{5}}$

Answer: D

Solution:

$$\lim_{x \rightarrow 0} \operatorname{cosec} x \left(\sqrt{2 \cos^2 x + 3 \cos x} - \sqrt{\cos^2 x + \sin x + 4} \right)$$

$$\lim_{x \rightarrow 0} \frac{\operatorname{cosec}(\cos^2 x + 3 \cos x - \sin x - 4)}{(\sqrt{2 \cos^2 x + 3 \cos x} + \sqrt{\cos^2 x + \sin x + 4})}$$

$$\lim_{x \rightarrow 0} \frac{1}{\sin x} \frac{(\cos^2 x + 3 \cos x - 4) - \sin x}{(\sqrt{2 \cos^2 x + 3 \cos x} + \sqrt{\cos^2 x + \sin x + 4})}$$

$$\lim_{x \rightarrow 0} \frac{(\cos^2 x + 4)(\cos x - 1) - \sin x}{\sin x (\sqrt{2 \cos^2 x + 3 \cos x} + \sqrt{\cos^2 x + \sin x + 4})}$$

$$\lim_{x \rightarrow 0} \frac{-2 \sin^2 \frac{x}{2} (\cos x + 4) - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2} (\sqrt{2 \cos^2 x + 3 \cos x} + \sqrt{\cos^2 x + \sin x + 4})}$$

$$\lim_{x \rightarrow 0} \frac{-(\sin^2 \frac{x}{2} (\cos x + 4) + \cos \frac{x}{2})}{\cos \frac{x}{2} (\sqrt{2 \cos^2 x + 3 \cos x} + \sqrt{\cos^2 x + \sin x + 4})}$$

$$= -\frac{1}{2\sqrt{5}}$$

Question 37

Let $[x]$ denote the greatest integer function, and let m and n respectively be the numbers of the points, where the function $f(x) = [x] + |x - 2|$, $-2 < x < 3$, is not continuous and not differentiable. Then $m + n$ is equal to :

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Options:

- A. 6
- B. 9
- C. 8
- D. 7

Answer: C

Solution:

$$f(x) = [x] + |x - 2| \quad -2 < x < 3$$

$$f(x) = \begin{cases} -x, & -2 < x < -1 \\ -x + 1, & -1 \leq x < 0 \\ -x + 2, & 0 \leq x < 1 \\ -x + 3, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \end{cases}$$

So $f(x)$ is not continuous at 4 points and not differentiable at 4 point

$$\text{So } m + n = 4 + 4 = 8$$

Question 38

The value of $\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!} \right)$ is :

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Options:

A.

$\frac{5}{3}$

B.

2

C.

$\frac{4}{3}$

D.

$\frac{7}{3}$

Answer: A

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 6 - 1}{(k+3)!} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(k+1)(k+2)(k+3) - 1}{(k+3)!} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(k+1)(k+2)(k+3)}{(k+3)!} - \frac{1}{(k+3)!} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k!} - \frac{1}{(k+3)!} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots + \frac{1}{n!} - \frac{1}{4!} - \frac{1}{5!} - \frac{1}{6!} \dots - \frac{1}{(n+3)!} \right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{6} - \frac{10}{6} = \frac{5}{3} \end{aligned}$$

Question 39

Let the function $f(x) = (x^2 - 1)|x^2 - ax + 2| + \cos|x|$ be not differentiable at the two points $x = \alpha = 2$ and $x = \beta$. Then the distance of the point (α, β) from the line $12x + 5y + 10 = 0$ is equal to :

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Options:

A.

5

B.

2

C. 4

D.

3

Answer: D

Solution:

$\cos|x|$ is always differentiable

$\therefore$ we have to check only for $|x^2 - ax + 2|$

$\therefore$ Not differentiable at

$$x^2 - ax + 2 = 0$$

One root is given, $\alpha = 2$

$$\therefore 4 - 2a + 2 = 0$$
$$a = 3$$

$\therefore$ other root $\beta = 1$

but for $x = 1$ $f(x)$ is differentiable

(Drop)

Question40

For $\alpha, \beta, \gamma \in \mathbf{R}$, if $\lim_{x \rightarrow 0} \frac{x^2 \sin \alpha x + (\gamma - 1)e^{x^2}}{\sin 2x - \beta x} = 3$, then $\beta + \gamma - \alpha$ is equal to :

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Options:

A. -1

B. 4

C. 6

D. 7

Answer: D

Solution:

At $x \rightarrow 0$

$$\sin 2x - \beta x \rightarrow 0$$

$$\Rightarrow \frac{0}{0} \text{ form}$$

$$\Rightarrow (\gamma - 1)e^0 + 0 \sin(\alpha x) \rightarrow 0$$

$$\Rightarrow (\gamma - 1) = 0$$

$$\Rightarrow \gamma = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 \sin(\alpha x)}{(\sin 2x - \beta x)} = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 \left(\alpha x - \frac{(\alpha x)^3}{3!} + \frac{(\alpha x)^5}{5!} - \dots \right)}{\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots \right] - \beta x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\alpha x^3 - \frac{\alpha^3 x^5}{3!} + \frac{\alpha^5 x^7}{5!} - \dots}{x(2 - \beta) - \frac{8x^3}{6} + \frac{32x^5}{360} - \dots} = 3$$

$$\Rightarrow 2 - \beta = 0 \text{ and } \frac{\alpha}{-\frac{8}{6}} = 3$$

$$\Rightarrow \beta = 2$$

$$\alpha = 3 \left(-\frac{8}{6} \right) = -4$$

$$\Rightarrow \gamma = 1, \beta = 2, \alpha = -4$$

$$\Rightarrow \beta + \gamma - \alpha = 7$$

Question41

If $\lim_{x \rightarrow 0} \frac{\cos(2x) + a \cos(4x) - b}{x^4}$ is finite, then $(a + b)$ is equal to :

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Options:

A. 0

B. $\frac{3}{4}$

C. -1

D. $\frac{1}{2}$

Answer: D

Solution:

To find the value of $a + b$ for which the following limit is finite:

$$\lim_{x \rightarrow 0} \frac{\cos(2x) + a \cos(4x) - b}{x^4}$$

we start by expanding the cosine functions using their Taylor series:

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots$$

$$\cos(4x) = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \dots$$

Substitute these expansions into the limit expression:

$$\lim_{x \rightarrow 0} \frac{\left(1 - \frac{4x^2}{2!} + \frac{(2x)^4}{4!} - \dots\right) + a\left(1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \dots\right) - b}{x^4}$$

To ensure that the limit is finite, the linear and quadratic terms must sum to zero. Let's equate the constant terms first.

For the limit to be finite:

$$1 + a - b = 0$$

Equating the coefficients of x^2 :

$$-2 - 8a = 0$$

Solving the second equation, we have:

$$8a = -2a = \frac{-1}{4}$$

Substitute a back into the first equation to find b :

$$1 + \frac{-1}{4} - b = 0 \Rightarrow b = 1 - \frac{1}{4} = \frac{3}{4}$$

Thus, the sum of a and b is:

$$a + b = \frac{-1}{4} + \frac{3}{4} = \frac{1}{2}$$

Therefore, the correct value of $a + b$ is $\frac{1}{2}$.

Question 42

Let $f(x) = \begin{cases} (1 + ax)^{1/x}, & x < 0 \\ 1 + b, & x = 0 \\ \frac{(x+4)^{1/2} - 2}{(x+c)^{1/3} - 2}, & x > 0 \end{cases}$ be continuous at $x = 0$. Then $e^a b c$ is equal to:

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Options:

A. 64

B. 48

C. 36

D. 72

Answer: B

Solution:

$$f(x) = \begin{cases} (1+ax)^{1/x} & , x < 0 \\ 1+b & , x = 0 \\ \frac{(x+4)^{1/2}-2}{(x+c)^{1/3}-2} & , x > 0 \end{cases}$$

Hence, $f(0) = 1 + 6$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \frac{(x+4)^{1/2}-2}{(x+c)^{1/3}-2} \left[\text{For } \frac{0}{0} \text{ form, } c = 8 \right]$$

$$= \lim_{x \rightarrow 0} \frac{2\left(1+\frac{x}{4}\right)^{1/2}-2}{\left(1+\frac{x}{8}\right)^{1/3}-2}$$

$$= \lim_{x \rightarrow 0} \frac{1+\frac{x}{8}-1}{1+\frac{x}{8} \cdot \frac{1}{3}-1} = \frac{\frac{1}{8}}{\frac{1}{8} \cdot \frac{1}{3}} = 3$$

$$\therefore 1+b=3$$

$$\Rightarrow b=2$$

$$\text{LHL} = \lim_{x \rightarrow 0} (1+ax)^{1/x} = \lim_{x \rightarrow 0} e^{\frac{1+ax-1}{x}} = e^a = 3$$

$$\therefore e^a \cdot bc = 3 \cdot 2.8 = 48$$

Question43

If $\lim_{x \rightarrow 1^+} \frac{(x-1)(6+\lambda \cos(x-1))+\mu \sin(1-x)}{(x-1)^3} = -1$, where $\lambda, \mu \in \mathbb{R}$, then $\lambda + \mu$ is equal to

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Options:

A. 20

B. 19

C. 18

D. 17

Answer: C

Solution:

$$\lim_{x \rightarrow 1^+} \frac{(x-1)(6+\lambda \cos(x-1))+\mu \sin(1-x)}{(x-1)^3} = -1$$

$$\text{Let } x-1=t$$

$$\lim_{t \rightarrow 0^+} \frac{6t + \lambda t \cos t - \mu \sin t}{t^3} = -1$$

$$= \lim_{t \rightarrow 0^+} \frac{6t + \lambda t \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) - \mu \left(t - \frac{t^3}{3!} + \dots\right)}{t^3} = -1$$

$$= \lim_{t \rightarrow 0^+} \frac{t(6 + \lambda - \mu) + t^3 \left(-\frac{\lambda}{2} + \frac{\mu}{6}\right) + \dots}{t^3} = -1$$

$$\therefore \lambda - \mu + 6 = 0 \quad \dots (i)$$

$$\frac{\mu}{6} - \frac{\lambda}{2} = -1 \quad \dots (ii)$$

Solving (i) and (ii)

$$\lambda = 6, \mu = 12$$

$$\lambda + \mu = 18$$

Question44

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and $f(2x) - f(x) = x$ for all $x \in \mathbb{R}$. If

$\lim_{n \rightarrow \infty} \left\{ f(x) - f\left(\frac{x}{2^n}\right) \right\} = G(x)$, then $\sum_{r=1}^{10} G(r^2)$ is equal to

JEE Main 2025 (Online) 4th April Morning Shift

Options:

- A. 215
- B. 420
- C. 385
- D. 540

Answer: C

Solution:

$$f(0) = 1, f(2x) - f(x) = x$$

$$\text{Replace } x \rightarrow \frac{x}{2}$$

$$f(x) - f\left(\frac{x}{2}\right) = \frac{x}{2} \quad \dots (1)$$

$$\text{Again, Replace } x \rightarrow \frac{x}{2}$$

$$f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^2}\right) = \frac{x}{2^2} \quad \dots (2)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}$$

$$\text{Adding (1) + (2) + (3) + } \dots + (n)$$

$$\text{We get } f(x) - f\left(\frac{x}{2^n}\right) = \frac{x}{2} + \frac{x}{2^2} + \dots + \frac{x}{2^n}$$

$$\lim_{n \rightarrow \infty} \left(f(x) - f\left(\frac{x}{2^n}\right) \right) = \lim_{n \rightarrow \infty} \left(\frac{x}{2} + \frac{x}{2^2} + \dots + \frac{x}{2^n} \right)$$

$$f(x) - f(0) = \frac{x}{2}$$

$$\Rightarrow G(x) = x$$

$$\Rightarrow G(r^2) = r^2$$

$$\Rightarrow \sum_{r=1}^{10} G(r^2) = \sum_{r=1}^{10} G(r^2)$$

$$= \frac{(10)(11)(21)}{6} = (55)7$$

$$\Rightarrow 385$$

Question 45

Let f be a differentiable function on $\mathbb{R}$ such that $f(2) = 1, f'(2) = 4$. Let $\lim_{x \rightarrow 0} (f(2+x))^{3/x} = e^\alpha$. Then the number of times the curve $y = 4x^3 - 4x^2 - 4(\alpha - 7)x - \alpha$ meets x -axis is :

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Options:

- A. 3
- B. 1
- C. 2
- D. 0

Answer: C

Solution:

$$\lim_{x \rightarrow 0} (f(2+x))^{3/x} = (1^\infty \text{ form})$$

$$e^{\lim_{x \rightarrow 0} \frac{3}{x} (f(2+x) - 1)} = e^{\lim_{x \rightarrow 0} 3f'(2+x)}$$

$$= e^{3f'(2)}$$

$$= e^{12}$$

$$\Rightarrow \alpha = 12$$

$$y = 4x^3 - 4x^2 - 4(12-7)x - 12$$

$$y = 4x^3 - 4x^2 - 20x - 12$$

$$y = 4(x^3 - x^2 - 5x - 3)$$

$$= 4(x+1)^2(x-3)$$

It meets the x -axis at two points

Question46

$$\lim_{x \rightarrow 0^+} \frac{\tan\left(5(x)^{\frac{1}{3}}\right) \log_e(1+3x^2)}{(\tan^{-1} 3\sqrt{x})^2 \left(e^{5(x)^{\frac{4}{3}}} - 1\right)} \text{ is equal to}$$

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Options:

A. $\frac{5}{3}$

B. 1

C. $\frac{1}{3}$

D. $\frac{1}{15}$

Answer: C

Solution:

$$\lim_{x \rightarrow 0^+} \frac{\tan\left(5(x)^{\frac{1}{3}}\right) \ln(1+3x^2)}{(\tan^{-1} 3\sqrt{x})^2 \left(e^{5x^{\frac{4}{3}}} - 1\right)}$$

$$\lim_{x \rightarrow 0^+} \frac{\frac{\tan\left(5(x)^{\frac{1}{3}}\right)}{5(x)^{\frac{1}{3}}} \cdot \frac{\ln(1+3x^2)}{3x^2} \times 5(x)^{\frac{1}{3}} (3x^2)}{\frac{(\tan^2(3\sqrt{x}))^2}{(3\sqrt{x})^2} \cdot \frac{e^{5x^{\frac{4}{3}}} - 1}{5x^{\frac{4}{3}}} \times 9x \times 5x^{\frac{4}{3}}}$$

$$\lim_{x \rightarrow 0^+} \frac{\frac{\tan\left(5(x)^{\frac{1}{3}}\right)}{5(x)^{\frac{1}{3}}} \cdot \frac{\ln(1+3x^2)}{3x^2} \times 15x^{\frac{2}{3}}}{\frac{(\tan^2(3\sqrt{x}))^2}{(3\sqrt{x})^2} \cdot \frac{e^{5x^{\frac{4}{3}}} - 1}{5x^{\frac{4}{3}}} \times 45x^{\frac{7}{3}}}$$

$$= \frac{1}{3}$$

Question47

Given below are two statements:

Statement I: $\lim_{x \rightarrow 0} \left(\frac{\tan^{-1} x + \log_e \sqrt{\frac{1+x}{1-x}} - 2x}{x^5} \right) = \frac{2}{5}$

Statement II: $\lim_{x \rightarrow 1} \left(x^{\frac{2}{1-x}} \right) = \frac{1}{e^2}$

In the light of the above statements, choose the correct answer from the options given below:

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Options:

A.

Statement I is false but Statement II is true

B.

Both Statement I and Statement II are false

C.

Both Statement I and Statement II are true

D.

Statement I is true but Statement II is false

Answer: C

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\tan^{-1} x + \frac{1}{2} [\ln(1+x) - \ln(1-x)] - 2x}{x^5} \\ &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots \right) + \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{x^3}{3} \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} \dots \right) \right] - 2x}{x^5} \\ &= \lim_{x \rightarrow 0} \frac{2x + \frac{2x^5}{5} \dots - 2x}{x^5} = \frac{2}{5} \\ & \lim_{x \rightarrow 1} x^{\frac{2}{1-x}} = e^{\lim_{x \rightarrow 1} \left(\frac{2}{1-x} \right) (x-1)} = e^{-2} \\ & \Rightarrow \text{Both statements correct} \end{aligned}$$

Question 48

Consider the function. $f(x) = \begin{cases} \frac{a(7x-12-x^2)}{b|x^2-7x+12|}, & x < 3 \\ 2^{\frac{\sin(x-3)}{x-[x]}}, & x > 3 \\ b, & x = 3 \end{cases}$

Where $[x]$ denotes the greatest integer less than or equal to x . If S denotes the set of all ordered pairs (a,b) such that $f(x)$ is continuous at $x = 3$, then the number of elements in S is :

[27-Jan-2024 Shift 1]

Options:

A.

2

B.

Infinitely many

C.

4

D.

