



Complex Numbers and Quadratic Equations

Question 1 If $x^2 + x + 1 = 0$, then the value of $(x + \frac{1}{x})^4 + (x^2 + \frac{1}{x^2})^4 + (x^3 + \frac{1}{x^3})^4 + \dots + (x^{25} + \frac{1}{x^{25}})^4$ is:

Options: **JEE Main 2026 (Online) 21st January Morning Shift**

- A. 162
- B. 145
- C. 128
- D. 175

Answer: B

Solution:

$$x^2 + x + 1 = 0 \begin{cases} W \\ W^2 \end{cases}$$

$$(w + \frac{1}{w})^4 + (w^2 + \frac{1}{w^2})^4 + (w^3 + \frac{1}{w^3})^4 + \dots + (w^{25} + \frac{1}{w^{25}})^4$$

$$\sum \left(w^k + \frac{1}{w^k} \right)^4$$

$$K = 3x \Rightarrow w^{3x} + \frac{1}{w^{3x}} = 2$$

$$k \neq 3x \Rightarrow w^k + \frac{1}{w^k} = -1$$

$$\sum_{k=1}^{25} \left(w^k + \frac{1}{w^k} \right)^4 \Rightarrow 8(1 + 1 + 2^4) + 1 = 145$$

Question 2

Let $z = (1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni)$, where $i = \sqrt{-1}$. If $|z|^2 = 44200$, then n is equal to _____

JEE Main 2026 (Online) 24th January Evening Shift

Answer: 5

Solution:

$$Z = (1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni)$$

$$|Z| = |(1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni)|$$

$$|Z| = |1 + i| \cdot |1 + 2i| \cdot |1 + 3i| \dots |1 + ni|$$

$$|Z|^2 = |1 + i|^2 \cdot |1 + 2i|^2 \cdot |1 + 3i|^2 \dots |1 + ni|^2$$

$$|Z|^2 = (1^2 + 1^2)(1^2 + 2^2)(1^2 + 3^2) \dots (1^2 + n^2) \{ \text{as } z = x + iy, |z|^2 = x^2 + y^2 \}$$

It is given that

$$|Z|^2 = 44200$$

$$(1 + 1^2)(1 + 2^2)(1 + 3^2) \dots (1 + n^2) = 44200$$

$$44200 = 2^3 \cdot 5^2 \cdot 13 \cdot 17$$

$$(1 + 1^2)(1 + 2^2)(1 + 3^2) \cdot (1 + 4^2)(1 + 5^2) \dots (1 + n^2) = 2^3 \cdot 5^2 \cdot 13 \cdot 17$$

solve some product terms

$$2 \cdot 5 \cdot (10) \cdot (17) \cdot (26) \dots (1 + n^2) = (2)(5)(2 \times 5)(17)(2 \times 13)$$

this matches for $n = 5$

so value of $n = 5$

Question3

If $x^2 + x + 1 = 0$, then the value of $\left(x + \frac{1}{x}\right)^4 + \left(x^2 + \frac{1}{x^2}\right)^4 + \left(x^3 + \frac{1}{x^3}\right)^4 + \dots + \left(x^{25} + \frac{1}{x^{25}}\right)^4$ is:

JEE Main 2026 (Online) 21st January Morning Shift

Options:

A.

162

B.

145

C.

128

D.

175

Answer: B

Solution:

$$x^2 + x + 1 = 0 \begin{cases} w \\ w^2 \end{cases}$$

$$\left(w + \frac{1}{w}\right)^4 + \left(w^2 + \frac{1}{w^2}\right)^4 + \left(w^3 + \frac{1}{w^3}\right)^4 + \dots + \left(w^{25} + \frac{1}{w^{25}}\right)^4$$

$$\sum \left(w^k + \frac{1}{w^k}\right)^4$$

$$K = 3x \Rightarrow w^{3x} + \frac{1}{w^{3x}} = 2$$

$$k \neq 3x \Rightarrow w^k + \frac{1}{w^k} = -1$$

$$\sum_{k=1}^{25} \left(w^k + \frac{1}{w^k}\right)^4 \Rightarrow 8(1 + 1 + 2^4) + 1 = 145$$

Question4

Let z be the complex number satisfying $|z - 5| \leq 3$ and having maximum positive principal argument.

Then $34 \left| \frac{5z-12}{5iz+16} \right|^2$ is equal to:

JEE Main 2026 (Online) 21st January Evening Shift

Options:

A.

20

B.

26

C.

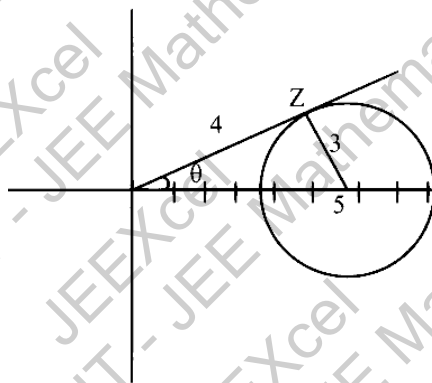
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D.

16

Answer: A

Solution:



$$\Rightarrow \arg(z) = \sin^{-1}\left(\frac{3}{5}\right) \tan^{-1}\left(\frac{3}{4}\right) \Rightarrow z = |z|e^{i\theta}$$

$$= 4 \times (\cos \theta + i \sin \theta) = 4 \times \left(\frac{4}{5} + i \frac{3}{5}\right)$$

$$= \frac{16}{5} + \frac{12i}{5} \Rightarrow 5z = 16 + 12i$$

$$\left(\frac{5z - 12}{5zi + 16}\right) = \frac{(4 + 12i)}{(4 + 16i)} = \frac{(1 + 3i)}{(1 + 4i)}$$

$$\Rightarrow \left(\frac{5z - 12}{5zi + 16}\right) = \frac{\sqrt{10}}{\sqrt{17}}$$

$$34 \left(\frac{5z - 12}{5zi + 16}\right)^2 = 34 \times \frac{10}{17} = 20$$

Question5

Let $S = \{z \in \mathbb{C} : 4z^2 + \bar{z} = 0\}$. Then $\sum_{z \in S} |z|^2$ is equal to:

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Options:

A.

$\frac{5}{64}$

B.

$\frac{1}{16}$

C.

$\frac{7}{64}$

D.

$$\frac{3}{16}$$

Answer: D

Solution:

$$4z^2 = \bar{z} = 0$$

$$\text{Let } z = x + iy$$

$$4(x + iy)^2 + x - iy = 0$$

$$4x^2 - 4y^2 + 8xyi + x - iy = 0$$

$$4x^2 - 4y^2 + x = 0 \text{ and } 8xy - 1 = 0$$

$$\Rightarrow y = 0 \text{ or } x = \frac{1}{8}$$

$$\text{If } y = 0, 4x^2 + x = 0$$

$$x = 0, \frac{-1}{4}$$

$$\therefore z_1 = 0 + 0i \quad |z_1|^2 = 0$$

$$z_2 = -\frac{1}{4} + 0i \quad |z_2|^2 = \frac{1}{16}$$

$$\text{If } x = \frac{1}{8}$$

$$4 \times \frac{1}{64} - 4y^2 + \frac{1}{8} = 0 \Rightarrow 4y^2 = \frac{3}{16} \Rightarrow y = \pm \frac{\sqrt{3}}{8}$$

$$\therefore z_3 = \frac{1}{8} + \frac{\sqrt{3}}{8}i \quad |z_3|^2 = \frac{1}{64} + \frac{3}{64} = \frac{1}{16}$$

$$z_4 = \frac{1}{8} - \frac{\sqrt{3}}{8}i \quad |z_4|^2 = \frac{1}{64} + \frac{3}{64} = \frac{1}{16}$$

$$\therefore \sum_{i=1}^4 |z_i|^2 = 0 + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$$

Question 6

Let $S = \{z : 3 \leq |2z - 3(1 + i)| \leq 7\}$ be a set of complex numbers.