1

Answer: D

Solution:

$$f(3^-) = \frac{a}{b} \frac{(7x-12-x^2)}{|x^2-7x+12|} \text{ (for } f(x) \text{ to be cont.)}$$

$$\Rightarrow f(3^-) = \frac{-a}{b} \frac{(x-3)(x-4)}{(x-3)(x-4)}; x < 3 \Rightarrow \frac{-a}{b}$$

$$\text{Hence } f(3^-) = \frac{-a}{b}$$

$$\text{Then } f(3^+) = 2^{x-3} \left(\frac{\sin(x-3)}{x-3} \right) = 2 \text{ and } f(3) = b.$$

$$\text{Hence } f(3) = f(3^-) = f(3^+)$$

$$\Rightarrow b = 2 = -\frac{a}{b}$$

$$b = 2, a = -4$$

Hence only 1 ordered pair $(-4, 2)$.

Question 49

Consider the function $f : (0, 2) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{2} + \frac{2}{x}$ and the function $g(x)$ defined by

$$g(x) = \begin{cases} \min\{f(t)\} & 0 < t \leq x \text{ and } 0 < x \leq 1 \\ \frac{3}{2} + x & 1 < x < 2 \end{cases} \quad \dots \text{Then}$$

[27-Jan-2024 Shift 2]

Options:

A.

g is continuous but not differentiable at $x = 1$

B.

g is not continuous for all $x \in (0, 2)$

C.

g is neither continuous nor differentiable at $x = 1$

D.

g is continuous and differentiable for all $x \in (0, 2)$

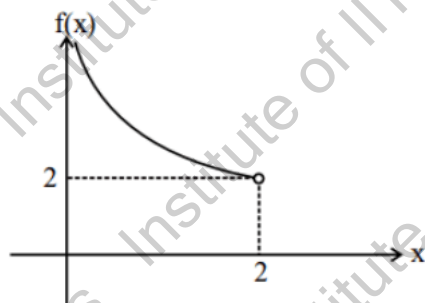
Answer: A

Solution:

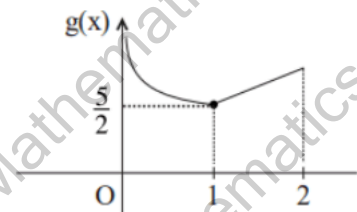
$$f: (0, 2) \rightarrow \mathbb{R}; f(x) = \frac{x}{2} + \frac{2}{x}$$

$$f(x) = \frac{1}{2} - \frac{2}{x^2}$$

$\therefore f(x)$ is decreasing in domain.



$$g(x) = \begin{cases} \frac{x}{2} + \frac{2}{x} & 0 < x \leq 1 \\ \frac{3}{2} + x & 1 < x < 2 \end{cases}$$



Question 50

Let $f(x) = \sqrt{\lim_{r \rightarrow x} \left\{ \frac{2r^2[(f(r))^2 - f(x)f(r)]}{r^2 - x^2} - r^3 e^{\frac{f(r)}{r}} \right\}}$ be differentiable in $(-\infty, 0) \cup (0, \infty)$ and $f(1) = 1$. Then the value of ea , such that $f(a) = 0$, is equal to _____

[29-Jan-2024 Shift 2]

Answer: 2

Solution:

$$f(1) = 1, f(a) = 0$$

$$f^2(x) = \lim_{r \rightarrow x} \left(\frac{2r^2(f^2(r) - f(x)f(r))}{r^2 - x^2} - r^3 e^{\frac{f(r)}{r}} \right)$$

$$= \lim_{r \rightarrow x} \left(\frac{2r^2 f(r)}{r+x} \frac{(f(r) - f(x))}{r-x} - r^3 e^{\frac{f(r)}{r}} \right)$$

$$f^2(x) = \frac{2x^2 f(x)}{2x} f'(x) - x^3 e^{\frac{f(x)}{x}}$$

$$y^2 = xy \frac{dy}{dx} - x^3 e^{\frac{y}{x}}$$

$$\frac{y}{x} = \frac{dy}{dx} - \frac{x^2}{y} e^{\frac{y}{x}}$$

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v = v + x \frac{dv}{dx} - \frac{x}{v} e^v$$

$$\frac{dv}{dx} = \frac{e^v}{v} \Rightarrow e^{-v} v dv = dx$$

Integrating both side

$$e^v(x+c) + 1 + v = 0$$

$$f(1) = 1 \Rightarrow x = 1, y = 1$$

$$\Rightarrow c = -1 - \frac{2}{e}$$

$$e^v \left(-1 - \frac{2}{e} + x \right) + 1 + v = 0$$

$$e^{\frac{y}{x}} \left(-1 - \frac{2}{e} + x \right) + 1 + \frac{y}{x} = 0$$

$$x = a, y = 0 \Rightarrow a = \frac{2}{e}$$

$$ae = 2$$

Question 51

If the function $f(x) = \begin{cases} \frac{1}{|x|} & , |x| \geq 2 \\ ax^2 + 2b & , |x| < 2 \end{cases}$ is differentiable on $\mathbb{R}$, then $48(a+b)$ is equal to _____

[30-Jan-2024 Shift 1]

Answer: 15

Solution:

$$f(x) = \begin{cases} \frac{1}{x}; & x \geq 2 \\ ax^2 + 2b; & -2 < x < 2 \\ -\frac{1}{x}; & x \leq -2 \end{cases}$$

$$\text{Continuous at } x = 2 \Rightarrow \frac{1}{2} = \frac{a}{4} + 2b$$

$$\text{Continuous at } x = -2 \Rightarrow \frac{1}{2} = \frac{a}{4} + 2b$$

Since, it is differentiable at $x = 2$

$$-\frac{1}{x^2} = 2ax$$

$$\text{Differentiable at } x = 2 \Rightarrow \frac{-1}{4} = 4a \Rightarrow a = \frac{-1}{16}, b = \frac{3}{8}$$

Question 52

Let $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a function satisfying $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y, f(y) \neq 0$. If $f'(1) = 2024$, then

[30-Jan-2024 Shift 2]

Options:

A.

$$xf'(x) - 2024f(x) = 0$$

B.

$$xf'(x) + 2024f(x) = 0$$

C.

$$xf'(x) + f(x) = 2024$$

D.

$$xf'(x) - 2023f(x) = 0$$

Answer: A

Solution:

$$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$$

$$f(1) = 2024$$

$$f(1) = 1$$

Partially differentiating w. r. t. x

$$f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = \frac{1}{f(y)} f'(x)$$

$$y \rightarrow x$$

$$f'(1) \cdot \frac{1}{x} = \frac{f'(x)}{f(x)}$$

$$2024f(x) = xf'(x) \Rightarrow xf'(x) - 2024f(x) = 0$$

Question 53

Let a and b be real constants such that the function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & x \leq 1 \\ bx + 2, & x > 1. \end{cases}$ be differentiable

on $\mathbb{R}$. Then, the value of $\int_{-2}^2 f(x) dx$ equals

[30-Jan-2024 Shift 2]

Options:

A.

15/6

B.

19/6

C.

21

D.

17

Answer: D

Solution:

f is continuous

$$\therefore 4 + a = b + 2$$

$$a = b - 2$$

$$f(x) = \begin{cases} 2x + 3, & x < 1 \\ b, & x > 1 \end{cases}$$

f is differentiable

$$\therefore b = 5$$

$$\therefore a = 3$$

$$\int_{-2}^1 (x^2 + 3x + 3) dx + \int_1^2 (5x + 2) dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 3x \right]_{-2}^1 + \left[\frac{5x^2}{2} + 2x \right]_1^2$$

$$= \left(\frac{1}{3} + \frac{3}{2} + 3 \right) - \left(\frac{-8}{3} + 6 - 6 \right) + \left(10 + 4 - \frac{5}{2} - 2 \right)$$

$$= 6 + \frac{3}{2} + 12 - \frac{5}{2} = 17$$

Question 54

Consider the function $f: (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = e^{-|\log_e x|}$. If m and n be respectively the number of points at which f is not continuous and f is not differentiable, then $m+n$ is

[31-Jan-2024 Shift 2]

Options:

- A.
0
B.
3
C.
1
D.
2

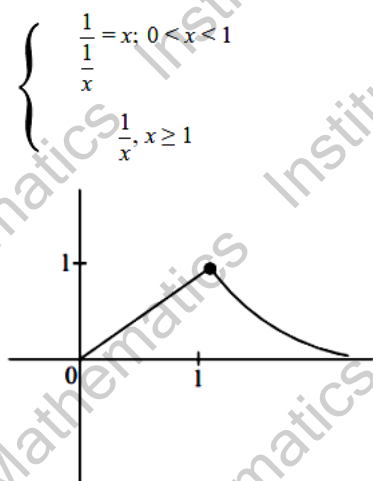
Answer: C

Solution:

$$f: (0, \infty) \rightarrow \mathbb{R}$$

$$f(x) = e^{-|\log_e x|}$$

$$f(x) = \frac{1}{e^{|\ln x|}} = \begin{cases} \frac{1}{e^{-\ln x}}; & 0 < x < 1 \\ \frac{1}{e^{\ln x}}; & x \geq 1 \end{cases}$$



$m = 0$ (No point at which function is not continuous)

$n = 1$ (Not differentiable)

$$\therefore m + n = 1$$

Question 55

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{a - b \cos 2x}{x^2} & ; \quad x < 0 \\ x^2 + cx + 2 & ; \quad 0 \leq x \leq 1 \\ 2x + 1 & ; \quad x > 1 \end{cases}$$

If f is continuous everywhere in $\mathbb{R}$ and m is the number of points where f is NOT differentiable then $m + a + b + c$ equals :

[1-Feb-2024 Shift 1]

Options:

A.

1

B.

4

C.

3

D.

2

Answer: D

Solution:

At $x = 1$, $f(x)$ is continuous therefore,

$$f(1) = f(1) = f(1^+)$$

$$f(1) = 3 + c \dots\dots(1)$$

$$f(1^-) = \lim_{h \rightarrow 0} 2(1+h) + 1$$

$$f(1^-) = \lim_{h \rightarrow 0} 3 + 2h = 3 \dots\dots(2)$$

from (1) & (2)

$$c = 0$$

at $x = 0$, $f(x)$ is continuous therefore,

$$f(0^-) = f(0) = f(0^+) \dots\dots(3)$$

$$f(0) = f(0^+) = 2 \dots\dots(4)$$

$f(0^-)$ has to be equal to 2

$$\lim_{h \rightarrow 0} \frac{a - b \cos(2h)}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b \left\{ 1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} + \dots \right\}}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b + b \left\{ \frac{2h^2}{3} - \frac{2}{3}h^4 \dots \right\}}{h^2}$$

for limit to exist $a - b = 0$ and limit is $2b \dots\dots(5)$

from (3), (4) & (5)

$$a = b = 1$$

checking differentiability at $x = 0$

$$\text{LHD} : \lim_{h \rightarrow 0} \frac{1 - \cos 2h - 2}{h^2 - h}$$

$$\lim_{h \rightarrow 0} \frac{1 - \left(1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} \dots\right) - 2h^2}{-h^3} = 0$$

$$\text{RHD} : \lim_{h \rightarrow 0} \frac{(0+h)^2 + 2 - 2}{h} = 0$$

Function is differentiable at every point in its domain

$$\therefore m = 0$$

$$m + a + b + c = 0 + 1 + 1 + 0 = 2$$

Question 56

Let $f(x) = 2|x^2 + 5|x| - 3|$, $x \in \mathbb{R}$. If m and n denote the number of points where f is not continuous and not differentiable respectively, then $m + n$ is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

5

B.

2

C.

0

D.

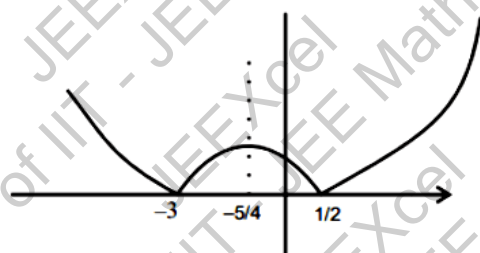
3

Answer: D

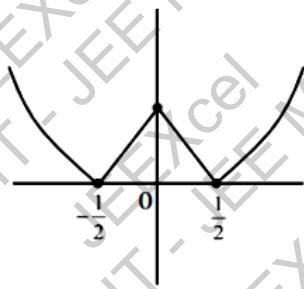
Solution:

$$f(x) = 2x^2 + 5|x| - 3|$$

Graph of $y = |2x^2 + 5x - 3|$



Graph of $f(x)$



Number of points of discontinuity $= 0 = m$

Number of points of non-differentiability $= 3 = n$

Question 57

Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$; Then at $x = 0$

[24-Jan-2023 Shift 1]

Options:

- A. f is continuous but not differentiable
- B. f is continuous but f' is not continuous
- C. f and f' both are continuous
- D. f' is continuous but not differentiable

Answer: B

Solution:

Continuity of $f(x)$: $f(0^+) = h^2 \cdot \sin \frac{1}{h} = 0$

$$f(0^-) = (-h)^2 \cdot \sin \left(\frac{-1}{h} \right) = 0$$

$$f(0) = 0$$

$f(x)$ is continuous

$$f'(0^+) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \frac{h^2 \cdot \sin \left(\frac{1}{h} \right) - 0}{h} = 0$$

$$f'(0^-) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \frac{h^2 \cdot \sin \left(\frac{1}{-h} \right) - 0}{-h} = 0$$

$f(x)$ is differentiable.

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$

$$f'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x)$ is not continuous (as $\cos\left(\frac{1}{x}\right)$ is highly oscillating at $x = 0$)

Question 58

If the function

$$f(x) = \begin{cases} (1 + |\cos x|) \frac{\lambda}{|\cos x|}, & 0 < x < \frac{\pi}{2} \\ \mu, & x = \frac{\pi}{2} \\ \frac{\cot 6x}{e^{\cot 4x}}, & \frac{\pi}{2} < x < \pi \end{cases}$$

is continuous at $x = \frac{\pi}{2}$, then $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda}$ is equal to
[25-Jan-2023 Shift 2]

Options:

- A. 11
- B. 8
- C. $2e^4 + 8$
- D. 10

Answer: D

Solution:

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cot 6x}{e^{\cot 4x}} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\sin 4x \cdot \cos 6x}{e^{\sin 6x \cdot \cos 4x}} = e^{2/3}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \left((1 + |\cos x|) \frac{\lambda}{|\cos x|} \right) = e^\lambda$$

$$\Rightarrow f(\pi/2) = \mu$$

$$\text{For continuous function} \Rightarrow e^{2/3} = e^\lambda = \mu$$

$$\lambda = \frac{2}{3}, \mu = e^{2/3}$$

$$\text{Now, } 9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda} = 10$$

Question59

Let $a \in \mathbb{Z}$ and $[t]$ be the greatest integer $\leq t$. Then the number of points, where the function $f(x) = [a + 13 \sin x]$, $x \in (0, \pi)$ is not differentiable, is
[6-Apr-2023 shift 1]

Answer: 25

Solution:

$$f(x) = [a + 13 \sin x] = a + [13 \sin x] \text{ in } (0, \pi)$$

$$x \in (0, \pi)$$

$$\Rightarrow 0 < 13 \sin x \leq 13$$

$$\Rightarrow [13 \sin x] = \{0, 1, 2, 3, \dots, 12, 13\}$$

$$\text{Total point of N.D.} = 25.$$

Question60

Let $f : (-2, 2) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x[x] & -2 < x < 0 \\ (x-1)[x] & 0 \leq x < 2 \end{cases}$$

where $[x]$ denotes the greatest integer function. If m and n respectively are the number of points in $(-2, 2)$ at which $y = |f(x)|$ is not continuous and not differentiable, then $m + n$ is equal to _____.
[10-Apr-2023 shift 1]

Answer: 4

Solution:

$$f(x) = \begin{cases} -2x & -2 < x < -1 \\ -x & -1 \leq x < 0 \\ 0 & 0 \leq x < 1 \\ x-1 & 1 \leq x < 2 \end{cases}$$

Clearly $f(x)$ is discontinuous at $x = -1$ also non differentiable.

$$\therefore m = 1$$

Now for differentiability

$$f'(x) = \begin{cases} -2 & -2 < x < -1 \\ -1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

Clearly $f(x)$ is non-differentiable at $x = -1, 0, 1$

Also, $|f(x)|$ remains same.

$$\therefore n = 3$$

$$\therefore m + n = 4$$

Question61

Let $f(x) = [x^2 - x] + |-x + [x]|$, where $x \in \mathbb{R}$ and $[t]$ denotes the greatest integer less than or equal to t . Then, f is :

[11-Apr-2023 shift 1]

Options:

- A. not continuous at $x = 0$ and $x = 1$
- B. continuous at $x = 0$ and $x = 1$
- C. continuous at $x = 1$, but not continuous at $x = 0$
- D. continuous at $x = 0$, but not continuous at $x = 1$

Answer: C

Solution:

Here $f(x) = [x(x-1)] + \{x\}$

$f(0^+) = -1 + 0 = -1$ $f(1^+) = 0 + 0 = 0$

$f(0) = 0$ $f(1) = 0$

$f(1^-) = -1 + 1 = 0$

$\therefore f(x)$ is continuous at $x = 1$, discontinuous at $x = 0$

Question 62

Let f and g be two functions defined by $f(x) = \begin{cases} x+1 & x < 0 \\ |x-1|, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x+1 & x < 0 \\ 1 & x \geq 0 \end{cases}$. Then $(g \circ f)(x)$ is

[11-Apr-2023 shift 2]

Options:

- A. continuous everywhere but not differentiable at $x = 1$
- B. continuous everywhere but not differentiable exactly at one point
- C. differentiable everywhere
- D. not continuous at $x = -1$

Answer: B

Solution:

$$f(x) = \begin{cases} x+1 & x < 0 \\ 1-x & 0 \leq x < 1 \\ x-1 & 1 \leq x \end{cases}$$

$$g(x) = \begin{cases} x+1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$g(f(x)) = \begin{cases} x+2 & x < -1 \\ 1 & x \geq -1 \end{cases}$$

$\therefore g(f(x))$ is continuous everywhere

$g(f(x))$ is not differentiable at $x = -1$

Differentiable everywhere else

Question 63

Let $[x]$ be the greatest integer $\leq x$. Then the number of points in the interval $(-2, 1)$, where the function $f(x) = |[x]| + \sqrt{x - [x]}$ is discontinuous, is _____.