Then $\min_{z \in S} \left| \left(z + \frac{1}{2}(5 + 3i) \right) \right|$ is equal to :

JEE Main 2026 (Online) 23rd January Morning Shift

Options:

A.

$$\frac{1}{2}$$

B.

$$\frac{3}{2}$$

C.

$$2$$

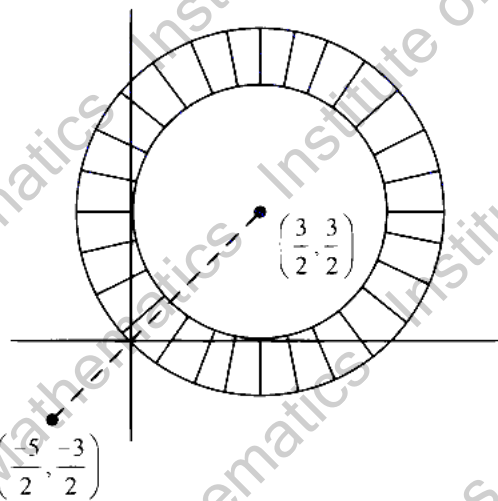
D.

$$\frac{3}{2}$$

Answer: D

Solution:

$$\frac{3}{2} \leq \left| z - \left(\frac{3}{2} + \frac{3}{2}i \right) \right| \leq \frac{7}{2}$$



$$\begin{aligned}\text{Min distance} &= \sqrt{16+9} - \frac{7}{2} \\ &= 5 - \frac{7}{2} \\ &= \frac{3}{2}\end{aligned}$$

Question 7

If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $i = \sqrt{-1}$, then $(z^{201} - i)^8$ is equal to

JEE Main 2026 (Online) 23rd January Evening Shift

Options:

A.

1

B.

0

C.

-1

D.

256

Answer: D

Solution:

$$z = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

Convert into Euler form $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ if $z = x + iy$.

$$r = |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

Since z lies in the first quadrant in the complex plane,

$$\text{So, } \theta = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{\pi}{6}$$

$$z = e^{i\frac{\pi}{6}}$$

$$\text{To calculate } (z^{201} - i)^8 = \left((e^{i\frac{\pi}{6}})^{201} - i\right)^8$$

$$= \left(e^{i\frac{201\pi}{6}} - i \right)^8$$

Using De Moivre's theorem, $e^{i\theta} = \cos \theta + i \sin \theta$

$$\begin{aligned} (z^{201} - i)^8 &= \left(\cos \left(\frac{201\pi}{6} \right) + i \sin \left(\frac{201\pi}{6} \right) - i \right)^8 \\ &= \left(\cos \left(\frac{67\pi}{2} \right) + i \sin \left(\frac{67\pi}{2} \right) - i \right)^8 \\ &= (0 + i(-1) - i)^8 \\ &= (-2i)^8 = 2^8(i)^8 \end{aligned}$$

$$\text{Since } i = \sqrt{-1} \implies i^2 = -1 \implies i^4 = 1 \implies i^8 = 1$$

$$\therefore (z^{201} - i)^8 = 2^8 = 256.$$

Question8

Let $S = \left\{ z \in \mathbb{C} : \left| \frac{z-6i}{z-2i} \right| = 1 \text{ and } \left| \frac{z-8+2i}{z+2i} \right| = \frac{3}{5} \right\}$.

Then $\sum_{z \in S} |z|^2$ is equal to :

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Options:

A.

413

B.

398

C.

385

D.

423

Answer: C

Solution:

$$\text{Given, } S = \left\{ z \in \mathbb{C} : \left| \frac{z-6i}{z-2i} \right| = 1 \text{ and } \left| \frac{z-8+2i}{z+2i} \right| = \frac{3}{5} \right\}$$

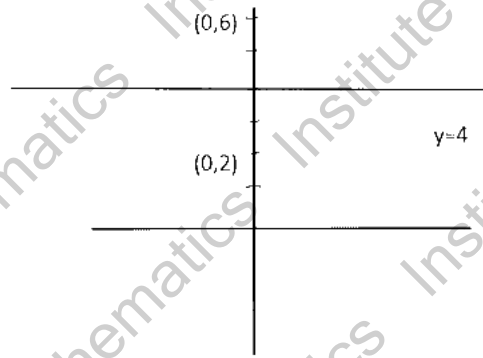
$$\left| \frac{z-6i}{z-2i} \right| = 1$$

$$\implies |z-6i| = |z-2i| \quad (1)$$

equation (1) represent z is a point which is equidistant from point $(0, 2)$ and $(0, 6)$ so z lies on the line segment which is perpendicular bisector of the line segment joining point $(0, 2)$ & $(0, 6)$.

If $z = x + iy$ then solution of equation (1) is $y = \frac{6+2}{2} = 4$ and $x \in \mathbb{R}$.

So, solution is $z = x + 4i$ & $x \in \mathbb{R}$.



Solving $\left| \frac{z-8+2i}{z+2i} \right| = \frac{3}{5}$

Substitute $z = x + 4i$

$$\frac{|x+4i-8+2i|}{|x+4i+2i|} = \frac{3}{5}$$

$$\Rightarrow 5|x+6i-8| = 3|x+6i|$$

$$\Rightarrow 5|(x-8)+6i| = 3|x+6i|$$

$$\Rightarrow 5\sqrt{(x-8)^2+6^2} = 3\sqrt{x^2+6^2}$$

Squaring

$$\Rightarrow 25(x^2+64-16x+36) = 9x^2+324$$

$$\Rightarrow 25x^2-400x+2500 = 9x^2+324$$

$$\Rightarrow 16x^2-400x+2176 = 0$$

$$\Rightarrow x^2-25x+136 = 0$$

$$\Rightarrow x^2-17x-8x+136 = 0$$

$$\Rightarrow x(x-17)-8(x-17) = 0$$

$$\Rightarrow (x-17)(x-8) = 0$$

$$\therefore x = 17, x = 8$$

So set S contain two complex number $z_1 = 17 + 4i$ and $z_2 = 8 + 4i$

To find $\sum_{z \in S} |z|^2 = |z_1|^2 + |z_2|^2$

$$= |17+4i|^2 + |8+4i|^2$$

$$= 17^2 + 4^2 + 8^2 + 4^2$$

$$= 289 + 16 + 64 + 16 = 385$$

Question9

Let z be a complex number such that $|z-6| = 5$ and $|z+2-6i| = 5$. Then the value of $z^3 + 3z^2 - 15z + 141$ is equal to :

JEE Main 2026 (Online) 28th January Morning Shift

Options:

A.

61

B.

37

C.

42

D.

50

Answer: D**Solution:**Let $z = x + iy$ $|z - (\alpha + i\beta)| = r$ represents a circle in the complex plane with centre at (α, β) and radius is r unit.

Cartesian equation

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

The equations $|z - 6| = 5$ and $|z - (-2 + 6i)| = 5$ describe two circles with the same radius ($r = 5$).

- **Circle 1 :** Center $C_1 = (6, 0)$
- **Circle 2 :** Center $C_2 = (-2, 6)$

Cartesian equation

$$(x - 6)^2 + y^2 = 25 \quad \text{and} \quad (x + 2)^2 + (y - 6)^2 = 25$$

Expanding both:

- $x^2 - 12x + 36 + y^2 = 25 \implies x^2 + y^2 - 12x + 11 = 0 \quad \dots (i)$
- $x^2 + 4x + 4 + y^2 - 12y + 36 = 25 \implies x^2 + y^2 + 4x - 12y + 15 = 0 \quad \dots (ii)$

Subtracting the first equation from the second to find the common chord:

$$(4x - (-12x)) - 12y + (15 - 11) = 0$$

$$16x - 12y + 4 = 0 \implies 4 - 3y + 1 = 0 \implies y = \frac{4x + 1}{3}$$

Substitute y back into the first circle equation:

$$(x - 6)^2 + \left(\frac{4x+1}{3}\right)^2 = 25$$

multiply both sides by 9

$$9(x^2 + 36 - 12x) + 16x^2 + 8x + 1 = 225$$

$$25x^2 - 100x + 100 = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)^2 = 0 \implies x = 2$$

put $x = 2$ in $y = (4x + 1)/3$

$$y = (4 \times 2 + 1)/3 = 9/3 = 3$$

so $z = 2 + 3i$ calculate $z^3 + 3z^2 - 15z + 141$ Compute z^3

$$z^3 = z \cdot z^2 = (2 + 3i)(-5 + 12i)$$

$$= -10 + 24i - 15i + 36i^2$$

$$= -10 + 9i - 36$$

$$= -46 + 9i$$

Compute z^2

$$(2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$$

so

$$z^3 = -46 + 9i$$

$$3z^2 = -15 + 36i$$

$$-15z = -30 - 45i$$

Add everything

$$(-46 + 9i) + (-15 + 36i) + (-30 - 45i) + 141$$

Real parts:

$$-46 - 15 - 30 + 141 = 50$$

Imaginary parts:

$$9i + 36i - 45i = 0$$

$$\text{So } z^3 + 3z^2 - 15z + 141 = 50$$

Question 10

Let

$$A = \{z \in \mathbb{C} : |z - 2| \leq 4\} \text{ and}$$

$$B = \{z \in \mathbb{C} : |z - 2| + |z + 2| = 5\}.$$

Then the max $\{|z_1 - z_2| : z_1 \in A \text{ and } z_2 \in B\}$ is :

JEE Main 2026 (Online) 28th January Evening Shift

Options:

A.

$$\frac{17}{2}$$

B.