[12-Apr-2023 shift 1]

Answer: 2

Solution:

Need to check at doubtful points
discont at $x = 1$ only

$$\text{at } x = -1 \Rightarrow f(-1^+) = 1 + 0 = 1$$

$$\Rightarrow f(-1^-) = 2 + 1 = 3$$

$$\text{at } x = 0 \Rightarrow f(0^+) = 0 + 0 = 0$$

$$\Rightarrow f(0^-) = 1 + 1 = 2$$

$$\text{at } x = 1 \Rightarrow f(1^+) = 1 + 0 = 1$$

$$\Rightarrow f(1^-) = 0 + 1 = 1$$

discont. at two points

Question 64

Let $[x]$ denote the greatest integer function and $f(x) = \max\{1 + x + [x], 2 + x, x + 2[x]\}$, $0 \leq x \leq 2$. Let m be the number of points in $[0, 2]$, where f is not continuous and n be the number of points in $(0, 2)$, where f is not differentiable. Then $(m + n)^2 + 2$ is equal to
[15-Apr-2023 shift 1]

Options:

A. 6

B. 3

C. 2

D. 11

Answer: B

Solution:

$$\text{Let } g(x) = 1 + x + [x] = \begin{cases} 1 + x; & x \in [0, 1) \\ 2 + x; & x \in [1, 2) \\ 5; & x = 2 \end{cases}$$

$$h(x) = x + 2[x] = \begin{cases} x; & x \in [0, 1) \\ x + 2; & x \in [1, 2) \\ 6; & x = 2 \end{cases}$$

$$r(x) = 2 + x$$

$$f(x) = \begin{cases} 2 + x; & x \in [0, 2) \\ 6; & x = 2 \end{cases}$$

$f(x)$ is discontinuous only at $x = 2 \Rightarrow m = 1$

$f(x)$ is differentiable in $(0, 2) \Rightarrow n = 0$

$$(m + n)^2 + 2 = 3$$

Question 65

$$\text{Let } f(x) = \begin{cases} \frac{\sin(x - [x])}{x - [x]} & x \in (-2, -1) \\ \max\{2x, 3[x]\} & |x| < 1 \\ 1 & \text{otherwise} \end{cases}$$

where $[t]$ denotes greatest integer $\leq t$. If m is the number of points

where f is not continuous and n is the number of points where f is not differentiable, then the ordered

pair (m, n) is :
[24-Jun-2022-Shift-2]

Options:

- A. (3, 3)
- B. (2, 4)
- C. (2, 3)
- D. (3, 4)

Answer: C

Solution:

$$f(x) = \begin{cases} \frac{\sin(x - [x])}{x[x]} & x \in (-2, -1) \\ \max\{2x, 3[x]\} & |x| < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{\sin(x+2)}{x+2} & x \in (-2, -1) \\ 0 & x \in (-1, 0] \\ 2x & x \in (0, 1) \\ 1 & \text{otherwise} \end{cases}$$

It clearly shows that $f(x)$ is discontinuous

At $x = -1, 1$ also non differentiable

$$\text{and at } x = 0, L.H.D = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = 0$$

$$R.H.D = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 2$$

$\therefore f(x)$ is not differentiable at $x = 0$

$$\therefore m = 2, n = 3$$

Question 66

Let $f(x) = [2x^2 + 1]$ and $g(x) = \begin{cases} 2x-3 & x < 0 \\ 2x+3 & x \geq 0 \end{cases}$, where $[t]$ is the greatest integer $\leq t$. Then, in the open

interval $(-1, 1)$, the number of points where $f \circ g$ is discontinuous is equal to _____

[25-Jun-2022-Shift-2]

Answer: 62

Solution:

$$f(g(x)) = [2g^2(x)] + 1$$

$$= \begin{cases} [2(2x-3)^2] + 1; & x < 0 \\ [2(2x+3)^2] + 1; & x \geq 0 \end{cases}$$

$\therefore f \circ g$ is discontinuous whenever $2(2x-3)^2$ or $2(2x+3)^2$ belongs to integer except $x = 0$.

$\therefore 62$ points of discontinuity.

Question67

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two real valued functions defined as $f(x) = \begin{cases} -|x+3| & x < 0 \\ e^x & x \geq 0 \end{cases}$

$$\text{and } g(x) = \begin{cases} x^2 + k_1x & x < 0 \\ 4x + k_2 & x \geq 0 \end{cases}$$

where k_1 and k_2 are real constants. If $(g \circ f)$ is differentiable at $x = 0$, then $(g \circ f)(-4) + (g \circ f)(4)$ is equal to
: **[26-Jun-2022-Shift-1]**

Options:

- A. $4(e^4 + 1)$
- B. $2(2e^4 + 1)$
- C. $4e^4$
- D. $2(2e^4 - 1)$

Answer: D

Solution:

$\because$ $g \circ f$ is differentiable at $x = 0$

So R.H.D = L.H.D

$$\frac{d}{dx}(4e^x + k_2) = \frac{d}{dx}((-|x+3|)^2 - k_1|x+3|)$$

$$\Rightarrow 4 = 6 - k_1 \Rightarrow k_1 = 2$$

$$\text{Also } f'(f(0^+)) = g'(f(0^-))$$

$$\Rightarrow 4 - k_2 = 9 - 3k_1 \Rightarrow k_2 = -1$$

$$\text{Now } g(f(-4)) + g(f(4))$$

$$= g(-1) - g(e^4) = (1 - k_1) + (4e^4 + k_2)$$

$$= 4e^4 - 2$$

$$= 2(2e^4 - 1)$$

Question68

Let $f(x) = \min\{1, 1 + x \sin x\}$, $0 \leq x \leq 2\pi$. If m is the number of points, where f is not differentiable and n is the number of points, where f is not continuous, then the ordered pair (m, n) is equal to
[26-Jun-2022-Shift-2]

Options:

- A. $(2, 0)$
- B. $(1, 0)$
- C. $(1, 1)$
- D. $(2, 1)$

Answer: B

Solution:

$$f(x) = \min\{1, 1 + x \sin x\}, 0 \leq x \leq \pi$$

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 1 + x \sin x & \pi \leq x \leq 2\pi \end{cases}$$

$$\text{Now at } x = \pi, \lim_{x \rightarrow \pi^-} f(x) = 1 = \lim_{x \rightarrow \pi^+} f(x)$$

$\therefore f(x)$ is continuous in $[0, 2\pi]$

$$\text{Now, at } x = \pi, \text{L.H.D} = \lim_{h \rightarrow 0} \frac{f(\pi - h) - f(\pi)}{-h} = 0$$

$$\text{R.H.D} = \lim_{h \rightarrow 0} \frac{f(\pi + h) - f(\pi)}{h} = 1 - \frac{(\pi + h) \sin h - 1}{h} = -\pi$$

$\therefore f(x)$ is not differentiable at $x = \pi$

$$\therefore (m, n) = (1, 0)$$

Question 69

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} [e^x], & x < 0 \\ ae^x + [x - 1], & 0 \leq x < 1 \\ b + [\sin(\pi x)], & 1 \leq x < 2 \\ [e^{-x}] - c, & x \geq 2. \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $[t]$ denotes greatest integer less than or equal to t . Then, which of the following statements is true?

[28-Jun-2022-Shift-1]

Options:

- A. There exists $a, b, c \in \mathbb{R}$ such that f is continuous on $\mathbb{R}$.
- B. If f is discontinuous at exactly one point, then $a + b + c = 1$
- C. If f is discontinuous at exactly one point, then $a + b + c \neq 1$
- D. f is discontinuous at at least two points, for any values of a, b and c

Answer: C

Solution:

$$f(x) = \begin{cases} 0 & x < 0 \\ ae^x - 1 & 0 \leq x < 1 \\ b & x = 1 \\ b - 1 & 1 < x < 2 \\ -c & x \geq 2 \end{cases}$$

To be continuous at $x = 0$

$$a - 1 = 0$$

to be continuous at $x = 1$

$$ae^{-1} = b = b - 1 \Rightarrow \text{not possible}$$

to be continuous at $x = 2$

$$b - 1 = -c \Rightarrow b + c = 1$$

If $a = 1$ and $b + c = 1$ then $f(x)$ is discontinuous at exactly one point.

Question 70

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} [x], & x < 0 \\ |1-x|, & x \geq 0. \end{cases} \text{ and}$$

$$g(x) = \begin{cases} e^x - x, & x < 0 \\ (x-1)^2 - 1, & x \geq 0. \end{cases}$$

where $[x]$ denote the greatest integer less than or equal to x . Then, the function fg is discontinuous at exactly :

[28-Jun-2022-Shift-2]

Options:

- A. one point
- B. two points
- C. three points
- D. four points

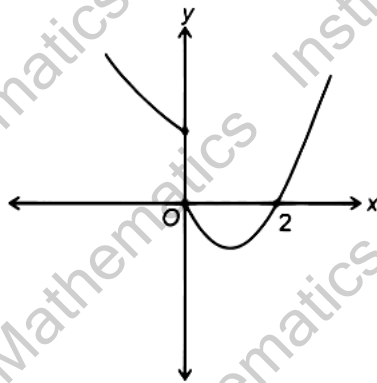
Answer: B

Solution:

$$f(x) = \begin{cases} [x], & x < 0 \\ |1-x|, & x \geq 0 \end{cases}$$

$$\text{and } g(x) = \begin{cases} e^x - x, & x < 0 \\ (x-1)^2 - 1, & x \geq 0 \end{cases}$$

$$f \circ g(x) = \begin{cases} [g(x)], & g(x) < 0 \\ |1-g(x)|, & g(x) \geq 0 \end{cases}$$



(graph of $y = g(x)$)

$$= \begin{cases} |1+x-e^x|, & x < 0 \\ 1, & x = 0 \\ [(x-1)^2 - 1], & 0 < x < 2 \\ |2-(x-1)^2|, & x \geq 2 \end{cases}$$

So, $x = 0, 2$ are the two points where fg is discontinuous.

Question 71

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by :

$$f(x) = \begin{cases} \max\{t^3 - 3t\}; & x \leq 2 \\ t \leq x \\ x^2 + 2x - 6; & 2 < x < 3 \\ [x - 3] + 9; & 3 \leq x \leq 5 \\ 2x + 1; & x > 5. \end{cases}$$

where $[t]$ is the greatest integer less than or equal to t . Let m be the number of points where f is not differentiable and $I = \int_{-2}^2 f(x) dx$. Then the ordered pair (m, I) is equal to :

[29-Jun-2022-Shift-1]

Options:

A. $\left(3, \frac{27}{4}\right)$

B. $\left(3, \frac{23}{4}\right)$

C. $\left(4, \frac{27}{4}\right)$

D. $\left(4, \frac{23}{4}\right)$

Answer: C

Solution:

$$f(x) = \begin{cases} x^3 - 3x & x \leq -1 \\ 2 & -1 < x < 2 \\ x^2 + 2x - 6 & 2 < x < 3 \\ 9 & 3 \leq x < 4 \\ 10 & 4 \leq x < 5 \\ 11 & x = 5 \\ 2x + 1 & x > 5 \end{cases}$$

Clearly $f(x)$ is not differentiable at $x = 2, 3, 4, 5 \Rightarrow m = 4$

$$I = \int_{-2}^1 (x^3 - 3x) dx + \int_{-1}^2 2 \cdot dx = \frac{27}{4}$$

Question 72

The number of points where the function

$$f(x) = \begin{cases} |2x^2 - 3x - 7| & \text{if } x \leq -1 \\ |4x^2 - 1| & \text{if } -1 < x < 1 \\ |x + 1| + |x - 2| & \text{if } x \geq 1. \end{cases}$$

$[t]$ denotes the greatest integer $\leq t$, is discontinuous is

[24-Jun-2022-Shift-1]

Answer: 7

Solution:

$$\therefore f(-1) = 2 \text{ and } f(1) = 3$$

$$\text{For } x \in (-1, 1), (4x^2 - 1) \in [-1, 3)$$

hence $f(x)$ will be discontinuous at $x = 1$ and also

$$\text{whenever } 4x^2 - 1 = 0, 1 \text{ or } 2$$

$$\Rightarrow x = \pm \frac{1}{2}, \pm \frac{1}{\sqrt{2}} \text{ and } \pm \frac{\sqrt{3}}{2}$$

So there are total 7 points of discontinuity.

Question 73

$$\text{Let } f(x) = \begin{cases} |4x^2 - 8x + 5|, & \text{if } 8x^2 - 6x + 1 \geq 0 \\ [4x^2 - 8x + 5], & \text{if } 8x^2 - 6x + 1 < 0 \end{cases}$$

where $[a]$ denotes the greatest integer less than or equal to a . Then the number of points in $\mathbb{R}$ where f is not differentiable is

[25-Jul-2022-Shift-1]

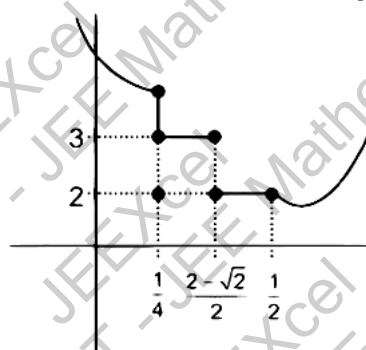
Answer: 3

Solution:

$$f(x) = \begin{cases} |4x^2 - 8x + 5| & \text{if } 8x^2 - 6x + 1 \geq 0 \\ [4x^2 - 8x + 5], & \text{if } 8x^2 - 6x + 1 < 0 \end{cases}$$

$$= \begin{cases} 4x^2 - 8x + 5 & \text{if } x \in \left[-\infty, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \infty\right) \\ [4x^2 - 8x + 5] & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right) \end{cases}$$

$$f(x) = \begin{cases} 4x^2 - 8x + 5 & \text{if } x \in \left(-\infty, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \infty\right) \\ 3 & \text{if } x \in \left(\frac{1}{4}, \frac{2-\sqrt{2}}{2}\right) \\ 2 & \text{if } x \in \left[\frac{2-\sqrt{2}}{2}, \frac{1}{2}\right) \end{cases}$$



$$\therefore \text{Non-diff at } x = \frac{1}{4}, \frac{2-\sqrt{2}}{2}, \frac{1}{2}$$

Question 74

If $f(x) = \begin{cases} x+a, & x \leq 0 \\ |x-4|, & x > 0. \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 0 \\ (x-4)^2+b, & x \geq 0. \end{cases}$ are continuous on $\mathbb{R}$, then $(g \circ f)(2) + (f \circ g)(-2)$ is equal to :
[26-Jul-2022-Shift-1]

Options:

- A. -10
- B. 10
- C. 8
- D. -8

Answer: D

Solution:

$\because f(x)$ and $g(x)$ are continuous on $\mathbb{R}$
 $\therefore a = 4$ and $b = 1 - 16 = -15$
 then $(g \circ f)(2) + (f \circ g)(-2)$
 $= g(2) + f(-1)$
 $= -11 + 3 = -8$

Question 75

If for $p \neq q \neq 0$, the function $f(x) = \frac{\sqrt[7]{p(729+x)} - 3}{\sqrt[3]{729+qx} - 9}$ is continuous at $x = 0$, then :
[27-Jul-2022-Shift-2]

Options:

- A. $7pqf(0) - 1 = 0$
- B. $63qf(0) - p^2 = 0$
- C. $21qf(0) - p^2 = 0$
- D. $7pqf(0) - 9 = 0$

Answer: B

Solution:

$f(x) = \frac{\sqrt[7]{p(729+x)} - 3}{\sqrt[3]{729+qx} - 9}$
 for continuity at $x = 0$, $\lim_{x \rightarrow 0} f(x) = f(0)$
 Now, $\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sqrt[7]{p(729+x)} - 3}{\sqrt[3]{729+qx} - 9}$
 $\Rightarrow p = 3$ (To make indeterminate form)
 So, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{(3^7 + 3x)^{\frac{1}{7}} - 3}{(729 + qx)^{\frac{1}{3}} - 9}$
 $= \lim_{x \rightarrow 0} \frac{3 \left[\left(1 + \frac{x}{3^6}\right)^{\frac{1}{7}} - 1 \right]}{9 \left[\left(1 + \frac{q}{729}x\right)^{\frac{1}{3}} - 1 \right]} = \frac{1}{3} \cdot \frac{\frac{1}{7} \cdot \frac{1}{3^6}}{\frac{1}{3} \cdot \frac{q}{729}}$
 $\therefore f(0) = \frac{1}{7q}$
 $\therefore$ Option (B) is correct.

Question 76

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$f(x) = \lim_{n \rightarrow \infty} \frac{\cos(2\pi x) - x^{2n} \sin(x-1)}{1 + x^{2n+1} - x^{2n}}$ is continuous for all x in :

[28-Jul-2022-Shift-2]

Options:

A. $\mathbb{R} - \{-1\}$

B. $\mathbb{R} - \{-1, 1\}$

C. $\mathbb{R} - \{1\}$

D. $\mathbb{R} - \{0\}$

Answer: B

Solution:

$$f(x) = \lim_{n \rightarrow \infty} \frac{\cos(2\pi x) - x^{2n} \sin(x-1)}{1 + x^{2n+1} - x^{2n}}$$

For $|x| < 1$, $f(x) = \cos 2\pi x$, continuous function

$$|x| > 1, f(x) = \lim_{n \rightarrow \infty} \frac{\frac{1}{x^{2n}} \cos 2\pi x - \sin(x-1)}{\frac{1}{x^{2n}} + x - 1}$$

$$= \frac{-\sin(x-1)}{x-1}, \text{ continuous}$$

$$\text{For } |x| = 1, f(x) = \begin{cases} 1 & \text{if } x = 1 \\ -(1 + \sin 2) & \text{if } x = -1. \end{cases}$$

Now,

$$\lim_{x \rightarrow 1^+} f(x) = -1, \lim_{x \rightarrow 1^-} f(x) = 1, \text{ so discontinuous at } x = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = 1, \lim_{x \rightarrow -1^-} f(x) = -\frac{\sin 2}{2}, \text{ so discontinuous at } x = -1$$

$\therefore f(x)$ is continuous for all $x \in \mathbb{R} - \{-1, 1\}$

Question 77

The number of points, where the function

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x-1| \cos |x-2| \sin |x-1| + (x-3)x^2 - 5x + 4$, is NOT differentiable, is :

[29-Jul-2022-Shift-1]

Options:

A. 1

B. 2

C. 3

D. 4

Answer: B

Solution:

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = |x-1| \cos |x-2| \sin |x-1| + (x-3)x^2 - 5x + 4$$

$$= |x-1| \cos |x-2| \sin |x-1| + (x-3)|x-1||x-4|$$

$$= |x-1| [\cos |x-2| \sin |x-1| + (x-3)|x-4|]$$

Sharp edges at $x = 1$ and $x = 4$

$\therefore$ Non-differentiable at $x = 1$ and $x = 4$

Question 78

Let the function $f(x) = \begin{cases} \frac{\log_e(1+5x) - \log_e(1+\alpha x)}{x} & ; \text{ if } x \neq 0 \\ 10 & ; \text{ if } x = 0. \end{cases}$ be continuous at $x = 0$.

Then α is equal to
[29-Jul-2022-Shift-2]

Options:

- A. 10
- B. -10
- C. 5
- D. -5

Answer: D

Solution:

$f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x)$$

$$\Rightarrow 10 = \lim_{x \rightarrow 0} \frac{\log_e(1+5x) - \log_e(1+\alpha x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\log(1+5x)}{5x} \times 5 - \frac{\log_e(1+\alpha x)}{\alpha x} \times \alpha$$

$$= 1 \times 5 - \alpha$$

$$\Rightarrow \alpha = 5 - 10 = -5$$

Question 79

If $[t]$ denotes the greatest integer $\leq t$, then the number of points, at which the function

$f(x) = 4 \left\lfloor 2x + 3 \right\rfloor + 9 \left\lfloor x + \frac{1}{2} \right\rfloor - 12[x + 20]$ is not differentiable in the open interval $(-20, 20)$, is _____.

[29-Jul-2022-Shift-2]

Answer: 79

Solution:

$$f(x) = 4 \left\lfloor 2x + 3 \right\rfloor + 9 \left\lfloor x + \frac{1}{2} \right\rfloor - 12[x + 20]$$

$$= 4 \left\lfloor 2x + 3 \right\rfloor + 9 \left\lfloor x + \frac{1}{2} \right\rfloor - 12[x] - 240$$

$$f(x) \text{ is non differentiable at } x = -\frac{3}{2}$$

and $f(x)$ is discontinuous at $\{-19, -18, \dots, 18, 19\}$

as well as $\left\{-\frac{39}{2}, -\frac{37}{2}, \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \dots, \frac{39}{2}\right\}$,

at same point they are also non differentiable

$\therefore$ Total number of points of non differentiability

$$= 39 + 40$$

$$= 79$$

Question80

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as If $f(x)$ is continuous on $\mathbb{R}$, then $a + b$ equals
[2021, 26 Feb. Shift-11]

Options:

- A. -3
- B. -1
- C. 3
- D. 1

Answer: B

Solution:

Given, $f(x)$ is continuous on $\mathbb{R}$.

If $f(x)$ is continuous, then f is continuous at $x = 1$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$$

$$\Rightarrow |a + 1 + b| = \sin \pi = 0$$

$$\Rightarrow a + b = -1 \dots (i)$$

Also, f is continuous at $x = -1$

$$\Rightarrow \lim_{x \rightarrow -1^-} f(x) = f(-1) = \lim_{x \rightarrow -1^+} f(x)$$

$$\Rightarrow 2 \sin\left(-\frac{\pi}{2}(-1)\right) = |a - 1 + b|$$

$$\Rightarrow 2 = |a + b - 1| \dots (ii)$$

Eq. (ii) is satisfied.

$$\therefore a + b = -1$$

Question81

Let f be any function defined on $\mathbb{R}$ and let it satisfy the condition $|f(x) - f(y)| \leq |x - y|^2$, $\forall (x, y) \in \mathbb{R}$ If $f(0) = 1$, then
[2021, 26 Feb. Shift-1]

Options:

- A. $f(x)$ can take any value in $\mathbb{R}$
- B. $f(x) < 0$, $\forall x \in \mathbb{R}$
- C. $f(x) = 0$, $\forall x \in \mathbb{R}$
- D. $f(x) > 0$, $\forall x \in \mathbb{R}$

Answer: D

Solution:

$$\text{Given, } |f(x) - f(y)| \leq |x - y|^2$$

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|$$

Now, taking the limit,

$$\lim_{x \rightarrow y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \rightarrow y} |x - y|$$

$$\Rightarrow |f'(y)| \leq 0 \text{ [using the definition of } f'(y)]$$

$$\Rightarrow f'(y) = 0 \text{ [since, modulus value can never be less than 0]}$$

On integrating it, we get

$$f(y) = c \text{ (constant)}$$

$$\text{Given, } f(0) = 1 \text{ gives } c = 1$$

$$\therefore f(y) = 1 \quad \forall y \in \mathbb{R}$$

From given options, $f(x) > 0 \quad \forall x \in \mathbb{R}$ is satisfied only.

Question82

Let $f(x)$ be a differentiable function at $x = a$ with $f'(a) = 2$ and $f(a) = 4$. Then, $\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$ equals

[2021, 26 Feb. Shift-II]

Options:

A. $2a + 4$

B. $4 - 2a$

C. $2a - 4$

D. $a + 4$

Answer: B

Solution:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a} &= \lim_{x \rightarrow a} \frac{xf(a) - af(x) + af(a) - af(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)f(a) - a[f(x) - f(a)]}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)f(a)}{x - a} - a \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= f(a) - af'(a) \\ &= 4 - a(2) \quad [\text{Given, } f(a) = 4, f'(a) = 2] \\ &= 4 - 2a \end{aligned}$$

Question83

A function f is defined on $[-3, 3]$ as $f(x) = \begin{cases} \min\{|x|, 2 - x^2\} & -2 \leq x \leq 2 \\ [x] & , 2 < |x| \leq 3. \end{cases}$

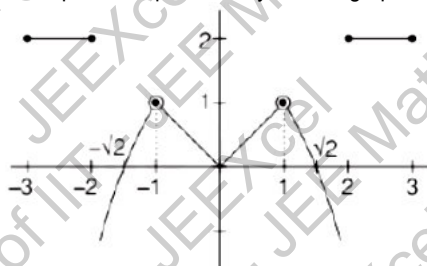
where, $[x]$ denotes the greatest integer $\leq x$. The number of points, where f is not differentiable in $(-3, 3)$ is

[2021, 25 Feb. Shift-II]

Answer: 5

Solution:

For this particular problem, try to draw graph in the region $(-3, 3)$, it will be as follows,



Thus, points of discontinuity will be at $-2, 2$ because the curve breaks at these points and at $-1, 0, 1$ because curve has sharp points.

∴ Point of discontinuity are $-2, -1, 0, 1, 2$ i.e. 5 points.

Question 84

The number of points at which the function $f(x) = |2x + 1| - 3|x + 2| + x^2 + x - 2|$ $x \in \mathbb{R}$ is not differentiable, is
[2021, 25 Feb. Shift-1]

Answer: 2

Solution:

Given,

$$f(x) = |2x + 1| - 3|x + 2| + x^2 + x - 2|$$

$$= |2x + 1| - 3|x + 2| + |x + 2| |x - 1|$$

Here, critical points are $x = -\frac{1}{2}, -2, 1$

$$\therefore f(x) = \begin{cases} x^2 + 2x + 3 & x < -2 \\ -x^2 - 6x - 5 & -2 < x < -\frac{1}{2} \\ -x^2 - 2x - 3 & -\frac{1}{2} < x < 1. \end{cases}$$

$$\text{Now, } f'(x) = \begin{cases} 2x + 2 & x < -2 \\ -2x - 6 & -2 < x < -\frac{1}{2} \\ -2x - 2 & -\frac{1}{2} < x < 1 \\ 2x & x > 1. \end{cases}$$

Now, $f'(x)$ at 1, -2 and $-1/2$.

For $x = 1$,

$$f'(x) = 2x = 2 \times 1 = 2$$

and $-2x - 2 = -(2 \times 1) - 2 = -4$ both are not equal.

$\therefore$ Non-differentiable at $x = 1$

Similarly, for $x = -2$,

$$f'(x) = 2x + 2 = 2 \times (-2) + 2 = -2$$

and $-2x - 6 = -2 \times (-2) - 6 = -2$ both are equal.

$\therefore$ Differentiable at $x = -2$

and for $x = -1/2$, $f'(x) = -2x - 6$

$$= -2 \times \left(-\frac{1}{2}\right) - 6 = -5 \text{ and}$$

$$-2x - 2 = -2 \times \left(-\frac{1}{2}\right) - 2 = -1 \text{ both are not equal.}$$

$\therefore$ Non-differentiable at $x = -1/2$

$\therefore$ The number of points at which $f(x)$ is non-differentiable is 2.

Question 85

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by $f(x) = [x - 1] \cos\left(\frac{2x-1}{2}\right) \pi$, where $[.]$ denotes the greatest integer function, then f is
[2021, 24 Feb. Shift-1]

Options:

- A. discontinuous only at $x = 1$
- B. discontinuous at all integral values of x except at $x = 1$
- C. continuous only at $x = 1$
- D. continuous for every real x

Answer: D

Solution:

Given, $f(x) = [x-1] \cos\left(\frac{2x-1}{2}\right) \pi$ where $[\cdot]$ is greatest integer function and $f: \mathbb{R} \rightarrow \mathbb{R}$

$\therefore$ It is a greatest integer function then we need to check its continuity at $x \in \mathbb{I}$ except these it is continuous.

Let $x = n$ where $n \in \mathbb{I}$

Then, $\text{LHL} = \lim_{x \rightarrow n^-} [x-1] \cos\left(\frac{2x-1}{2}\right) \pi$

$$= (n-2) \cos\left(\frac{2n-1}{2}\right) \pi = 0$$

$\text{RHL} = \lim_{x \rightarrow n^+} [x-1] \cos\left(\frac{2x-1}{2}\right) \pi$

$$= (n-2) \cos\left(\frac{2n-1}{2}\right) \pi = 0$$

and $f(n) = 0$.

Here, $\lim_{x \rightarrow n} f(x) = \lim_{x \rightarrow n^+} f(x) = f(n)$

$\therefore$ It is continuous at every integers.

Therefore, the given function is continuous for all real x .

Question 86

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by $f(x) = [x-1] \cos\left(\frac{2x-1}{2}\right) \pi$, where $[\cdot]$ denotes the greatest integer function, then f is:
24 Feb 2021 Shift 1

Options:

- A. discontinuous at all integral values of x except at $x = 1$
- B. continuous only at $x = 1$
- C. continuous for every real x
- D. discontinuous only at $x = 1$

Answer: C

Solution:

For $x = n, n \in \mathbb{Z}$

$\text{LHL} = \lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} [x-1] \cos\left(\frac{2x-1}{2}\right) \pi = 0$

$\text{RHL} = \lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} [x-1] \cos\left(\frac{2x-1}{2}\right) \pi = 0$

$f(n) = 0$

$\Rightarrow \text{LHL} = \text{RHL} = f(n)$

$\Rightarrow f(x)$ is continuous for every real x .

Question 87

If $f(x) = \begin{cases} \frac{1}{|x|} & |x| \geq 1; ax^2 + b, & |x| < 1. \end{cases}$ is differentiable at every point of the domain, then the values of a and b are respectively
[2021, 18 March shift-1]

Options:

A. $\frac{1}{2}, \frac{1}{2}$

B. $\frac{1}{2}, -\frac{3}{2}$

C. $\frac{5}{2}, -\frac{3}{2}$

D. $-\frac{1}{2}, \frac{3}{2}$

Answer: D

Solution:

Given, $f(x) = \begin{cases} \frac{1}{|x|} & |x| \geq 1; ax^2 + b, & |x| < 1. \end{cases}$

$\Rightarrow f(x) = \begin{cases} \frac{1}{|x|} & x \leq -1 \text{ or } x \geq 1 \\ ax^2 + b & -1 < x < 1. \end{cases}$

$\Rightarrow f(x) = \begin{cases} \frac{-1}{x} & x \leq -1 \\ ax^2 + b & -1 < x < 1 \\ \frac{1}{x} & x \geq 1. \end{cases}$

Given, $f(x)$ is differentiable at every point of domain. $\therefore f'(x) = \begin{cases} \frac{1}{x^2} & x < -1 \\ 2ax & -1 < x < 1 \\ \frac{-1}{x^2} & x > 1. \end{cases}$

$\because f(x)$ is differentiable at $x = 1$

$\therefore (\text{LHD at } x = 1) = (\text{RHD at } x = 1)$

$\Rightarrow f'(1^-) = f'(1^+)$

$\Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$

As, we know that, a function is differentiable at $x = a$, if it is continuous at $x = a$.

Hence, $f(x)$ is also continuous at $x = 1$.

i.e., $(\text{LHL at } x = 1) = (\text{RHL at } x = 1) = f(1)$

$\Rightarrow a + b = 1$

$\Rightarrow \left(-\frac{1}{2}\right) + b = 1$

$\Rightarrow b = \frac{3}{2}$

Hence, $a = -\frac{1}{2}, b = \frac{3}{2}$

Note You can also (or apply) continuity and differentiability at $x = -1$.

Question 88

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x} & \text{if } x < 0 \\ b & \text{if } x = 0 \\ \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & \text{if } x > 0. \end{cases}$$

If f is continuous at $x = 0$, then the value of $a + b$ is equal to
[2021, 18 March Shift-II]

Options:

A. $-\frac{5}{2}$

B. -2

C. -3

D. $-\frac{3}{2}$

Answer: D

Solution:

$$\text{Given, } f(x) = \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x} & x < 0 \\ b & x = 0 \\ \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & x > 0. \end{cases}$$

$\therefore f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) \dots (i)$$

$$\therefore f(0) = b \dots (ii)$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left(\frac{\sin(a+1)x + \sin 2x}{2x} \right) \\ \Rightarrow \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left(\frac{\sin(a+1)x}{2x} + \frac{\sin 2x}{2x} \right) \\ &= \lim_{x \rightarrow 0^-} \left(\frac{\sin(a+1)x}{(a+1)x} \times \left(\frac{a+1}{2} \right) + \frac{\sin 2x}{2x} \right) \\ &= \frac{a+1}{2} + 1 \dots (iii) \end{aligned}$$

$$\begin{aligned} \text{Again, } \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left(\frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} \right) \\ &= \lim_{x \rightarrow 0^+} \frac{(\sqrt{x+bx^3} - \sqrt{x})(\sqrt{x+bx^3} + \sqrt{x})}{bx^{5/2}(\sqrt{x+bx^3} + \sqrt{x})} \\ &= \lim_{x \rightarrow 0^+} \frac{(x+bx^3-x)}{bx^{5/2}(\sqrt{x+bx^3} + \sqrt{x})} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x}(\sqrt{1+bx^2} + 1)} \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \frac{1}{2} \dots (iv)$$

From Eq. (i), (ii), (iii) and (iv)

$$\frac{1}{2} = b = \frac{a+1}{2} + 1$$

$$\Rightarrow b = \frac{1}{2}, a = -2$$

$$\therefore a+b = -\frac{3}{2}$$

Question 89

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) \neq 0$ for any $x \in \mathbb{R}$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then $\lim_{h \rightarrow 0} \frac{1}{h}(f(h) - 1)$ is equal to.....