8

C.

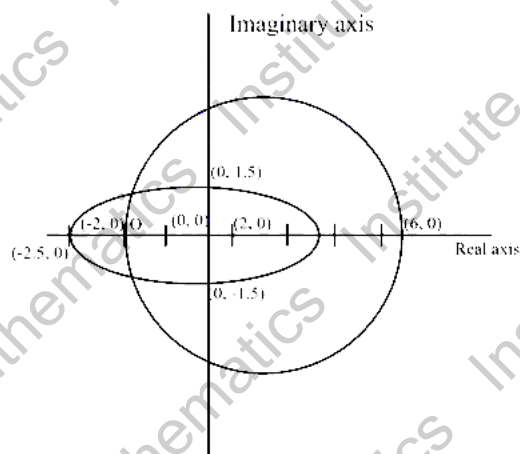
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D.

$$\frac{15}{2}$$

Answer: A

Solution:



Analyze Set A :

The set A is defined as $\{z \in \mathbb{C} : |z - 2| \leq 4\}$.

- This represents a circle (and its interior) centered at (2, 0) with a radius $r = 4$.
- The circle extends from $x = -2$ to $x = 6$ along the real axis.

Analyze Set B:

The set B is defined as $\{z \in \mathbb{C} : |z - 2| + |z + 2| = 5\}$.

- By the definition of an ellipse, the sum of distances from two fixed points (foci) is constant.
- Here, the foci are $F_1(2, 0)$ and $F_2(-2, 0)$, and the length of the major axis $2a = 5$, so $a = 2.5$.
- By the definition of an ellipse, the sum of distances from two fixed points (foci) is constant.
- Here, the foci are $F_1(2, 0)$ and $F_2(-2, 0)$, and the length of the major axis $2a = 5$, so $a = 2.5$.
- The distance between foci is $2ae = 4$, so $e = \frac{4}{5} = 0.8$.
- The semi-minor axis b can be found using $b^2 = a^2(1 - e^2) = (2.5)^2(1 - 0.64) = 6.25(0.36) = 2.25$, so $b = 1.5$.
- The ellipse is centered at the origin $(0, 0)$ and extends from $x = -2.5$ to $x = 2.5$ along the real axis.

To find $\max\{|z_1 - z_2| : z_1 \in A \text{ and } z_2 \in B\}$ This means maximize the distance between a point z_1 in the circle and a point z_2 on the ellipse.

The furthest point in set A (the circle) is at the rightmost edge, $z_1 = (2 + 4, 0) = (6, 0)$.

The furthest point in set B (the ellipse) from z_1 is at its leftmost edge, $z_2 = (-2.5, 0)$.

Maximum Distance $|z_1 - z_2| = |6 - (-2.5)| = |6 + 2.5| = 8.5 = \frac{17}{2}$

Question 11

Let x and y be real numbers such that $50\left(\frac{2x}{1+3i} - \frac{y}{1-2i}\right) = 31 + 17i, i = \sqrt{-1}$. Then the value of $10(x - 3y)$ is :

JEE Main 2026 (Online) 2nd April Morning Shift

Options:

- A. 20
- B. 31
- C. 35
- D. 75

Answer: D

Solution:

$$50\left(\frac{2x}{1+3i} - \frac{y}{1-2i}\right) = 31 + 17i$$

We need to find $10(x - 3y)$.

First, simplify the fractions by rationalising the denominators.

$$\begin{aligned}\frac{2x}{1+3i} &= \frac{2x(1-3i)}{(1+3i)(1-3i)} \\ &= \frac{2x(1-3i)}{1+9} = \frac{2x(1-3i)}{10} \\ &= \frac{x}{5}(1-3i)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{y}{1-2i} &= \frac{y(1+2i)}{(1-2i)(1+2i)} \\ &= \frac{y(1+2i)}{1+4} = \frac{y(1+2i)}{5}\end{aligned}$$

Now substitute into the given equation:

$$50\left(\frac{x}{5}(1-3i) - \frac{y}{5}(1+2i)\right) = 31 + 17i$$

Take $\frac{1}{5}$ common inside:

$$50 \cdot \frac{1}{5} [x(1-3i) - y(1+2i)] = 31 + 17i$$

$$10 [x(1-3i) - y(1+2i)] = 31 + 17i$$

Expand:

$$10[x - 3xi - y - 2yi] = 31 + 17i$$

$$10[(x - y) + (-3x - 2y)i] = 31 + 17i$$

So,

$$10(x - y) + 10(-3x - 2y)i = 31 + 17i$$

Now compare real and imaginary parts:

$$10(x - y) = 31$$

$$10(-3x - 2y)i = 17i$$

Thus,

$$x - y = \frac{31}{10}$$

$$-3x - 2y = \frac{17}{10}$$

Now solve these equations.

From the first equation,

$$x = y + \frac{31}{10}$$

Substitute into the second:

$$-3\left(y + \frac{31}{10}\right) - 2y = \frac{17}{10}$$

$$-3y - \frac{93}{10} - 2y = \frac{17}{10}$$

$$-5y = \frac{17}{10} + \frac{93}{10}$$

$$-5y = \frac{110}{10} = 11$$

$$y = -\frac{11}{5}$$

Then

$$x = y + \frac{31}{10} = -\frac{11}{5} + \frac{31}{10}$$

$$x = -\frac{22}{10} + \frac{31}{10} = \frac{9}{10}$$

Now find:

$$10(x - 3y) = 10\left(\frac{9}{10} - 3\left(-\frac{11}{5}\right)\right)$$

$$= 10\left(\frac{9}{10} + \frac{33}{5}\right)$$

$$= 10\left(\frac{9}{10} + \frac{66}{10}\right)$$

$$= 10 \cdot \frac{75}{10} = 75$$

Therefore, the value of $10(x - 3y)$ is

So, the correct option is .

Question 12

Let the circles $C_1 : |z| = r$ and $C_2 : |z - 3 - 4i| = 5$, $z \in \mathbb{C}$, be such that C_2 lies within C_1 . If z_1 moves on C_1 , z_2 moves on C_2 and $\min |z_1 - z_2| = 2$, then $\max |z_1 - z_2|$ is equal to :

JEE Main 2026 (Online) 2nd April Evening Shift

Options:

A.

12

B.

17

C.

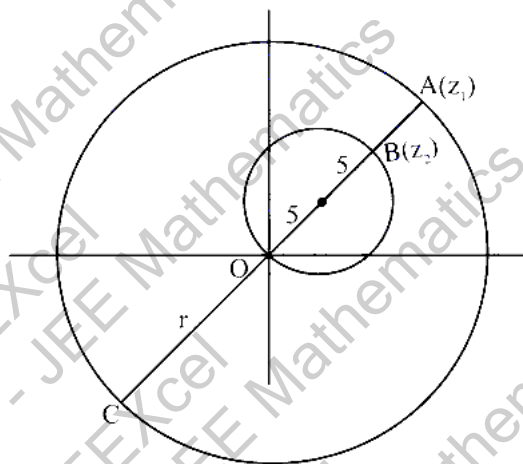
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D.

24

Answer: C

Solution:



$$|z| = r$$

$$OA = r$$

Step 1: Circle $C_1 : |z| = r$ has centre $O(0, 0)$ and radius r (so $OA = r$).

Step 2: Circle $C_2 : |z - 3 - 4i| = 5$ has centre at the point $(3, 4)$ and radius 5.

Step 3: Distance between the centres is

$$OC = \sqrt{3^2 + 4^2} = 5$$

Step 4: Since C_2 lies completely inside C_1 , the minimum distance between a point z_1 on C_1 and a point z_2 on C_2 happens when the two points are on the same line through the centres, on the nearest sides.

So,

$$\text{minimum } |z_1 - z_2| = r - (OC + 5) = r - (5 + 5) = r - 10$$

Given minimum $|z_1 - z_2| = 2$, we get

$$\Rightarrow r - 10 = 2 \Rightarrow r = 12$$

Step 5: The maximum distance $|z_1 - z_2|$ happens when z_1 and z_2 are on opposite sides along the same line through the centres.

Then the farthest distance is

$$|z_1 - z_2| = r + (OC + 5) = r + 10$$

Substituting $r = 12$,

$$|z_1 - z_2| = 12 + 10 = 22$$

Question 13

Let z be a complex number such that $|z + 2| = |z - 2|$ and $\arg\left(\frac{z+3}{z-i}\right) = \frac{\pi}{4}$. Then $|z|^2$ is equal to:

JEE Main 2026 (Online) 4th April Morning Shift

Options:

A.

9

B.

4

C.

5

D.

1

Answer: A

Solution:

Let

$$z = x + iy$$

where $x, y \in \mathbb{R}$.

We are given:

$$|z + 2| = |z - 2|$$

and

$$\arg\left(\frac{z+3}{z-i}\right) = \frac{\pi}{4}$$

We will use these one by one.

1. Using $|z + 2| = |z - 2|$

Substitute $z = x + iy$:

$$|x + iy + 2| = |x + iy - 2|$$

So,

$$\sqrt{(x+2)^2 + y^2} = \sqrt{(x-2)^2 + y^2}$$

Squaring both sides:

$$(x+2)^2 + y^2 = (x-2)^2 + y^2$$

$$x^2 + 4x + 4 = x^2 - 4x + 4$$

$$8x = 0$$

$$x = 0$$

Hence,

$$z = iy$$

So the point lies on the imaginary axis.

2. Using the argument condition

Now,

$$\frac{z+3}{z-i} = \frac{3+iy}{i(y-1)}$$

We are given that its argument is $\frac{\pi}{4}$.

A complex number has argument $\frac{\pi}{4}$ when its real and imaginary parts are equal and positive in sign together.

Let us simplify:

$$\frac{3+iy}{i(y-1)}$$

Since $\frac{1}{i} = -i$, we get

$$\frac{3+iy}{i(y-1)} = \frac{-i(3+iy)}{y-1}$$

Now expand:

$$-i(3+iy) = -3i - i^2y = y - 3i$$

So,

$$\frac{z+3}{z-i} = \frac{y-3i}{y-1} = \frac{y}{y-1} - i \frac{3}{y-1}$$

Thus real part is

$$\frac{y}{y-1}$$

and imaginary part is

$$-\frac{3}{y-1}$$

Since argument is $\frac{\pi}{4}$, we need

$$\frac{\text{imaginary part}}{\text{real part}} = \tan \frac{\pi}{4} = 1$$

So,

$$\frac{-\frac{3}{y-1}}{\frac{y}{y-1}} = 1$$

$$-\frac{3}{y} = 1$$

$$y = -3$$

Therefore,

$$z = -3i$$

3. Find $|z|^2$

$$|z|^2 = 0^2 + (-3)^2 = 9$$

So,

$$\boxed{9}$$

Hence, the correct option is **Option A**.

Question 14

Let $S = \{z \in \mathbb{C} : z^2 + 4z + 16 = 0\}$. Then $\sum_{z \in S} |z + \sqrt{3}i|^2$ is equal to :

JEE Main 2026 (Online) 4th April Evening Shift

Options:

A.

42

B.

23

C.

27

D.

38

Answer: D

Solution:

Solve the quadratic equation first:

$$z^2 + 4z + 16 = 0$$

Using the quadratic formula,

$$z = \frac{-4 \pm \sqrt{16 - 64}}{2} = \frac{-4 \pm \sqrt{-48}}{2} = \frac{-4 \pm 4\sqrt{3}i}{2} = -2 \pm 2\sqrt{3}i$$

So the set is

$$S = \{-2 + 2\sqrt{3}i, -2 - 2\sqrt{3}i\}$$

Now we need to find

$$\sum_{z \in S} |z + \sqrt{3}i|^2$$

That is,

$$|-2 + 2\sqrt{3}i + \sqrt{3}i|^2 + |-2 - 2\sqrt{3}i + \sqrt{3}i|^2$$

Simplify each term:

1. For $z = -2 + 2\sqrt{3}i$,

$$z + \sqrt{3}i = -2 + 3\sqrt{3}i$$

Hence,

$$|-2 + 3\sqrt{3}i|^2 = (-2)^2 + (3\sqrt{3})^2 = 4 + 27 = 31$$

2. For $z = -2 - 2\sqrt{3}i$,

$$z + \sqrt{3}i = -2 - \sqrt{3}i$$

Hence,

$$|-2 - \sqrt{3}i|^2 = (-2)^2 + (-\sqrt{3})^2 = 4 + 3 = 7$$

Therefore,

$$\sum_{z \in S} |z + \sqrt{3}i|^2 = 31 + 7 = 38$$

So the correct answer is:

38

Option D

Question 15

Let $a, b \in \mathbb{C}$. Let α, β be the roots of the equation $x^2 + ax + b = 0$. If $\beta - \alpha = \sqrt{11}$ and $\beta^2 - \alpha^2 = 3i\sqrt{11}$, then $(\beta^3 - \alpha^3)^2$ is equal to:

JEE Main 2026 (Online) 5th April Morning Shift

Options:

A.

160

B.

176

C.

194

D.

187

Answer: B

Solution:

Given:

- α, β are roots of $x^2 + ax + b = 0$
- $\beta - \alpha = \sqrt{11}$

- $\beta^2 - \alpha^2 = 3i\sqrt{11}$

We have to find $(\beta^3 - \alpha^3)^2$.

Now,

$$\beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$$

Substituting the given values,

$$3i\sqrt{11} = \sqrt{11}(\beta + \alpha)$$

Since $\sqrt{11} \neq 0$, we get

$$\beta + \alpha = 3i$$

Now use the identity

$$\beta^3 - \alpha^3 = (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2)$$

So first let us find $\beta^2 + \alpha\beta + \alpha^2$.

We know

$$(\beta + \alpha)^2 = \beta^2 + \alpha^2 + 2\alpha\beta$$

Also,

$$\alpha\beta = \frac{(\alpha + \beta)^2 - (\beta - \alpha)^2}{4}$$

Substitute the values:

$$\alpha + \beta = 3i, \quad \beta - \alpha = \sqrt{11}$$

Thus,

$$\alpha\beta = \frac{(3i)^2 - (\sqrt{11})^2}{4} = \frac{-9 - 11}{4} = \frac{-20}{4} = -5$$

Now,

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

So,

$$\alpha^2 + \beta^2 = (3i)^2 - 2(-5) = -9 + 10 = 1$$

Hence,

$$\beta^2 + \alpha\beta + \alpha^2 = 1 + (-5) = -4$$

Therefore,

$$\beta^3 - \alpha^3 = (\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2) = \sqrt{11} \cdot (-4) = -4\sqrt{11}$$

Now square it:

$$(\beta^3 - \alpha^3)^2 = (-4\sqrt{11})^2 = 16 \cdot 11 = 176$$

So the correct answer is:

$$\boxed{176}$$

Hence, Option B is correct.

Question 16

Let $z_1, z_2 \in \mathbb{C}$ be the distinct solutions of the equation $z^2 + 4z - (1 + 12i) = 0$.

Then $|z_1|^2 + |z_2|^2$ is equal to :

JEE Main 2026 (Online) 5th April Evening Shift

Options:

A.

B.

22

C.

29

D.

34

Answer: D

Solution:

Given the quadratic equation

$$z^2 + 4z - (1 + 12i) = 0$$

let its roots be z_1 and z_2 .