[2021, 18 March Shift-II]

Answer: 3

Solution:

Method 1

Given, $f(x+y) = f(x) \cdot f(y) \forall x, y \in \mathbb{R}$

$$\therefore f(x) = a^x$$

$$\Rightarrow f'(x) = a^x \cdot \log(a)$$

$$\text{Now, } f'(0) = \log(a)$$

$$\Rightarrow 3 = \log(a)$$

$$\Rightarrow a = e^3$$

$$\therefore f(x) = (e^3)^x = e^{3x}$$

$$f(h) = e^{3h}$$

$$\text{Now, } \lim_{h \rightarrow 0} \left(\frac{f(h) - 1}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{e^{3h} - 1}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{e^{3h} - 1}{3h} \times 3 \right)$$

$$= 3 \times 1 = 3$$

Method(2)

$$\text{Let } L = \lim_{h \rightarrow 0} \frac{1}{h}(f(h) - 1) \left(\frac{0}{0} \text{ form} \right)$$

$$f(x+y) = f(x) + f(y)$$

$$\text{Put } x = y = 0$$

$$\therefore f(0) = f(0) + f(0)$$

$$\Rightarrow [f(0)]^2 = f(0)$$

$$\Rightarrow [f(0)]^2 - f(0) = 0$$

$$\Rightarrow f(0)[f(0) - 1] = 0$$

$$\Rightarrow f(0) = 0, f(0) = 1$$

Rejected because $f(x) \neq 0, \forall x \in \mathbb{R}$

$$\therefore f(0) = 1$$

Using L-Hospital Rule,

$$L = \lim_{h \rightarrow 0} \frac{f'(h) - 0}{1}$$

$$= f'(0) = 3$$

Question90

If the function $f(x) = \frac{\cos(\sin x) - \cos x}{x^4}$ is continuous at each point in its domain and $f(0) = \frac{1}{k}$, then k is

[2021, 17 March Shift-1]

Answer: 6

Solution:

$$f(x) = \frac{\cos(\sin x) - \cos x}{x^4}$$

As, $f(x)$ is continuous everywhere, so

$$f(0) = \lim_{x \rightarrow 0} f(x)$$

$$= \lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$$

On expanding the numerator and only identifying the coefficient of x^4 will give us the required limit.

$$\begin{aligned}
 \cos(\sin x) &= \left(1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24}\right) \\
 &= 1 - \frac{1}{2} \left(x - \frac{x^3}{6}\right)^2 + \frac{1}{24} (x)^4 \\
 &= 1 - \frac{1}{2} \left(x^2 - \frac{x^4}{3}\right) + \frac{x^4}{24} \\
 &= 1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} \\
 \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\
 \therefore \frac{\cos(\sin x) - \cos x}{x^4} \\
 &= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)}{x^4} \\
 &= \frac{1}{6} \\
 \therefore f(0) &= \frac{1}{6} = \frac{1}{k} \\
 \text{Hence, } k &= 6.
 \end{aligned}$$

Question91

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by Then, f is $f(x) = \begin{cases} \left[2 - \sin\left(\frac{1}{x}\right)\right] |x|, & x \neq 0 \\ 0 & x = 0 \end{cases}$.

[2021, 17 March Shift-II]

Options:

- A. monotonic on $(-\infty, 0) \cup (0, \infty)$
- B. not monotonic on $(-\infty, 0)$ and $(0, \infty)$
- C. monotonic on $(0, \infty)$ only
- D. monotonic on $(-\infty, 0)$ only

Answer: B

Solution:

Method (1)

$$\text{Given, } f(x) = \begin{cases} \left[2 - \sin\left(\frac{1}{x}\right)\right] |x|, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here, $f'(x)$ is an oscillating function which is non-monotonic in $(-\infty, 0) \cup (0, \infty)$.

Method (II)

$$\therefore f(x) = \begin{cases} -\left(2 - \sin \frac{1}{x}\right)x & x < 0 \\ 0 & x = 0 \\ \left(2 - \sin \frac{1}{x}\right)x & x > 0. \end{cases}$$

From above we observe that, $f(x)$ is continuous and $f\left(\frac{1}{\pi}\right) = f\left(\frac{2}{\pi}\right) = \frac{2}{\pi}$ So, $f(x)$ is non-monotonic in $(0, \infty)$.

Further, $\lim_{x \rightarrow -\infty} f(x) \rightarrow \infty$ and $\lim_{x \rightarrow \infty} f(x) \rightarrow \infty$

and $f(0) = 0$

Hence, $f(x)$ is non-monotonic on $(-\infty, 0) \cup (0, \infty)$.

Question92

Let the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x+2 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

$$\text{and } g(x) = \begin{cases} x^3 & x < 1 \\ 3x-2 & x \geq 1. \end{cases}$$

Then, the number of points in $\mathbb{R}$, where $(f \circ g)(x)$ is not differentiable is equal to
[2021, 16 March Shift-1]

Options:

- A. 3
- B. 1
- C. 0
- D. 2

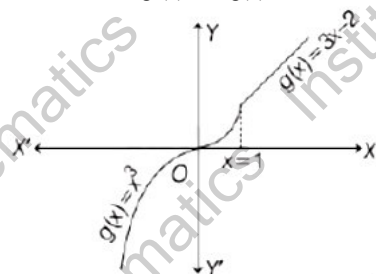
Answer: B

Solution:

$$f(x) = \begin{cases} x+2 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

$$g(x) = \begin{cases} x^3 & x < 1 \\ 3x-2 & x \geq 1. \end{cases}$$

$$f[g(x)] = \begin{cases} g(x)+2 & g(x) < 0 \\ g^2(x) & g(x) \geq 0. \end{cases}$$



When $g(x) < 0 \Rightarrow g(x) = x^3, x < 0$

When $g(x) \geq 0 \Rightarrow g(x) = \begin{cases} x^3 & 0 \leq x < 1 \\ 3x-2 & x \geq 1. \end{cases}$

$$f[g(x)] = \begin{cases} x^3+2 & x < 0 \\ x^6 & 0 \leq x < 1 \\ (3x-2)^2 & x \geq 1. \end{cases}$$

As, polynomial function is continuous everywhere in its domain. So, $f[g(x)]$ will be continuous everywhere at $x < 0$, $0 < x < 1$ and $x > 1$. We will check the behaviour of $f \circ g(x)$ only at boundary points which is $x = 0$ and $x = 1$.

$$\lim_{x \rightarrow 0^-} x^6 = 0 \rightarrow 0^-(x^3+2) = 2$$

Clearly, $L^+ H L \neq R H L$ at $x = 0$

So, $f \circ g(x)$ is discontinuous at $x = 0$.

$$\lim_{x \rightarrow 1^-} (3x-2)^2 = 1 \quad \lim_{x \rightarrow 1^-} x^6 = 1$$

Also $f(1) = 1$ $f \circ g(x)$ is continuous at $x = 1$

Derivative test at $x = 1$,

$$L H D = \lim_{h \rightarrow 0} \frac{f(1) - f(1-h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1 - (1-h)^6}{h}$$

$$= \lim_{h \rightarrow 0} 6(1-h)^5 = 6$$

$$R H D = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{[3(1+h)-2]^2 - 1}{h} \\
 &= \lim_{h \rightarrow 0} 2[3(1+h)-2] \cdot 3 = 6 \\
 &\therefore f(x) \text{ is continuous and differentiable at } x = 1.
 \end{aligned}$$

Question 93

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x+a & x < 0 \\ |x-1| & x \geq 0. \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x+1 & x < 0 \\ (x-1)^2 + b & x \geq 0. \end{cases}$$

where a, b are non-negative real numbers. If $(g \circ f)(x)$ is continuous for all $x \in \mathbb{R}$, then $a + b$ is equal to.....

[2021, 16 March Shift-II]

Answer: 1

Solution:

$$g(x) = \begin{cases} x+1 & x < 0 \\ (x-1)^2 + b & x \geq 0. \end{cases}$$

$$g[f(x)] = \begin{cases} f(x)+1 & f(x) < 0 \\ [f(x)-1]^2 + b, & f(x) \geq 0. \end{cases}$$

$$f(x) < 0$$

$$\text{Case I } x+a < 0 \text{ and } x < 0 \Rightarrow x < -a$$

$$\text{Case II } |x-1| < 0 \text{ and } x \geq 0 \Rightarrow \text{Not possible } f(x) \geq 0$$

$$\text{Case I } x+a \geq 0 \text{ and } x < 0 \Rightarrow x \in [-a, 0)$$

$$\text{Case II } |x-1| \geq 0 \text{ and } x \geq 0 \Rightarrow x \geq 0$$

$$g[f(x)] = \begin{cases} x+a+1 & x < -a \\ (x+a-1)^2 + b & -a \leq x < 0 \\ (|x-1|-1)^2 + b & x \geq 0. \end{cases}$$

This is continuous function.

Since, $g[f(x)]$ is continuous for all $x \in \mathbb{R}$

So, $g[f(x)]$ will be continuous at $x = -a$ and $x = 0$

Now, at $x = -a$

$$\text{LHL} = \text{RHL} = \text{value of function}$$

$$\Rightarrow 1 = 1 + b = 1 + b \Rightarrow b = 0$$

At $x = 0$

$$\text{LHL} = \text{RHL} = \text{value of function}$$

$$\Rightarrow (a-1)^2 + b = b$$

$$\Rightarrow (a-1)^2 = 0$$

$$\Rightarrow a = 1$$

$$\text{Hence, } a+b = 1$$

Question 94

$$\text{Let } \alpha \in \mathbb{R} \text{ be such that the function } f(x) = \begin{cases} \frac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3} & x \neq 0 \\ \alpha & x = 0 \end{cases} \text{ is continuous at } x = 0, \text{ where}$$

$\{x\} = x - [x]$, $[x]$ is the greatest integer less than or equal to x .

**Then,
[2021, 16 March Shift-II]**

Options:

A. $\alpha = \frac{\pi}{\sqrt{2}}$

B. $\alpha = 0$

C. no such α exists

D. $\alpha = \frac{\pi}{4}$

Answer: C

Solution:

$$\text{Given, } f(x) = \begin{cases} \frac{\cos^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3} & x \neq 0 \\ \alpha & x = 0 \end{cases}$$

$$\{x\} = x - [x]$$

So, when $x \rightarrow 0^+$

$$\Rightarrow \{x\} = x - 0 = x$$

And, when $x \rightarrow 0^-$

$$\Rightarrow \{x\} = x + 1$$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0} \frac{\cos^{-1}[1 - (1+x)^2] \sin^{-1}[1 - (1+x)]}{(1+x) - (1+x)^3}$$

$$= \lim_{x \rightarrow 0} \frac{\cos^{-1}(-x^2 - 2x) \sin^{-1}(-x)}{(1+x)(1+1+x)(1-1-x)}$$

$$= \lim_{x \rightarrow 0} \frac{\cos^{-1}(-x^2 - 2x)}{(1+x)(x+2)}$$

$$= \frac{\cos^{-1}(0)}{1 \cdot 2} = \frac{\pi}{4}$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x)$$

$$= \lim_{x \rightarrow 0} \frac{\cos^{-1}(1 - x^2) \sin^{-1}(1 - x)}{x(1-x)(1+x)}$$

$$= \frac{\pi}{2} \lim_{x \rightarrow 0} \frac{\cos^{-1}(1 - x^2)}{x}$$

Applying L-Hospital Rule,

$$= \frac{\pi}{2} \lim_{x \rightarrow 0} \frac{(-1)(-2x)}{\sqrt{1 - (1 - x^2)^2}}$$

$$= \frac{\pi}{2} \cdot 2 \lim_{x \rightarrow 0} \frac{x}{\sqrt{2x^2 - x^4}}$$

$$= \pi \lim_{x \rightarrow 0} \frac{1}{\sqrt{2 - x^2}}$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\text{LHL} = \frac{\pi}{4} \text{ and } \text{RHL} = \frac{\pi}{\sqrt{2}}$$

Hence, $\text{LHL} \neq \text{RHL}$

So, the function will be discontinuous for every value of $\alpha \in \mathbb{R}$.

$\therefore$ No such α exist.

Question 95

Let $f : S \rightarrow S$, where $S = (0, \infty)$ be a twice differentiable function, such that $f(x+1) = xf(x)$. If $g : S \rightarrow \mathbb{R}$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to

[2021, 16 March Shift-II]

Options:

- A. $\frac{205}{144}$
- B. $\frac{197}{144}$
- C. $\frac{187}{144}$
- D. 1

Answer: A

Solution:

We have, $f : S \rightarrow S, S = (0, \infty)$

$$f(x+1) = x \cdot f(x)$$

$$g : S \rightarrow R$$

$$g(x) = \log_e f(x)$$

To find $g''(5) - g''(1)$

$$\Rightarrow g'(x+1) = \log_e f(x+1)$$

$$\Rightarrow g'(x+1) = \log[x \cdot f(x)]$$

$$\Rightarrow g'(x+1) = \log x + \log f(x)$$

$$\Rightarrow g'(x+1) = \log x + g'(x)$$

$$\Rightarrow g'(x+1) - g'(x) = \log x$$

$$\Rightarrow g''(x+1) - g''(x) = \frac{1}{x}$$

$$\Rightarrow g''(x+1) - g''(x) = \frac{-1}{x^2}$$

$$x=1, g''(2) - g''(1) = -1 \dots (i)$$

$$x=2, g''(3) - g''(2) = -1/4 \dots (ii)$$

$$x=3, g''(4) - g''(3) = -1/9 \dots (iii)$$

$$x=4, g''(5) - g''(4) = -1/16 \dots (iv)$$

Adding Eqs. (i), (ii), (iii) and (iv),

$$g''(5) - g''(1) = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16}$$

$$= -\left(\frac{144+36+16+9}{144}\right)$$

$$= -\frac{205}{144}$$

$$\text{So, } |g''(5) - g''(1)| = \frac{205}{144}$$

Question 96

Let $f : [0, 3] \rightarrow R$ be defined by $f(x) = \min\{x - [x], 1 + [x] - x\}$ where $[x]$ is the greatest integer less than or equal to x . Let P denote the set containing all x in $(0, 3)$, where f is discontinuous and Q denote the set containing all x in $(0, 3)$, where f is not differentiable.

Then the sum of number of elements in P and Q is equal to

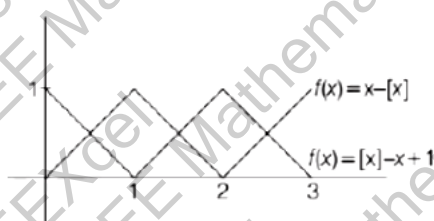
[2021, 27 July Shift-1]

Answer: 5

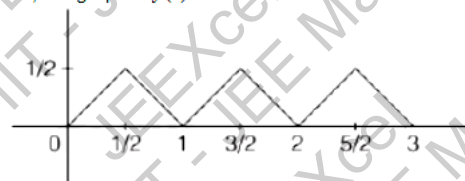
Solution:

$$f(x) = \min\{x - [x], 1 + [x] - x\}$$

$$f(x) = \min\{\{x\}, 1 - \{x\}\}$$



So, the graph of $f(x)$ will be



f is continuous everywhere for $0 \leq x \leq 3$. But f is non-differentiable at $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and $x = 1, 2$.

So, if set A denotes the points of discontinuity, then $n(A) = 0$.