We need to find

$$|z_1|^2 + |z_2|^2$$

Using Vieta's formulas for

$$az^2 + bz + c = 0,$$

we have

$$z_1 + z_2 = -\frac{b}{a} = -4$$

and

$$z_1 z_2 = \frac{c}{a} = -(1 + 12i) = -1 - 12i$$

Now solve the equation to identify the roots.

$$z^2 + 4z - (1 + 12i) = 0$$

Using the quadratic formula,

$$z = \frac{-4 \pm \sqrt{16 + 4(1 + 12i)}}{2}$$

$$z = \frac{-4 \pm \sqrt{20 + 48i}}{2}$$

Now observe that

$$(6 + 4i)^2 = 36 + 48i - 16 = 20 + 48i$$

So,

$$\sqrt{20 + 48i} = 6 + 4i$$

Hence the roots are

$$z_1 = \frac{-4 + (6 + 4i)}{2} = 1 + 2i$$

and

$$z_2 = \frac{-4 - (6 + 4i)}{2} = -5 - 2i$$

Now compute their moduli squared:

$$|z_1|^2 = |1 + 2i|^2 = 1^2 + 2^2 = 5$$

$$|z_2|^2 = |-5 - 2i|^2 = (-5)^2 + (-2)^2 = 25 + 4 = 29$$

Therefore,

$$|z_1|^2 + |z_2|^2 = 5 + 29 = 34$$

So the correct answer is

34

Hence, Option D is correct.

Question 17

Let the set of all values of $k \in \mathbb{R}$ such that the equation $z(\bar{z} + 2 + i) + k(2 + 3i) = 0, z \in \mathbb{C}$, has at least one solution, be the interval $[\alpha, \beta]$. Then $9(\alpha + \beta)$ is equal to:

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Options:

A.

-10

B.

-8

C.

$10\sqrt{13}$

D.

$8\sqrt{13}$

Answer: A

Solution:

Let

$$z = x + iy \quad \text{where } x, y \in \mathbb{R}.$$

Then

$$\bar{z} = x - iy.$$

The given equation is

$$z(\bar{z} + 2 + i) + k(2 + 3i) = 0.$$

We have to find all real values of k for which this equation has at least one complex solution z .

First, simplify $z(\bar{z} + 2 + i)$:

$$z(\bar{z} + 2 + i) = z\bar{z} + z(2 + i).$$

Now,

$$z\bar{z} = |z|^2 = x^2 + y^2$$

and

$$z(2 + i) = (x + iy)(2 + i).$$

Let us multiply:

$$(x + iy)(2 + i) = 2x + ix + 2iy + i^2y$$

Since $i^2 = -1$,

$$(x + iy)(2 + i) = (2x - y) + i(x + 2y).$$

So,

$$z(\bar{z} + 2 + i) = x^2 + y^2 + (2x - y) + i(x + 2y).$$

Hence the given equation becomes

$$(x^2 + y^2 + 2x - y) + i(x + 2y) + k(2 + 3i) = 0.$$

Now separate real and imaginary parts:

$$(x^2 + y^2 + 2x - y + 2k) + i(x + 2y + 3k) = 0.$$

Therefore,

$$x^2 + y^2 + 2x - y + 2k = 0 \quad \text{and} \quad x + 2y + 3k = 0.$$

From the second equation,

$$3k = -(x + 2y)$$

so

$$k = -\frac{x+2y}{3}.$$

Substitute this into the first equation:

$$x^2 + y^2 + 2x - y + 2\left(-\frac{x+2y}{3}\right) = 0.$$

Multiply by 3:

$$3x^2 + 3y^2 + 6x - 3y - 2x - 4y = 0$$

$$3x^2 + 3y^2 + 4x - 7y = 0.$$

Now complete squares:

$$3\left(x^2 + \frac{4}{3}x\right) + 3\left(y^2 - \frac{7}{3}y\right) = 0.$$

$$3\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9}\right] + 3\left[\left(y - \frac{7}{6}\right)^2 - \frac{49}{36}\right] = 0.$$

$$3\left(x + \frac{2}{3}\right)^2 - \frac{4}{3} + 3\left(y - \frac{7}{6}\right)^2 - \frac{49}{12} = 0.$$

$$3\left(x + \frac{2}{3}\right)^2 + 3\left(y - \frac{7}{6}\right)^2 = \frac{4}{3} + \frac{49}{12}.$$

$$3\left(x + \frac{2}{3}\right)^2 + 3\left(y - \frac{7}{6}\right)^2 = \frac{16+49}{12} = \frac{65}{12}.$$

So,

$$\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{7}{6}\right)^2 = \frac{65}{36}.$$

This is a circle with centre

$$\left(-\frac{2}{3}, \frac{7}{6}\right)$$

and radius

$$\frac{\sqrt{65}}{6}.$$

Now,

$$k = -\frac{x+2y}{3}.$$

So we need the range of the linear expression $x + 2y$ on this circle.

Let

$$S = x + 2y.$$

For a circle, maximum and minimum of a linear expression occur at the extreme points in the direction of its normal vector.

At the centre,

$$S_c = -\frac{2}{3} + 2 \cdot \frac{7}{6} = -\frac{2}{3} + \frac{7}{3} = \frac{5}{3}.$$

The vector corresponding to $x + 2y$ is $(1, 2)$, whose magnitude is

$$\sqrt{1^2 + 2^2} = \sqrt{5}.$$

Hence the maximum variation from the centre is

$$\text{radius} \times \sqrt{5} = \frac{\sqrt{65}}{6} \cdot \sqrt{5} = \frac{\sqrt{325}}{6} = \frac{5\sqrt{13}}{6}.$$

Therefore,

$$S_{\min} = \frac{5}{3} - \frac{5\sqrt{13}}{6}, \quad S_{\max} = \frac{5}{3} + \frac{5\sqrt{13}}{6}.$$

Since

$$k = -\frac{S}{3},$$

the interval for k is

$$\left[-\frac{S_{\max}}{3}, -\frac{S_{\min}}{3}\right].$$

So if this interval is $[\alpha, \beta]$, then

$$\alpha + \beta = -\frac{S_{\max} + S_{\min}}{3} = -\frac{2 \cdot \frac{5}{3}}{3} = -\frac{10}{9}.$$

Hence,

$$9(\alpha + \beta) = 9\left(-\frac{10}{9}\right) = -10.$$

Therefore, the correct answer is

$$\boxed{-10}.$$

So, Option A is correct.

Question 18

Let $S = \left\{ z \in \mathbb{C} : z^2 + \sqrt{6}iz - 3 = 0 \right\}$. Then $\sum_{z \in S} z^8$ is equal to :

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Options:

A.

162

B.

184

C.

262

D.

324

Answer: A

Solution:

$$\begin{aligned} z &= \frac{-\sqrt{6}i \pm \sqrt{-6+12}}{2} \\ &= \frac{-\sqrt{6}i \pm \sqrt{6}}{2} \\ &= \frac{\sqrt{6}}{2}(\pm 1 - i) \end{aligned}$$

$$\alpha = \frac{-\sqrt{6}}{2}(1+i), \beta = \frac{\sqrt{6}}{2}(1-i)$$

$$\alpha = \frac{-\sqrt{6}}{2} \times \sqrt{2}e^{i\pi/4}, \beta = +\frac{\sqrt{6}}{2} \times \sqrt{2}e^{-i\pi/4}$$

$$\alpha = -\sqrt{3}e^{i\pi/4}, \beta = +\sqrt{3}e^{-i\pi/4}$$

$$\begin{aligned} \alpha^8 + \beta^8 &= 81 \cdot e^{i2\pi} + 81 \cdot e^{i(-2\pi)} \\ &= 81(1+1) = 162 \end{aligned}$$

Question 19

The number of values of $z \in \mathbb{C}$, satisfying the equations $|z - (4+8i)| = \sqrt{10}$ and $|z - (3+5i)| + |z - (5+11i)| = 4\sqrt{5}$, is :

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Options:

A.

0

B.

2

C.

1

D.