And if set B denotes the points of non-differentiable, then

$n(B) = 5$

$\therefore n(A) + n(B) = 0 + 5 = 5$

Question 97

Let $f : \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \rightarrow \mathbb{R}$ be defined as $f(x) = \begin{cases} (1 + |\sin x|) \frac{3a}{|\sin x|} & -\frac{\pi}{4} < x < 0 \\ b & x = 0 \\ \frac{\cot 4x}{e^{\cot 2x}} & 0 < x < \frac{\pi}{4} \end{cases}$

If f is continuous at $x = 0$, then the value of $6a + b^2$ is equal to
[2021, 27 July Shift I]

Options:

- A. $1 - e$
- B. $e - 1$
- C. $1 + e$
- D. e

Answer: C

Solution:

$$f : \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} (1 + |\sin x|) \frac{3a}{|\sin x|} & -\frac{\pi}{4} < x < 0 \\ b & x = 0 \\ \frac{\cot 4x}{e^{\cot 2x}} & 0 < x < \frac{\pi}{4} \end{cases}$$

Given $f(x)$ is continuous at $x = 0$

LHL at $x = 0$

Put $x = 0 - h$

we get $\lim_{h \rightarrow 0} (1 - \sinh) \frac{3a}{\sinh}$

$$\lim_{eh \rightarrow 0} \frac{(1 - \sinh - 1)}{-\sinh} \cdot \frac{3a}{-\sinh} = e^{3a}$$

$$\lim_{x \rightarrow 0^+} (1 + |\sin x|) \frac{3a}{|\sin x|}$$

$$= e^{3a} \lim_{x \rightarrow 0^+} \frac{|\sin x|}{|\sin x|} = e^{3a}$$

RHL at $x = 0$

$$\lim_{x \rightarrow 0^+} e^{\frac{\cot 4x}{\cot 2x}}$$

Put $x = 0 + h$

$$\text{we get } \lim_{h \rightarrow 0} e^{\frac{\cot 4h}{\cot 2h}}$$

$$\lim_{h \rightarrow 0} \frac{\cos 4h}{e \cos 2h} \times \frac{\sin 2h}{\sin 4h}$$

$$e^{\frac{\cos 4h}{\cos 2h} \times \frac{\sin 2h}{\sin 4h} \times 2h} = e^{1/2}$$

As, $f(x)$ is continuous at $x = 0$.

So, LHL = $f(0)$ = RHL

$$e^{3a} = b = e^{\frac{1}{2}}$$

$$\therefore a = \frac{1}{6}, b = \sqrt{e}$$

$$\therefore 6a + b^2 = 6\left(\frac{1}{6}\right) + (\sqrt{e})^2 = 1 + e$$

Question 98

Let $f : [0, \infty) \rightarrow [0, 3]$ be a function defined by $f(x) = \begin{cases} \max\{\sin t : 0 \leq t \leq x\} & 0 \leq x < \pi \\ 2 + \cos x & x > \pi. \end{cases}$

Then which of the following is true?
[2021, 27 July Shift-11]

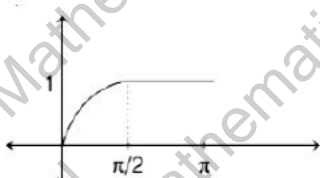
Options:

- A. f is continuous everywhere but not differentiable exactly at one point in $(0, \infty)$
- B. f is differentiable everywhere in $(0, \infty)$
- C. f is not continuous exactly at two points in $(0, \infty)$
- D. f is continuous everywhere but not differentiable exactly at two points in $(0, \infty)$

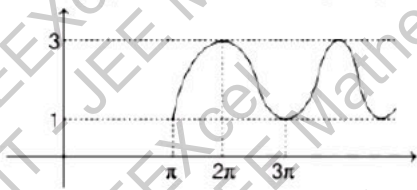
Answer: B

Solution:

Graph of $\max(\sin t : 0 \leq t \leq x)$ in $x \in [0, \pi]$

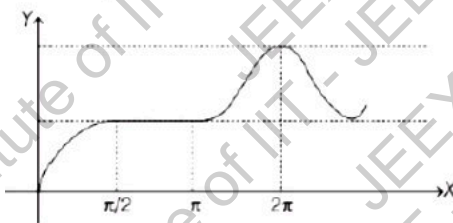


and graph of $2 + \cos x$ for $x \in [\pi, \infty]$



So, graph of

$$f(x) = \begin{cases} \max[\sin t : 0 \leq t \leq x], & 0 \leq x \leq \pi \\ 2 + \cos x, & x > \pi \end{cases}$$



So, $f(x)$ is differentiable everywhere in $(0, \infty)$.

Question99

Let $f : (a, b) \rightarrow \mathbb{R}$ be twice differentiable function such that $f(x) = \int_a^x g(t) dt$ for a differentiable function $g(x)$. If $f(x) = 0$ has exactly five distinct roots in (a, b) , then $g(x)g'(x) = 0$ has at least [2021, 27 July Shift-II]

Options:

- A. twelve roots in (a, b)
- B. five roots in (a, b)
- C. seven roots in (a, b)
- D. three roots in (a, b)

Answer: C

Solution:

We have, $f(x) = \int_a^x g(t) dt$

So, $f'(x) = g(x)$ and $f''(x) = g'(x)$

$$\left\{ \begin{array}{l} \because f(x) = \int_a^{h(x)} g(t) \cdot dt \Rightarrow f'(x) \\ = F[h(x)] \cdot h'(x) - F[g(x)] \cdot g'(x) \end{array} \right\}$$

Now, $g(x)g'(x) = 0$

$\Rightarrow f''(x)f'(x) = 0$

If $f(x)$ has five roots, then $f'(x)$ has atleast 4 roots and $f''(x)$ has atleast 3 roots.

So, $f''(x) \cdot f'(x) = 0$ has atleast 7 roots. Hence, the minimum number of roots of the equation $g(x)g'(x) = 0$ is 7.

Question100

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{\lambda x^2 - 5x + 6}{\mu(5x - x^2 - 6)} & x < 2 \\ \frac{\tan(x-2)}{e^{x-[x]}} & x > 2 \\ \mu & x = 2. \end{cases}$$

where, $[x]$ is the greatest integer less than or equal to x . If f is continuous at $x = 2$, then $\lambda + \mu$ is equal to
[2021, 25 July Shift-1]

Options:

- A. $e e(-e+1)$
- B. $e(e-2)$
- C. 1
- D. $2e-1$

Answer: A

Solution:

$$\text{We have } f(x) = \begin{cases} \frac{\lambda |x^2 - 5x + 6|}{\mu(5x - x^2 - 6)} & x < 2 \\ \frac{\tan(x-2)}{e^{x-[x]}} & x > 2 \\ \mu & x = 2. \end{cases}$$

$f(x)$ is continuous at $x = 2$.

$$\therefore \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\therefore \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{\lambda}{\mu} \frac{|(x-3)(x-2)|}{-(x-3)(x-2)}$$

$$= \lim_{x \rightarrow 2^-} \frac{\lambda}{\mu} \frac{(x-3)(x-2)}{-(x-3)(x-2)} = -\frac{\lambda}{\mu}$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} e^{-\frac{\tan(x-2)}{x-2}} = e$$

As, $f(x)$ is continuous.

So, LHL = $f(2)$ = RHL

$$-\frac{\lambda}{\mu} = \mu = e$$

$$\lambda = -e^2$$

$$\mu = e$$

$$\lambda + \mu = e(-e+1)$$

Question 101

Let $f : [0, \infty) \rightarrow [0, \infty)$ be defined as $f(x) = \int_0^x [y] dy$ where, $[x]$ is the greatest integer less than or equal to x . Which of the following is true?
[2021, 25 July Shift-1]

Options:

- A. f is continuous at every point in $[0, \infty)$ and differentiable except at the integer points
- B. f is both continuous and differentiable except at the integer points in $[0, \infty)$
- C. f is continuous everywhere except at the integer points in $[0, \infty)$
- D. f is differentiable at every point in $[0, \infty)$

Answer: A

Solution:

$$f: [0, \infty) \rightarrow [0, \infty)$$

$$f(x) = \int_0^x [y] dy$$

$$\text{Let } x = 1 + f, 0 < f < 1$$

$$f(x) = \int_0^1 [y] dy + \int_1^2 [y] dy + \int_2^3 [y] dy + \dots + \int_{1-f}^1 [y] dy$$

$$+ \int_{1+f}^1 [y] dy$$

$$f(x) = 0 + 1 + 2 + \dots + (1-1) + 1 \cdot f$$

$$= \frac{(1-1)(1-1+1)}{2} + 1 \cdot f$$

$$= \frac{1(1-1)}{2} + 1 \cdot f$$

$$f(x) = \frac{[x]([x]-1)}{2} + [x][x]$$

$$f(x) = \frac{[x]([x]-1)}{2} + [x](x - [x])$$

$$f(1) = \frac{1(1-1)}{2}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} \frac{1(1-1)}{2} + 1(1+h-1)$$

$$= \frac{1(1-1)}{2}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} \frac{1(1-1)(1-2)}{2} + (1-1)(1+h-1+1)$$

$$= \frac{(1-1)(1-2)}{2} + (1-1)$$

$$= \frac{(1-1)}{2}$$

$\therefore f(x)$ is continuous and differentiable except at integer points.

Question 102

$$\text{If } f(x) = \begin{cases} \int_0^x (5 + |1-t|) dt & x > 2 \\ 5x + 1 & x \leq 2. \end{cases}, \text{ then}$$

[2021, 25 July Shift-11]

Options:

- A. $f(x)$ is not continuous at $x = 2$
- B. $f(x)$ is everywhere differentiable
- C. $f(x)$ is continuous but not differentiable at $x = 2$
- D. $f(x)$ is not differentiable at $x = 1$

Answer: C

Solution:

$$f(x) = \begin{cases} \int_0^x (5 + |1-t|) dt, & x > 2; \\ 5x + 1, & x \leq 2. \end{cases}$$

$$\int_0^x 5 + |1-t| dt$$

$$= \int_0^1 5 + (1-t) dt + \int_1^x 5 + (t-1) dt$$

$$= \int_0^1 (6-t) dt + \int_1^x (4+t) dt$$

$$= \left[6t - \frac{t^2}{2} \right]_0^1 + \left[4t + \frac{t^2}{2} \right]_1^x = 1 + 4x + \frac{x^2}{2}$$

$$\Rightarrow f(x) = \begin{cases} 1 + 4x + \frac{x^2}{2} & x > 2 \\ 5x + 1 & x \leq 2 \end{cases}$$

At $x = 2$

$$\text{LHL} = \lim_{x \rightarrow 2^-} (5x + 1) = 11$$

$$\text{RHL} = \lim_{x \rightarrow 2^+} \left(1 + 4x + \frac{x^2}{2} \right) = 1 + 8 + 2 = 11$$

$$\therefore f(2) = 11$$

So, $f(x)$ is continuous at $x = 2$.

$$f'(x) = \begin{cases} 4 + x & x > 2 \\ 5 & x \leq 2 \end{cases}$$

$$\text{Now, LHD at } x = 2 \text{ is } \frac{d}{dx}(5x + 1)_{x=2} = 5$$

$$\text{RHD at } x = 2 \text{ is } 4 + 2 = 6$$

Here, $\text{LHD} \neq \text{RHD}$

So, function is not differentiable at $x = 2$.

Question 103

Consider the function $f(x) = \begin{cases} \frac{P(x)}{\sin(x-2)} & x \neq 2 \\ 7 & x = 2 \end{cases}$

where, $P(x)$ is a polynomial such that $P'''(x)$ is always a constant and $P(3) = 9$. If $f(x)$ is continuous at $x = 2$, then $P(5)$ is equal to

[2021, 25 July Shift-11]

Answer: 39

Solution:

$$f(x) = \begin{cases} P(x)/\sin(x-2) & x \neq 2 \\ 7 & x = 2 \end{cases}$$

Given, that $P'''(x)$ is always a constant.

$\Rightarrow P(x)$ is a 2 degree polynomial.

$f(x)$ is continuous at $x = 2$

$$\lim_{x \rightarrow 2^+} P(x)/\sin(x-2) = 7$$

$$\Rightarrow \lim_{x \rightarrow 2^+} (x-2)(ax+b)/\sin(x-2) = 7$$

$$\Rightarrow 2a + b = 7 \dots (i)$$

Now, $P(x) = (x-2)(ax+b)$

$P(3) = 9$ (given)

$$\Rightarrow 3a + b = 9$$

Subtracting Eq. (ii) from Eq. (i),

$$a = 2$$

From Eq. (i), $b = 3$

Hence, $P(x) = (x-2)(2x+3)$

$$\text{So, } P(5) = (5-2)(2 \times 5 + 3) = 3 \times 13 = 39$$

Question 104

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \begin{cases} \frac{x^3}{(1 - \cos 2x)^2} \log_e \left(\frac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2} \right) & x \neq 0 \\ \alpha & x = 0 \end{cases}$

If f is continuous at $x = 0$, then α is equal to

[2021, 22 July Shift-II]

Options:

- A. 1
- B. 3
- C. 0
- D. 2

Answer: A

Solution:

$$f(x) = \begin{cases} \frac{x^3}{(1 - \cos 2x)^2} \log_e \left[\frac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2} \right] & x \neq 0 \\ \alpha & x = 0. \end{cases}$$

For continuity, $\lim_{x \rightarrow 0} \frac{x^3}{4\sin^4 x} [\log_e (1 + 2xe^{-2x}) - \log_e (1 - xe^{-x})^2] = \alpha$ (by expansion) ... (i)

$$\therefore \log(1 + 2xe^{-2x}) = 2xe^{-2x} - \frac{(2xe^{-2x})^2}{2} + \dots \text{ and } \log(1 - xe^{-x}) = -xe^{-x} - \frac{(xe^{-x})^2}{2} - \dots$$

On putting the values in Eq. (i), we get

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{4} \cdot \frac{x}{x} \right) \left(\frac{x^3}{\sin^4 x} \right) [2xe^{-2x} - 2(-xe^{-x})] \\ = \lim_{x \rightarrow 0} \left(\frac{1}{4x} \right) \left(\frac{x}{\sin x} \right)^4 (2xe^{-2x} + 2xe^{-x}) \\ = \lim_{x \rightarrow 0} \left(\frac{1}{4x} \right) \cdot \left(\frac{x}{\sin x} \right)^4 \cdot 2x \cdot (e^{-2x} + e^{-x}) \\ = \left(\frac{1}{4} \right) \cdot (1) \cdot (2) \cdot (2) \Rightarrow \alpha = 1 \end{aligned}$$

Question 105

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined $f(x) = \begin{cases} 3\left(1 - \frac{|x|}{2}\right) & \text{if } |x| \leq 2; \\ 0, & \text{if } |x| > 2. \end{cases}$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = f(x+2) - f(x-2)$

If n and m denote the number of points in $\mathbb{R}$,

where g is not continuous and not differentiable respectively, then $n + m$ is equal to

[2021, 22 July Shift-II]

Answer: 4

Solution:

$$f(x) = \begin{cases} 3\left(1 - \frac{|x|}{2}\right) & \text{if } |x| \leq 2; \\ 0, & \text{if } |x| > 2. \end{cases}$$

$$g(x) = f(x+2) + f(x-2)$$

$$f(x) = \begin{cases} 0 & x < -2 \\ \frac{3}{2}(1+x) & -2 \leq x < 0 \\ \frac{3}{2}(1-x) & 0 \leq x < 2 \\ 0 & x > 2. \end{cases}$$

$$f(x+2) = \begin{cases} 0 & x < -4 \\ \frac{3}{2}(3+x) & -4 \leq x < -2 \\ \frac{3}{2}(-1-x) & -2 \leq x < 0 \\ 0 & x > 4. \end{cases}$$

$$f(x-2) = \begin{cases} 0 & x < 0 \\ \frac{3}{2}(x-1) & 0 \leq x < 2 \\ \frac{3}{2}(-1-x) & 2 \leq x < 4 \\ 0 & x > 4. \end{cases}$$

$$g(x) = f(x+2) + f(x-2)$$

$$= \begin{cases} \frac{3x}{2} + 6 & -4 \leq x \leq 2 \\ -\frac{3x}{2} & -2 < x < 2 \\ \frac{3x}{2} - 6 & 2 \leq x \leq 4 \\ 0 & x > 4. \end{cases}$$

So, $n = 0$ and $m = 4$
 $\therefore m + n = 4$

Question 106

Let a function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \begin{cases} \sin x - e^x & \text{if } x \leq 0 \\ a + [-x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \geq 1. \end{cases}$

where, $[x]$ is the greatest integer less than or equal to x . If f is continuous on $\mathbb{R}$, then $(a + b)$ is
[2021, 20 July Shift-1]

Options:

- A. 4
- B. 3
- C. 2
- D. 5

Answer: B

Solution:

$$f(x) = \begin{cases} \sin x - e^x & x < 0 \\ a + [-x] & 0 < x < 1 \\ 2x - b & x \geq 1. \end{cases}$$

$f(x)$ is continuous.

$$\text{So, } \lim_{x \rightarrow 0^-} f(x) = 0 - e^0 = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = a - 1$$

$$\Rightarrow a - 1 = -1 \Rightarrow a = 0$$

$$\lim_{h \rightarrow 0} f(1) = \lim_{h \rightarrow 0} a + [-1 - h] = a - 1$$

$$\lim_{h \rightarrow 0} f^{-1}(1^+) = 2(1 + h) - b = 2 - b$$

$$\therefore 2 - b = a - 1 \Rightarrow b = 2 + 1 = 3$$

$$\therefore a + b = 3$$

Question 107

Let a function $g : [0, 4] \rightarrow \mathbb{R}$ be defined as $g(x) = \begin{cases} \max_{0 \leq t \leq x} (t^3 - 6t^2 + 9t - 3) & 0 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4. \end{cases}$

then the number of points in the interval $(0, 4)$ where $g(x)$ is not differentiable, is
[2021, 20 July Shift-II]

Answer: 1

Solution:

$$\text{Let } f(x) = x^3 - 6x^2 + 9x - 3$$

$$f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0 \text{ gives}$$

$$3x^2 - 12x + 9 = 0$$

$$\Rightarrow 3(x-1)(x-3) = 0$$

$$\therefore x = 1 \text{ or } x = 3$$

$$\text{Now, } f(1) = 1 \text{ and } f(3) = -3$$

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq 1 \\ 1 & 1 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4. \end{cases}$$

$g(x)$ is continuous.

$$g'(x) = \begin{cases} 3(x-1)(x-3) & 0 \leq x \leq 1 \\ 0 & 1 \leq x < 3 \\ -1 & 3 < x \leq 4. \end{cases}$$

$g(x)$ is non-differentiable at $x = 3$.