4

Answer: B

Solution:

$$|z - (3 + 5i)| + |z - (5 + 11i)| = 4\sqrt{5}$$

So, $S_1(3, 5)$ & $S_2(5, 11)$ are two foci of ellipse, where $2ae = S_1 S_2 = 2\sqrt{10}$ & $2a = 4\sqrt{5}$

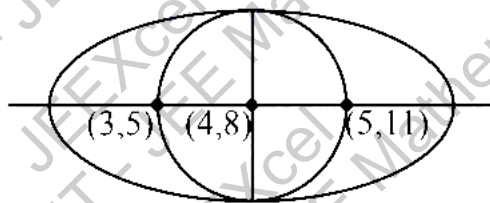
$$\Rightarrow a = 2\sqrt{5}, e = \frac{1}{\sqrt{2}} \text{ \& } b^2 = a^2(1 - e^2) \text{ gives } b^2 = 10$$

\& center of an ellipse is mid point of both focus

$\therefore$ center $(4, 8)$

$$\text{Equation of circle is } |Z - (4 + 8i)| = \sqrt{10}$$

$\Rightarrow$ Ellipse \& given circle are concentric \& radius of given circle is equal to length of minor axis of given ellipse



$\Rightarrow$ Two points which are common to both.

Question 20

Let α, β be the roots of the equation $x^2 - ax - b = 0$ with $\text{Im}(\alpha) < \text{Im}(\beta)$. Let $P_n = \alpha^n - \beta^n$. If $P_3 = -5\sqrt{7}i$, $P_4 = -3\sqrt{7}i$, $P_5 = 11\sqrt{7}i$ and $P_6 = 45\sqrt{7}i$, then $|\alpha^4 + \beta^4|$ is equal to _____.

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Answer: 31

Solution:

We begin with the equations for the roots:

$$\alpha + \beta = a$$

$$\alpha\beta = -b$$

Given:

$$P_6 = aP_5 + bP_4$$

$$P_5 = aP_4 + bP_3$$

Using the given values:

For P_6 :

$$45\sqrt{7}i = a \times 11\sqrt{7}i + b(-3\sqrt{7}i)$$

Simplifying, we obtain:

$$45 = 11a - 3b \quad (\text{Equation 1})$$

For P_5 :

$$11\sqrt{7}i = a(-3\sqrt{7}i) + b(-5\sqrt{7}i)$$

Simplifying, we obtain:

$$11 = -3a - 5b \quad (\text{Equation 2})$$

Solving these linear equations, we find:

$$a = 3$$

$$b = -4$$

Now, we calculate $|\alpha^4 + \beta^4|$ using the relation:

$$\alpha^4 + \beta^4 = \sqrt{(\alpha^4 - \beta^4)^2 + 4(\alpha^4\beta^4)}$$

From $b = -4$, we know:

$$\alpha\beta = -b = 4 \Rightarrow \alpha^4\beta^4 = (\alpha\beta)^4 = 4^4 = 256$$

Substitute into the relation:

$$\alpha^4 + \beta^4 = \sqrt{(-63)^2 + 1024}$$

$$= \sqrt{961} = 31$$

Thus, $|\alpha^4 + \beta^4|$ is equal to 31.

Question21

Let integers $a, b \in [-3, 3]$ be such that $a + b \neq 0$. Then the number of all possible ordered pairs (a, b) ,

for which $\left| \frac{z-a}{z+b} \right| = 1$ and $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1, z \in \mathbb{C}$, where ω and ω^2 are the roots of $x^2 + x + 1 = 0$, is equal to _____.

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Answer: 10

Solution:

$$a, b \in I, -3 \leq a, b \leq 3, a + b \neq 0$$

$$|z-a| = |z+b|$$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ z & \omega & z+\omega^2 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ z & \omega & z+\omega^2-\omega \\ \omega^2 & 1-\omega^2 & z+\omega-\omega^2 \end{vmatrix} = 1$$

$$\Rightarrow z^3 = 1$$

$$\Rightarrow z = \omega, \omega^2, 1$$

Now

$$|1-a| = |1+b|$$

$$\Rightarrow 10 \text{ pairs}$$

Question22

Let $A = \{z \in \mathbb{C} : |z - 2 - i| = 3\}$, $B = \{z \in \mathbb{C} : \operatorname{Re}(z - iz) = 2\}$ and $S = A \cap B$. Then $\sum_{z \in S} |z|^2$ is equal to _____.

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Answer: 22

Solution:

$$\begin{aligned} \text{Let } z &= x + iy \\ |z - 2 - i| &= 3 \Rightarrow (x - 2)^2 + (y - 1)^2 = 3^2 \\ \operatorname{Re}(z - iz) &= \operatorname{Re}(x + iy - ix + y) = x + y \Rightarrow x + y = 2 \\ \Rightarrow A &= \{(x, y) : (x - 2)^2 + (y - 1)^2 = 3^2, x, y \in \mathbb{R}\}, \\ B &= \{(x, y) : x + y = 2\} \\ \Rightarrow x - 2 &= -y \Rightarrow y^2 + (y - 1)^2 = 3^2 \\ \Rightarrow 2y^2 - 2y - 8 &= 0 \Rightarrow y^2 - y - 4 = 0 \\ y_1 + y_2 &= 1, y_1 y_2 = -4 \\ \Rightarrow y_1^2 + y_2^2 &= (y_1 + y_2)^2 - 2y_1 y_2 = 9 \\ \Rightarrow x_1 + x_2 &= 4(y_1 + y_2) = 3, \\ x_1 x_2 &= (2 - y_1)(2 - y_2) = 4 - 2(y_1 + y_2) + y_1 y_2 = -2 \\ \Rightarrow x_1^2 + x_2^2 &= (x_1 + x_2)^2 - 2x_1 x_2 = 13 \\ \therefore S &= \{(x_1, y_1), (x_2, y_2)\} \\ \Rightarrow \sum_{z \in S} |z|^2 &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) = 22 \end{aligned}$$

Question23

If α is a root of the equation $x^2 + x + 1 = 0$ and $\sum_{k=1}^n \left(\alpha^k + \frac{1}{\alpha^k}\right)^2 = 20$, then n is equal to _____.

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Answer: 11

Solution:

$$\begin{aligned} \alpha \text{ is root of equation } 1 + x + x^2 &= 0, \alpha = \omega \text{ or } \omega^2 \\ \left(\alpha^k + \frac{1}{\alpha^k}\right)^2 &= \alpha^{2k} + \frac{1}{\alpha^{2k}} + 2 = \omega^k + \frac{1}{\omega^k} + 2 \\ \Rightarrow \omega^k + \frac{1}{\omega^k} + 2 &= \begin{cases} 4, 3 \text{ divides } k \\ 1, 3 \text{ does not divide } k \end{cases} \\ \therefore \sum_{k=1}^n \left(\alpha^k + \frac{1}{\alpha^k}\right)^2 &= 20 \\ \Rightarrow (1 + 1 + 4) + (1 + 1 + 4) + (1 + 1 + 4) + (1 + 1) &= 20 \\ \Rightarrow n &= 11 \end{aligned}$$

Question24

Let z_1, z_2 and z_3 be three complex numbers on the circle $|z| = 1$ with $\arg(z_1) = \frac{-\pi}{4}$, $\arg(z_2) = 0$ and $\arg(z_3) = \frac{\pi}{4}$. If $|z_1 \bar{z}_2 + z_2 \bar{z}_3 + z_3 \bar{z}_1|^2 = \alpha + \beta\sqrt{2}$, $\alpha, \beta \in \mathbb{Z}$, then the value of $\alpha^2 + \beta^2$ is :

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Options:

- A. 41
- B. 29
- C. 24
- D. 31

Answer: B

Solution:

To solve the problem, we start with the given information about the complex numbers z_1, z_2 , and z_3 , which lie on the unit circle $|z| = 1$. Their arguments are as follows:

$$\arg(z_1) = -\frac{\pi}{4}$$

$$\arg(z_2) = 0$$

$$\arg(z_3) = \frac{\pi}{4}$$

Thus, the complex numbers can be represented as:

$$z_1 = e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$z_2 = e^{i0} = 1$$

$$z_3 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Next, calculate the conjugates needed:

$$\bar{z}_2 = 1$$

$$\bar{z}_3 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\bar{z}_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

We need to evaluate:

$$z_1 \bar{z}_2 + z_2 \bar{z}_3 + z_3 \bar{z}_1$$

Calculate each term separately:

$$z_1 \bar{z}_2 = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \cdot 1 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$z_2 \bar{z}_3 = 1 \cdot \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$z_3 \bar{z}_1 = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \cdot \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \frac{(1+i)^2}{2} = \frac{1+2i-1}{2} = i$$

Sum the evaluated terms:

$$z_1 \bar{z}_2 + z_2 \bar{z}_3 + z_3 \bar{z}_1 = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) + i$$

Simplify:

$$= \sqrt{2} + i - \sqrt{2}i = \sqrt{2} + i(1 - \sqrt{2})$$

Calculate the modulus squared:

$$|\sqrt{2} + i(1 - \sqrt{2})|^2 = (\sqrt{2})^2 + (1 - \sqrt{2})^2$$

$$= 2 + (1 - 2\sqrt{2} + 2) = 5 - 2\sqrt{2}$$

Thus, the expression $|z_1 \bar{z}_2 + z_2 \bar{z}_3 + z_3 \bar{z}_1|^2$ simplifies as follows:

$$\alpha = 5$$

$$\beta = -2$$

Finally, compute $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = 5^2 + (-2)^2 = 25 + 4 = 29$$

Therefore, the value of $\alpha^2 + \beta^2$ is 29.

Question 25

Let the curve $z(1+i) + \bar{z}(1-i) = 4, z \in C$, divide the region $|z-3| \leq 1$ into two parts of areas α and β . Then $|\alpha - \beta|$ equals :

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Options:

A. $1 + \frac{\pi}{3}$

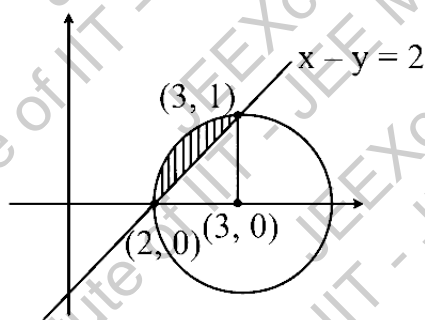
B. $1 + \frac{\pi}{6}$

C. $1 + \frac{\pi}{2}$

D. $1 + \frac{\pi}{4}$

Answer: C

Solution:



$$\text{Let } z = x + iy$$

$$(x + iy)(1 + i) + (x - iy)(1 - i) = 4$$

$$x + ix + iy - y + x - ix - iy - y = 4$$

$$2x - 2y = 4$$

$$x - y = 2$$

$$|z - 3| \leq 1$$

$$(x - 3)^2 + y^2 \leq 1$$

$$\text{Area of shaded region} = \frac{\pi \cdot 1^2}{4} - \frac{1}{2} \cdot 1 \cdot 1 = \frac{\pi}{4} - \frac{1}{2}$$

$$\text{Area of unshaded region inside the circle}$$

$$= \frac{3}{4}\pi \cdot 1^2 + \frac{1}{2} \cdot 1 \cdot 1 = \frac{3\pi}{4} + \frac{1}{2}$$

$$\therefore \text{ difference of area} = \left(\frac{3\pi}{4} + \frac{1}{2} \right) - \left(\frac{\pi}{4} - \frac{1}{2} \right)$$

$$= \frac{\pi}{2} + 1$$

Question 26

Let $\left| \frac{z-i}{2z+i} \right| = \frac{1}{3}, z \in C$, be the equation of a circle with center at C . If the area of the triangle, whose vertices are at the points $(0, 0)$, C and $(\alpha, 0)$ is 11 square units, then α^2 equals:

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Options:

A. $\frac{121}{25}$

B. 100

C. $\frac{81}{25}$

D. 50

Answer: B

Solution:

$$\left| \frac{\bar{z} - i}{2\bar{z} + i} \right| = \frac{1}{3}$$

$$\left| \frac{\bar{z} - i}{\bar{z} + \frac{i}{2}} \right| = \frac{2}{3}$$

$$3|x - iy - i| = 2 \left| x - iy + \frac{i}{2} \right|$$

$$9(x^2 + (y+1)^2) = 4(x^2 + (y - 1/3)^2)$$

$$9x^2 + 9y^2 + 18y + 9 = 4x^2 + 4y^2 - 4y + 1$$

$$5x^2 + 5y^2 + 22y + 8 = 0$$

$$x^2 + y^2 + \frac{22}{5}y + \frac{8}{5} = 0$$

$$\text{centre} \Rightarrow \left(0, -\frac{11}{5} \right)$$

$$\left| \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 0 & -11/5 & 1 \\ \alpha & 0 & 1 \end{vmatrix} \right| = 11$$

$$\Rightarrow \left(-\frac{11}{5}\alpha \right)^2 = (11 \times 2)^2$$

$$\Rightarrow \alpha^2 = 100$$

Question 27

The number of complex numbers z , satisfying $|z| = 1$ and $\left| \frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right| = 1$, is :

JEE Main 2025 (Online) 23rd January Evening Shift

Options:

A. 8

B. 10

C. 4

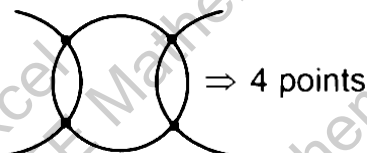
D. 6

Answer: A

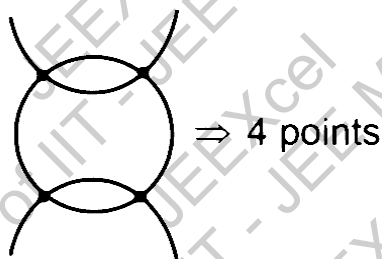
Solution:

$$\begin{aligned}
 |z| &= 1 \\
 \left| \frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right| &= 1 \\
 \Rightarrow |z^2 + (\bar{z})^2| &= 1 \\
 \text{Let } z &= x + iy \\
 \Rightarrow |(x + iy)^2 + (x - iy)^2| &= 1 \\
 \Rightarrow |2x^2 - 2y^2| &= 1 \\
 \Rightarrow |x^2 - y^2| &= \frac{1}{2} \\
 \Rightarrow x^2 - y^2 &= \pm \frac{1}{2} \\
 \text{and } x^2 + y^2 &= 1
 \end{aligned}$$

$$\text{Case I: } x^2 - y^2 = \frac{1}{2}$$



$$\text{Case II: } x^2 - y^2 = -\frac{1}{2}$$



Hence, we get 8 complex numbers.

Question 28

If α and β are the roots of the equation $2z^2 - 3z - 2i = 0$, where $i = \sqrt{-1}$, then $16 \cdot \operatorname{Re} \left(\frac{\alpha^{19} + \beta^{19} + \alpha^{11} + \beta^{11}}{\alpha^{15} + \beta^{15}} \right) \cdot \operatorname{Im} \left(\frac{\alpha^{19} + \beta^{19} + \alpha^{11} + \beta^{11}}{\alpha^{15} + \beta^{15}} \right)$ is equal to

JEE Main 2025 (Online) 24th January Morning Shift

Options:

- A. 441
- B. 312
- C. 409
- D. 398

Answer: A

Solution:

Sol. $2z^2 - 32 - 2i = 0$

$$2\left(z - \frac{i}{z}\right) = 3$$

$$\alpha - \frac{i}{\alpha} = \frac{3}{2}$$

$$\Rightarrow \alpha^2 - \frac{1}{\alpha^2} - 2i = \frac{9}{4}$$

$$\Rightarrow \alpha^2 - \frac{1}{\alpha^2} - 2i = \frac{9}{4}$$

$$\Rightarrow \frac{9}{4} + 2i = \alpha^2 - \frac{1}{\alpha^2}$$

$$\Rightarrow \frac{81}{16} - 4 + 9i = \alpha^4 + \frac{1}{\alpha^4} - 2$$

$$\Rightarrow \frac{49}{16} + 9i = \alpha^4 + \frac{1}{\alpha^4}$$

Similarly

$$\Rightarrow \frac{49}{16} + 9i = \beta^4 + \frac{1}{\beta^4}$$

$$\Rightarrow \frac{\alpha^{19} + \beta^{19} + \alpha^{11} + \beta^{11}}{\alpha^{15} + \beta^{15}} = \frac{\alpha^{15}\left(\alpha^4 + \frac{1}{\alpha^4}\right) + \beta^{15}\left(\beta^4 + \frac{1}{\beta^4}\right)}{\alpha^{15} + \beta^{15}}$$

$$= \frac{(\alpha^{15} + \beta^{15})\left(\frac{49}{16} + 9i\right)}{(\alpha^{15} + \beta^{15})}$$

$$\text{Real} = \frac{49}{16}$$

$$\text{Im} = 9$$

$$\text{Ans. 441}$$

Question 29

Let O be the origin, the point A be $z_1 = \sqrt{3} + 2\sqrt{2}i$, the point $B(z_2)$ be such that $\sqrt{3}|z_2| = |z_1|$ and $\arg(z_2) = \arg(z_1) + \frac{\pi}{6}$. Then