So, the number of points in $(0, 4)$ where $g(x)$ is not differentiable is 1.

Question 108

The function $f(x) = x^2 - 2x - 3 \cdot e^{9x^2 - 12x + 4}$ is not differentiable at exactly
[2021, 31 Aug. Shift-1]

Options:

- A. four points
- B. three points
- C. two points
- D. one point

Answer: C

Solution:

$$f(x) = \begin{cases} (x-3)(x+1)e^{(3x-2)^2} & x > 3 \\ -(x-3)(x+1)e^{(3x-2)^2} & -1 \leq x \leq 3 \\ (x-3)(x+1)e^{(3x-2)^2} & x < -1. \end{cases}$$

At $x = -1$, let LHD be α , then it's clear that RHD be $-\alpha$.

Similarly, at $x = 3$, if LHD is β , then RHD at $x = 3$ will be $-\beta$.

So, $f(x)$ is not differentiable at $x = -1$, $x = 3$

At all other points $f(x)$ will be differentiable.

Question 109

If the function $f(x) =$

$$\begin{cases} \frac{1}{x} \log_e \left(\frac{1 + \frac{x}{a}}{1 - \frac{x}{b}} \right) & x < 0 \\ k & x = 0 \\ \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 1} - 1} & x > 0. \end{cases}$$

is continuous at $x = 0$, then $\frac{1}{a} + \frac{1}{b} + \frac{4}{k}$ is equal to

[2021,31 Aug. Shift-1]

Options:

- A. -5-2
- B. 5
- C. -4
- D. 4

Answer: A

Solution:

$f(x)$ is continuous at $x = 0$

LHL at $x = 0 = f(0) = \text{RHL at } x = 0$

$$\lim_{x \rightarrow 0^-} \frac{\ln \left(\frac{1 + \frac{x}{a}}{1 - \frac{x}{b}} \right)}{x} = \lim_{x \rightarrow 0^-} \frac{\left(\frac{1}{a} \right) \ln \left(1 + \frac{x}{a} \right)}{\left(\frac{1}{a} \right) x}$$

$$= \lim_{x \rightarrow 0^-} \frac{\left(-\frac{1}{b} \right) \ln \left(1 - \frac{x}{b} \right)}{\left(-\frac{1}{b} \right) x} = \left(\frac{1}{a} + \frac{1}{b} \right)$$

$$\lim_{x \rightarrow 0^+} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 1} - 1}$$

$$\lim_{x \rightarrow 0^+} \frac{-2\sin^2 x}{\sqrt{x^2 + 1} - 1} = \lim_{x \rightarrow 0^+} - \left(\frac{2\sin^2 x}{x^2} \right)$$

$$(\sqrt{x^2 + 1} + 1) = -4$$

$$\Rightarrow \frac{1}{a} + \frac{1}{b} = -4 = k$$

$$\left(\frac{1}{a} + \frac{1}{b} \right) + \left(\frac{4}{k} \right) = -4 - 1 = -5$$

Question 110

Let $a, b \in \mathbb{R}$, $b \neq 0$. Define a function

$f(x) = \begin{cases} \sin \frac{\pi}{2}(x-1), & \text{for } x \leq 0 \\ \frac{\tan 2x - \sin 2x}{bx^3}, & \text{for } x > 0 \end{cases}$ If f is continuous at $x = 0$,

then $10 - ab$ is equal to

[2021, 26 Aug. Shift-1]

Answer: 14

Solution:

For continuity

LHL at 0 = $f(0) =$ RHL at 0

$$\text{LHL} = \lim_{x \rightarrow 0^-} a \sin \frac{\pi}{2}(x-1)$$

$$= -a \sin \frac{\pi}{2} = -a \dots (i)$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \frac{\tan 2x - \sin 2x}{bx^3}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin 2x(1 - \cos 2x)}{bx^3 \cdot \cos 2x}$$

$$= \lim_{x \rightarrow 0^+} 2 \left(\frac{\sin 2x}{2x} \right) \left(\frac{2\sin^2 x}{x^2} \right) \cdot \frac{1}{b \cos 2x} = \frac{4}{b} \dots (ii)$$

From Eqs. (i) and (ii), we get

$$-a = \frac{4}{b}$$

$$\Rightarrow ab = -4$$

$$\Rightarrow 10 - ab = 14$$

Question 111

Let $[t]$ denote the greatest integer less than or equal to t . Let $f(x) = x - [x]$, $g(x) = 1 - x + [x]$, and $h(x) = \min\{f(x), g(x)\}$, $x \in [-2, 2]$. Then h is
[2021, 26 Aug. Shift-II]

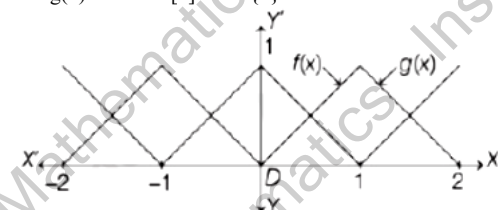
Options:

- A. continuous in $[-2, 2]$ but not differentiable at more than four points in $(-2, 2)$
- B. not continuous at exactly three points in $[-2, 2]$
- C. continuous in $[-2, 2]$ but not differentiable at exactly three points in $(-2, 2)$
- D. not continuous at exactly four points in $[-2, 2]$

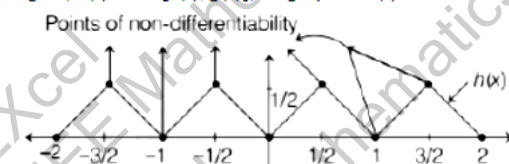
Answer: A

Solution:

We have, $f(x) = x - [x] = \{x\}$
and $g(x) = 1 - x + [x] = 1 - \{x\}$



Again, $h(x) = \min\{f(x), g(x)\}$, so graph of $h(x)$ will be



From graph, it is clear that $h(x)$ is continuous in $[-2, 2]$ but not differentiable at $x = -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$ in $(-2, 2)$.

Question 112

Let $x^k + y^k = a^k$, ($a, k > 0$) and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then k is:

[Jan. 7, 2020 (I)]

Options:

A. $\frac{3}{2}$

B. $\frac{4}{3}$

C. $\frac{2}{3}$

D. $\frac{1}{3}$

Answer: C

Solution:

$$k \cdot x^{k-1} + k \cdot y^{k-1} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$$

$$\Rightarrow k - 1 = -\frac{1}{3}$$

$$\Rightarrow k = 1 - \frac{1}{3} = \frac{2}{3}$$

Question 113

The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is:

[Jan. 7, 2020 (II)]

Options:

A. $\frac{4 - \sqrt{5}}{3}$

B. $\frac{4 - \sqrt{7}}{3}$

C. $\frac{2}{3}$

D. $\frac{\sqrt{7} - 2}{3}$

Answer: B

Solution:

Since, $f(x)$ is a polynomial function.

$\therefore$ It is continuous and differentiable in $[0, 1]$

Here, $f(0) = 11$, $f(1) = 1 - 4 + 8 + 11 = 16$

$$f'(x) = 3x^2 - 8x + 8$$

$$\therefore f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{16 - 11}{1}$$

$$= 3c^2 - 8c + 8$$

$$\Rightarrow 3c^2 - 8c + 3 = 0$$

$$\Rightarrow c = \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3}$$

$$\therefore c = \frac{4 - \sqrt{7}}{3} \in (0, 1)$$

Question 114

Let $[t]$ denote the greatest integer $\leq t$ and $\lim_{x \rightarrow 0} x \left\lfloor \frac{4}{x} \right\rfloor = A$. Then the function, $f(x) = [x^2] \sin(\pi x)$ is discontinuous, when x is equal to :
[Jan. 9, 2020 (II)]

Options:

- A. $\sqrt{A+1}$
- B. $\sqrt{A+5}$
- C. $\sqrt{A+21}$
- D. $\sqrt{A}$

Answer: A

Solution:

$$\lim_{x \rightarrow 0} x \left\lfloor \frac{4}{x} \right\rfloor = A \Rightarrow \lim_{x \rightarrow 0} x \left[\frac{4}{x} - \left\{ \frac{4}{x} \right\} \right] = A$$

$$\Rightarrow \lim_{x \rightarrow 0} 4 - x \left\{ \frac{4}{x} \right\} = A \Rightarrow 4 - 0 = A$$

As, $f(x) = [x^2] \sin(\pi x)$ will be discontinuous at non-integers. And, when $x = \sqrt{A+1} \Rightarrow x = \sqrt{5}$, which is not an integer.

Hence, $f(x)$ is discontinuous when x is equal to $\sqrt{A+1}$

Question 115

If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

$$f(x) = \begin{cases} \frac{1}{x} \log_e \left(\frac{1+3x}{1-2x} \right), & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$$

is continuous, then k is equal to _____.
[NA Jan. 7, 2020 (II)]

Answer: 5

Solution:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{1}{x} \ln \left(\frac{1+3x}{1-2x} \right) \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{\ln(1+3x)}{x} - \frac{\ln(1-2x)}{x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{3 \ln(1+3x)}{3x} - \frac{2 \ln(1-2x)}{-2x} \right)$$

$$= 3 + 2 = 5$$

$\therefore f(x)$ will be continuous

$\therefore k = f(0) = \lim_{x \rightarrow 0} f(x) = 5$

Question 116

If c is a point at which Rolle's theorem holds for the function, $f(x) = \log_e \left(\frac{x^2 + a}{7x} \right)$ in the interval $[3, 4]$,

where $a \in \mathbb{R}$, then $f''(c)$ is equal to:

[Jan. 8, 2020 (I)]

Options:

A. $-\frac{1}{12}$

B. $\frac{1}{12}$

C. $-\frac{1}{24}$

D. $\frac{\sqrt{3}}{7}$

Answer: B

Solution:

Since, Rolle's theorem is applicable

$$\therefore f(a) = f(b)$$

$$f(3) = f(4) \Rightarrow a = 12$$

$$f'(x) = \frac{x^2 - 12}{x(x^2 + 12)}$$

As $f'(c) = 0$ (by Rolle's theorem)

$$x = \pm\sqrt{12}, \therefore c = \sqrt{12}, \therefore f''(c) = \frac{1}{12}$$

Question 117

$$\text{If } f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & : x < 0 \\ b & : x = 0 \\ \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & : x > 0 \end{cases}$$

is continuous at $x = 0$, then $a + 2b$ is equal to:

[Jan. 9, 2020 (I)]

Options:

A. 1

B. -1

C. 0

D. -2

Answer: C

Solution:

$$\text{LHL} = \lim_{x \rightarrow 0} \frac{\sin(a+2)x + \sin x}{x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin(a+2)x}{(a+2)x} \right) (a+2) + \lim_{x \rightarrow 0} \frac{\sin x}{x} = a+3$$

$$f(0) = b$$

$$\text{RHL} = \lim_{h \rightarrow 0} \left(\frac{(1+3h)^{\frac{1}{3}} - 1}{h} \right) = 1$$

$\therefore$ Function $f(x)$ is continuous
 $\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$
 $\therefore a + 3 = 1 \Rightarrow a = -2$
 and $b = 1$
 Hence, $a + 2b = 0$

Question 118

Let f and g be differentiable functions on $\mathbb{R}$ such that $f \circ g$ is the identity function. If for some $a, b \in \mathbb{R}$, $g'(a) = 5$ and $g(a) = b$, then $f'(b)$ is equal to:

[Jan. 9, 2020 (II)]

Options:

A. $\frac{1}{5}$

B. 1

C. 5

D. $\frac{2}{5}$

Answer: A

Solution:

It is given that functions f and g are differentiable and $f \circ g$ is identity function.

$$\therefore (f \circ g)(x) = x \Rightarrow f(g(x)) = x$$

Differentiating both sides, we get

$$f'(g(x)) \cdot g'(x) = 1$$

Now, put $x = a$, then

$$f'(g(a)) \cdot g'(a) = 1$$

$$f'(b) \cdot 5 = 1$$

$$f'(b) = \frac{1}{5}$$

Question 119

Let S be the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$, which are continuous on $[0, 1]$ and differentiable on $(0, 1)$. Then for every f in S , there exists a $c \in (0, 1)$, depending on f , such that:

[Jan. 8, 2020 (II)]

Options:

A. $|f(c) - f(1)| < (1 - c) |f'(c)|$

B. $\frac{f(1) - f(c)}{1 - c} = f'(c)$

C. $|f(c) + f(1)| < (1 + c) |f'(c)|$

D. Bonus

Answer: D

Solution:

For a constant function $f(x)$, option (1), (3) and (4) doesn't hold and by LMVT theorem, option (2) is incorrect.

Question 120

Let the function, $f : [-7, 0] \rightarrow \mathbb{R}$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \geq 2$, for all $x \in (-7, 0)$, then for all such functions f , $f'(-1) + f(0)$ lies in the interval:
[Jan. 7, 2020 (I)]

Options:

- A. $(-\infty, 20]$
- B. $[-3, 11]$
- C. $(-\infty, 11]$
- D. $[-6, 20]$

Answer: A

Solution:

From, LMVT for $x \in [-7, -1]$

$$\frac{f(-1) - f(-7)}{(-1) - (-7)} \leq 2 \Rightarrow \frac{f(-1) + 3}{6} \leq 2 \Rightarrow f(-1) \leq 9$$

From, LMVT for $x \in [-7, 0]$

$$\frac{f(0) - f(-7)}{(0) - (-7)} \leq 2$$

$$\frac{f(0) + 3}{7} \leq 2 \Rightarrow f(0) \leq 11$$

$$\therefore f(0) + f(-1) \leq 20$$

Question 121

Let S be the set of points where the function, $f(x) = |2 - |x - 3||$, $x \in \mathbb{R}$, is not differentiable. Then $\sum_{x \in S} f(f(x))$ is equal to _____.

[NA Jan. 7, 2020 (I)]

Answer: 3

Solution:

$\because f(x)$ is non differentiable at $x = 1, 3, 5$

$\because x - 3$ is not differentiable at $x = 3$

$$\sum f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$$

$$= 1 + 1 + 1 = 3$$

Question 122

If $x = 2 \sin \theta - \sin 2\theta$ and $y = 2 \cos \theta - \cos 2\theta$, $\theta \in [0, 2\pi]$, then $\frac{dy}{dx}$ at $\theta = \pi$ is :

[Jan. 9, 2020 (II)]

Options:

- A. $\frac{3}{4}$
- B. $\frac{3}{8}$
- C. $\frac{3}{2}$

D. $-\frac{3}{4}$

Answer: B

Solution:

It is given that

$$x = 2 \sin \theta - \sin 2\theta \dots (i)$$

$$y = 2 \cos \theta - \cos 2\theta \dots (ii)$$

Differentiating (i) w.r.t. θ , we get

$$\frac{dx}{d\theta} = 2 \cos \theta - 2 \cos 2\theta$$

Differentiating (ii) w.r.t. θ ; we get

$$\frac{dy}{d\theta} = -2 \sin \theta + 2 \sin 2\theta$$

From (ii)+(i), we get

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} \\ &= \frac{2 \sin \frac{\theta}{2} \cdot \cos \frac{3\theta}{2}}{2 \sin \frac{\theta}{2} \cdot \sin \frac{3\theta}{2}} = \cot \frac{3\theta}{2} \dots (iii) \end{aligned}$$

Again, differentiating eqn. (iii), we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-3}{2} \operatorname{cosec} \frac{3\theta}{2} \cdot \frac{d\theta}{dx} \\ \frac{d^2y}{dx^2} &= \frac{-3 \operatorname{cosec} \frac{3\theta}{2}}{2(\cos \theta - \cos 2\theta)} \\ \frac{d^2y}{dx^2}(\theta = \pi) &= \frac{3}{4(-1-1)} = -\frac{3}{8} \end{aligned}$$

Question 123

If $y(\alpha) = \sqrt{2 \left(\frac{\tan \alpha + \cot \alpha}{1 + \tan^2 \alpha} \right) + \frac{1}{\sin^2 \alpha}}$, $\alpha \in \left(\frac{3\pi}{4}, \pi \right)$, then $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$ is:

[Jan. 7, 2020 (I)]

Options:

A. 4

B. $\frac{4}{3}$

C. -4

D. $-\frac{1}{4}$

Answer: A

Solution:

$$\begin{aligned} y(\alpha) &= \sqrt{\frac{2 \sin \alpha + \cos \alpha}{\cos \alpha \sec^2 \alpha}} = \sqrt{\frac{2 \cos^2 \alpha}{\sin \alpha \cos \alpha} + \frac{1}{\sin^2 \alpha}} \\ &= \sqrt{2 \cot \alpha + \operatorname{cosec}^2 \alpha} = \sqrt{2 \cot \alpha + 1 + \cot^2 \alpha} \\ &= |1 + \cot \alpha| = -1 - \cot \alpha \left[\because \alpha \in \left(\frac{3\pi}{4}, \pi \right) \right] \\ \frac{dy}{d\alpha} &= \operatorname{cosec}^2 \alpha \Rightarrow \left(\frac{dy}{d\alpha} \right)_{\alpha = \frac{5\pi}{6}} = 4 \end{aligned}$$

Question 124

Let $y = y(x)$ be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$.

Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$, is equal to:

[Jan. 7, 2020 (II)]

Options:

A. $-\frac{\sqrt{5}}{4}$

B. $-\frac{\sqrt{5}}{2}$

C. $\frac{2}{\sqrt{5}}$

D. $\frac{\sqrt{5}}{2}$

Answer: B

Solution:

$$\text{Given, } x = \frac{1}{2}, y = -\frac{1}{4} \Rightarrow xy = -\frac{1}{8}$$

$$y \cdot \frac{1 \cdot (-2x)}{2\sqrt{1-x^2}} + y' \sqrt{1-x^2}$$

$$= - \left\{ 1 \cdot \sqrt{1-y^2} + \frac{x \cdot (-2y)}{2\sqrt{1-y^2}} y' \right\}$$

$$\Rightarrow -\frac{xy}{\sqrt{1-x^2}} + y' \sqrt{1-x^2} = -\sqrt{1-y^2} + \frac{xy \cdot y'}{\sqrt{1-y^2}}$$

$$\Rightarrow y' \left(\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \right) = \frac{xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}$$

$$\Rightarrow y' \left(\frac{\sqrt{3}}{2} + \frac{1}{8 \cdot \frac{\sqrt{15}}{4}} \right) = \frac{-1}{8 \cdot \frac{\sqrt{3}}{2}} - \frac{\sqrt{15}}{4}$$

$$\Rightarrow y' \left(\frac{\sqrt{45}+1}{2\sqrt{15}} \right) = -\frac{(1+\sqrt{45})}{4\sqrt{3}}$$

$$\therefore y' = -\frac{\sqrt{5}}{2}$$

Question 125

For all twice differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, with $f(0) = f(1) = f'(0) = 0$

[Sep. 06, 2020 (II)]

Options:

A. $f''(x) \neq 0$ at every point $x \in (0,1)$

B. $f''(x) = 0$, for some $x \in (0,1)$

C. $f''(0) = 0$

D. $f''(x) = 0$, at every point $x \in (0,1)$

Answer: B

Solution:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, with $f(0) = f(1) = 0$ and $f'(0) = 0$

$\therefore f(x)$ is differentiable and continuous and

$f(0) = f(1) = 0$

Then by Rolle's theorem, $f'(c) = 0$, $c \in (0,1)$

Now again

$\therefore f'(c) = 0$, $f''(c) = 0$

Then, again by Rolle's theorem,
 $f''(x) = 0$ for some $x \in (0, 1)$

Question 126

If $y^2 + \log_e(\cos^2 x) = y$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then :

[Sep. 03, 2020 (I)]

Options:

- A. $y''(0) = 0$
- B. $|y'(0)| + |y''(0)| = 1$
- C. $|y''(0)| = 2$
- D. $|y'(0)| + |y''(0)| = 3$

Answer: C

Solution:

$$y^2 + 2\log_e(\cos x) = y \dots\dots(i)$$

$$\Rightarrow 2yy' - 2\tan x = y' \dots\dots(ii)$$

From (i), $y(0) = 0$ or 1

$$\therefore y'(0) = 0$$

Again differentiating (ii) we get,

$$2(y')^2 + 2yy'' - 2\sec^2 x = y''$$

Put $x = 0$, $y(0) = 0, 1$ and $y'(0) = 0$

we get, $|y''(0)| = 2$.

Question 127

Let $f(x) = x \cdot \left[\frac{x}{2} \right]$, for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to _____.

[NA Sep. 05, 2020 (I)]

Answer: 0

Solution:

We know $[x]$ discontinuous for $x \in \mathbb{Z}$

$f(x) = x \left[\frac{x}{2} \right]$ may be discontinuous where $\frac{x}{2}$ is an integer.

So, points of discontinuity are,

$x = \pm 2, \pm 4, \pm 6, \pm 8$ and 0

but at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = 0 = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

So, $f(x)$ will be discontinuous at $x = \pm 2, \pm 4, \pm 6$ and ± 8 .

Question 128

If a function $f(x)$ defined by

$$f(x) = \begin{cases} ae^x + be^{-x}, & -1 \leq x < 1 \\ cx^2, & 1 \leq x \leq 3 \\ ax^2 + 2cx, & 3 < x \leq 4 \end{cases}$$

be continuous for some $a, b, c \in \mathbb{R}$ and $f'(0) + f'(2) = e$, then the value of a is :
[Sep. 02, 2020 (I)]

Options:

A. $\frac{1}{e^2 - 3e + 13}$

B. $\frac{e}{e^2 - 3e - 13}$

C. $\frac{e}{e^2 + 3e + 13}$

D. $\frac{e}{e^2 - 3e + 13}$

Answer: D

Solution:

Since, function $f(x)$ is continuous at $x = 1, 3$

$$\therefore f(1) = f(1^+)$$

$$\Rightarrow ae + be^{-1} = c \dots (i)$$

$$f(3) = f(3^+)$$

$$\Rightarrow 9c = 9a + 6c \Rightarrow c = 3a \dots (ii) \text{ From (i) and (ii),}$$

$$b = ae(3 - e) \dots (iii)$$

$$f'(x) = \begin{cases} ae^x - be^{-x} & -1 < x < 1 \\ 2cx & 1 < x < 3 \\ 2ax + 2c & 3 < x < 4 \end{cases}$$

$$f'(0) = a - b, f'(2) = 4c$$

$$\text{Given, } f'(0) + f'(2) = e$$

$$a - b + 4c = e$$

From eqs. (i), (ii), (iii) and (iv),

$$a - 3ae + ae^2 + 12a = e$$

$$\Rightarrow 13a - 3ae + ae^2 = e$$

$$\Rightarrow a = \frac{e}{e^2 - 3e + 13}$$

Question 129

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \max\{x, x^2\}$. Let S denote the set of all points in $\mathbb{R}$, where f is not differentiable. Then:
[Sep. 06, 2020 (II)]

Options:

A. $\{0, 1\}$

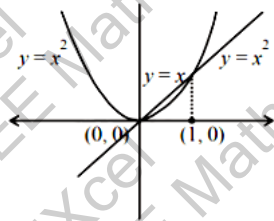
B. $\{0\}$

C. π (an empty set)

D. $\{1\}$

Answer: A

Solution:



$$f(x) = \max\{x, x^2\}$$

$$\Rightarrow f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x < 1 \\ x^2, & x \geq 1 \end{cases}$$

$\therefore f(x)$ is not differentiable at $x = 0, 1$

Question 130

If the function $f(x) = \begin{cases} k_1(x-\pi)^2 - 1, & x \leq \pi \\ k_2 \cos x, & x > \pi \end{cases}$ is twice differentiable, then the ordered pair (k_1, k_2) is equal to:

[Sep. 05, 2020 (I)]

Options:

A. $(\frac{1}{2}, 1)$

B. $(1, 0)$

C. $(\frac{1}{2}, -1)$

D. $(1, 1)$

Answer: A

Solution:

$f(x)$ is differentiable then, $f(x)$ is also continuous.

$$\therefore \lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^+} f(x) = f(\pi)$$

$$\Rightarrow -1 = -K_2 \Rightarrow K_2 = 1$$

$$\therefore f'(x) = \begin{cases} 2K_1(x-\pi) & : x \leq \pi \\ -K_2 \sin x & : x > \pi \end{cases}$$

$$\text{Then, } \lim_{x \rightarrow \pi^-} f'(x) = \lim_{x \rightarrow \pi^+} f'(x) = 0$$

$$f''(x) = \begin{cases} 2K_1 & ; x \leq \pi \\ -K_2 \cos x & ; x > \pi \end{cases}$$

$$\text{Then, } \lim_{x \rightarrow \pi^-} f''(x) = \lim_{x \rightarrow \pi^+} f''(x)$$

$$\Rightarrow 2K_1 = K_2 \Rightarrow K_1 = \frac{1}{2}$$

$$\text{So, } (K_1, K_2) = (\frac{1}{2}, 1)$$

Question 131

Let f be a twice differentiable function on $(1, 6)$. If $f(2) = 8, f'(2) = 5, f'(x) \geq 1$ and $f''(x) \geq 4$, for all $x \in (1, 6)$, then

[Sep. 04, 2020 (I)]

Options:

- A. $f(5) + f'(5) \leq 26$
 B. $f(5) + f'(5) \geq 28$
 C. $f'(5) + f''(5) \leq 20$
 D. $f(5) \leq 10$

Answer: B

Solution:

Let f be twice differentiable function

$$\because f'(x) \geq 1$$

$$\Rightarrow \frac{f(5) - f(2)}{3} \geq 1$$

$$\Rightarrow f(5) \geq 3 + f(2)$$

$$\Rightarrow f(5) \geq 3 + 8 \Rightarrow f(5) \geq 11$$

and also $f''(x) \geq 4$

$$\Rightarrow \frac{f'(5) - f'(2)}{3} \geq 4 \Rightarrow f'(5) \geq 12 + f'(2)$$

$$\Rightarrow f'(5) \geq 17$$

$$\text{Hence, } f(5) + f'(5) \geq 28$$

Question 132

Suppose a differentiable function $f(x)$ satisfies the identity $f(x+y) = f(x) + f(y) + xy^2 + x^2y$, for all real x and y . If $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$, then $f'(3)$ is equal to _____.

[NA Sep. 04, 2020 (I)]

Answer: 10

Solution:

$$f(x+y) = f(x) + f(y) + xy^2 + x^2y$$

Differentiate w.r.t. x :

$$f'(x+y) = f'(x) + 0 + y^2 + 2xy$$

Put $y = -x$

$$f'(0) = f'(x) + x^2 - 2x^2 \dots\dots(i)$$

$$\because \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1 \Rightarrow f(0) = 0$$

$$\therefore f'(0) = 1 \dots\dots(ii)$$

From equations (i) and (ii),

$$f'(x) = (x^2 + 1) \Rightarrow f'(3) = 10$$

Question 133

The function

$$f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1}x, & |x| \leq 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$$

is :

[Sep. 04, 2020 (II)]

Options:

- A. continuous on $\mathbb{R} - \{1\}$ and differentiable on $\mathbb{R} - \{-1, 1\}$

B. both continuous and differentiable on $\mathbb{R} - \{1\}$.

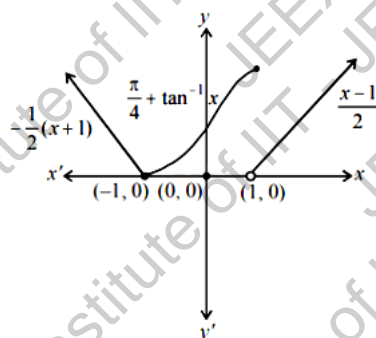
C. continuous on $\mathbb{R} - \{-1\}$ and differentiable on $\mathbb{R} - \{-1, 1\}$

D. both continuous and differentiable on $\mathbb{R} - \{-1\}$.

Answer: A

Solution:

$$f(x) = \begin{cases} \frac{-x-1}{2}, & x < -1 \\ \frac{\pi}{4} + \tan^{-1}x, & -1 \leq x \leq 1 \\ \frac{1}{2}(x-1), & x > 1 \end{cases}$$



It is clear from above graph that,
 $f(x)$ is discontinuous at $x = 1$,
i.e. continuous on $\mathbb{R} - \{1\}$
 $f(x)$ is non-differentiable at $x = -1, 1$
i.e. differentiable on $\mathbb{R} - \{-1, 1\}$.

Question 134

The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = \frac{1}{2}$ is :

[Sep. 05, 2020 (II)]

Options:

A. $\frac{2\sqrt{3}}{5}$

B. $\frac{\sqrt{3}}{12}$

C. $\frac{2\sqrt{3}}{3}$

D. $\frac{\sqrt{3}}{10}$

Answer: D

Solution:

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan \theta \Rightarrow \theta = \tan^{-1}x$

$\therefore u = \tan^{-1}\left(\frac{\sec \theta - 1}{\tan \theta}\right) = \tan^{-1}\left(\tan \frac{\theta}{2}\right)$

$= \frac{\theta}{2} = \frac{1}{2} \tan^{-1}x$

$\therefore \frac{du}{dx} = \frac{1}{2} \times \frac{1}{(1+x^2)}$

$$\text{Let } v = \tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right)$$

$$\text{Put } x = \sin \phi \Rightarrow \phi = \sin^{-1} x$$

$$v = \tan^{-1} \left(\frac{2 \sin \phi \cos \phi}{\cos 2\phi} \right) = \tan^{-1}(\tan 2\phi)$$

$$= 2\phi = 2\sin^{-1} x$$

$$\frac{dv}{dx} = 2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1-x^2}{4(1+x^2)}$$

$$\therefore \left(\frac{du}{dv} \right)_{\left(x=\frac{1}{2}\right)} = \frac{\sqrt{3}}{10}$$

Question 135

If $(a + \sqrt{2}b \cos x)(a - \sqrt{2}b \cos y) = a^2 - b^2$, where $a > b > 0$ then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is

[Sep. 04, 2020 (I)]

Options:

A. $\frac{a-2b}{a+2b}$

B. $\frac{a-b}{a+b}$

C. $\frac{a+b}{a-b}$

D. $\frac{2a+b}{2a-b}$

Answer: C

Solution:

$$(a + \sqrt{2}b \cos x)(a - \sqrt{2}b \cos y) = a^2 - b^2$$

Differentiating both sides,

$$(-\sqrt{2}b \sin x)(a - \sqrt{2}b \cos y) + (a + \sqrt{2}b \cos x)$$

$$(\sqrt{2}b \sin y) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sqrt{2}b \sin x)(a - \sqrt{2}b \cos y)}{(a + \sqrt{2}b \cos x)(\sqrt{2}b \sin y)}$$

$$\therefore \left[\frac{dy}{dx} \right] \left(\frac{\pi}{4}, \frac{\pi}{4} \right) = \frac{a-b}{a+b} \Rightarrow \frac{dx}{dy} = \frac{a+b}{a-b}$$

Question 136

If $y = \sum_{k=1}^6 k \cos^{-1} \left\{ \frac{3}{5} \cos kx - \frac{4}{5} \sin kx \right\}$, then $\frac{dy}{dx}$ at $x = 0$ is _____.

[NA Sep. 02, 2020 (II)]

Answer: 91

Solution:

$$y = \sum_{k=1}^6 k \cos^{-1} \left\{ \frac{3}{5} \cos kx - \frac{4}{5} \sin kx \right\}$$

$$\text{Let } \cos a = \frac{3}{5} \text{ and } \sin a = \frac{4}{5}$$

$$\begin{aligned}
 \therefore y &= \sum_{k=1}^6 k \cos^{-1} \{ \cos a \cos kx - \sin a \sin kx \} \\
 &= \sum_{k=1}^6 k \cos^{-1} (\cos(kx + a)) \\
 &= \sum_{k=1}^6 k(kx + a) = \sum_{k=1}^6 (k^2 x + ak) \\
 \therefore \frac{dy}{dx} &= \sum_{k=1}^6 k^2 = \frac{6(7)(13)}{6} = 91
 \end{aligned}$$

Question 137

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} 5, & \text{if } x \leq 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \leq x < 5 \\ 30, & \text{if } x \geq 5 \end{cases}$$

Then, f is :

[Jan 09, 2019 (I)]

Options:

- A. continuous if $a = 5$ and $b = 5$
- B. continuous if $a = -5$ and $b = 10$
- C. continuous if $a = 0$ and $b = 5$
- D. not continuous for any values of a and b

Answer: D

Solution:

Let $f(x)$ is continuous at $x = 1$, then

$$f(1^-) = f(1) = f(1^+)$$

$$\Rightarrow 5 = a + b \dots (1)$$

Let $f(x)$ is continuous at $x = 3$, then

$$f(3^-) = f(3) = f(3^+)$$

$$\Rightarrow a + 3b = b + 15 \dots (2)$$

Let $f(x)$ is continuous at $x = 5$, then

$$f(5^-) = f(5) = f(5^+)$$

$$\Rightarrow b + 25 = 30$$

$$\Rightarrow b = 30 - 25 = 5$$

From (1), $a = 0$

But $a = 0$, $b = 5$ do not satisfy equation (2)

Hence, $f(x)$ is not continuous for any values of a and b