JEE Main 2025 (Online) 28th January Morning Shift

Options:

A. area of triangle ABO is $\frac{11}{4}$

B. area of triangle ABO is $\frac{11}{\sqrt{3}}$

C. ABO is a scalene triangle

D. ABO is an obtuse angled isosceles triangle

Answer: D

Solution:

$$z_1 = \sqrt{3} + 2\sqrt{2}i \quad \& \quad \frac{|z_2|}{|z_1|} = \frac{1}{\sqrt{3}}$$

$$\text{given } \arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{6}$$

$$z_2 = \frac{|z_2|}{|z_1|} \cdot z_1 e^{i(\frac{\pi}{6})}$$

$$z_2 = \frac{1}{\sqrt{3}} \cdot \frac{(\sqrt{3} + 2\sqrt{2}i)(\sqrt{3} + i)}{2}$$

$$z_2 = \frac{(3 - 2\sqrt{2}) + i(2\sqrt{6} + \sqrt{3})}{2\sqrt{3}}$$

Now,

$$z_1 - z_2 = \frac{(3+2\sqrt{2})+i(2\sqrt{6}-\sqrt{3})}{2\sqrt{3}}$$

$$|z_1 - z_2| = |z_2| \Rightarrow \triangle ABO \text{ is isosceles with angles } \frac{\pi}{6}, \frac{\pi}{6} \& \frac{2\pi}{3}$$

Question30

If $\alpha + i\beta$ and $\gamma + i\delta$ are the roots of $x^2 - (3 - 2i)x - (2i - 2) = 0$, $i = \sqrt{-1}$, then $\alpha\gamma + \beta\delta$ is equal to:

JEE Main 2025 (Online) 28th January Evening Shift

Options:

A.

2

B.

-6

C.

6

D.

-2

Answer: A

Solution:

$$\begin{aligned} x^2 - (3 - 2i)x - (2i - 2) &= 0 \\ x &= \frac{(3 - 2i) \pm \sqrt{(3 - 2i)^2 - 4(1)(-(2i - 2))}}{2(1)} \\ &= \frac{(3 - 2i) \pm \sqrt{9 - 4 - 12i + 8i - 8}}{2} \\ &= \frac{3 - 2i \pm \sqrt{-3 - 4i}}{2} \\ &= \frac{3 - 2i \pm \sqrt{(1)^2 + (2i)^2 - 2(1)(2i)}}{2} \\ &= \frac{3 - 2i \pm (1 - 2i)}{2} \\ &\Rightarrow \frac{3 - 2i + 1 - 2i}{2} \text{ or } \frac{3 - 2i - 1 + 2i}{2} \\ &\Rightarrow 2 - 2i \text{ or } 1 + 0i \\ \text{So } \alpha\gamma + \beta\delta &= 2(1) + (-2)(0) = 2 \end{aligned}$$

Question31

Let $|z_1 - 8 - 2i| \leq 1$ and $|z_2 - 2 + 6i| \leq 2$, $z_1, z_2 \in \mathbb{C}$. Then the minimum value of $|z_1 - z_2|$ is :

JEE Main 2025 (Online) 29th January Morning Shift

Options:

A.

3

B.

10

C.

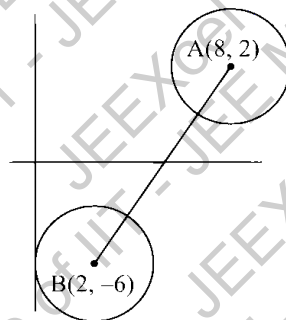
7

D.

13

Answer: C

Solution:



$$\begin{aligned}\therefore AB &= \sqrt{100} = 10 \\ \therefore |Z_1 - Z_2|_{\min} &= 10 - 2 - 1 = 7\end{aligned}$$

Question 32

Let z be a complex number such that $|z| = 1$. If $\frac{2+k^2z}{k+\bar{z}} = kz, k \in \mathbb{R}$, then the maximum distance of $k + ik^2$ from the circle $|z - (1 + 2i)| = 1$ is :

JEE Main 2025 (Online) 2nd April Morning Shift

Options:

A. $\sqrt{5} + 1$

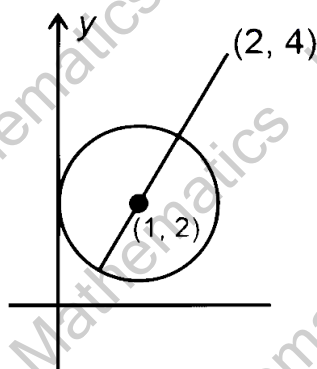
B. 3

C. $\sqrt{3} + 1$

D. 2

Answer: A

Solution:



$$\begin{aligned}\frac{2+k^2z}{k+\bar{z}} &= kz \\ \Rightarrow 2+k^2z &= k^2z+k\bar{z} \\ \Rightarrow 2+k|z|^2 &\quad (z\bar{z}=|z|^2, |z|=1) \\ \Rightarrow 2 &= k \\ \therefore k+k^2i &= 2+4i \\ \therefore \text{The maximum distance is} \\ &= \sqrt{(4-2)^2 + (2-1)^2} + \text{radius} \\ &= \sqrt{(2)^2 + (1)^2} + 1 \\ &= \sqrt{5} + 1\end{aligned}$$

Question33

Let $z \in C$ be such that $\frac{z^2+3i}{z-2+i} = 2+3i$. Then the sum of all possible values of z^2 is :

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

- A. $-19+2i$
- B. $-19-2i$
- C. $19-2i$
- D. $19+2i$

Answer: B

Solution:

$$\begin{aligned}\frac{z^2+3i}{z-2+i} &= 2+3i \\ z^2+3i &= (z-2+i)(2+3i) \\ z^2+3i &= 2z-4+2i+3iz-6i-3 \\ z^2+3i &= (2z-7)+i(3z-4) \\ z^2-(2+3i)z+(7+7i) &= 0\end{aligned}$$

This is a quadratic in z .

$$z_1 + z_2 = 2 + 3i$$

$$z_1 + z_2 = 7 + 7i$$

$$z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1z_2$$

$$= (2 + 3i)^2 - 2(7 + 7i)$$

$$= 4 - 9 + 12i - 14 - 14i$$

$$= -19 - 2i$$

Question 34

If $z_1, z_2, z_3 \in \mathbb{C}$ are the vertices of an equilateral triangle, whose centroid is z_0 , then $\sum_{k=1}^3 (z_k - z_0)^2$

JEE Main 2025 (Online) 3rd April Evening Shift

Options:

A. 0

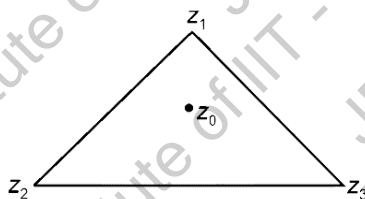
B. 1

C. i

D. $-i$

Answer: A

Solution:



$$z_0 = \frac{z_1 + z_2 + z_3}{3}$$

$$\sum_{k=1}^3 (z_k - z_0)^2 = (z_1 - z_0)^2 + (z_2 - z_0)^2 + (z_3 - z_0)^2$$

Let z_0 is origin $\Rightarrow z_1, z_2, z_3$ lies on a circle having $|z_0 - z_i| = R$

$$\therefore z_1 = Re^{i2\pi/3}, z_2 = Re^{i4\pi/3}, z_3 = Re^{i6\pi/3}$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = R^2 [e^{i4\pi/3} + e^{i8\pi/3} + e^{i12\pi/3}]$$

$$= 0$$

$$\therefore \sum_{k=1}^3 (z_k - z_0)^2 = 0$$

Question 35

Let the product of $\omega_1 = (8 + i) \sin \theta + (7 + 4i) \cos \theta$ and $\omega_2 = (1 + 8i) \sin \theta + (4 + 7i) \cos \theta$ be $\alpha + i\beta$, $i = \sqrt{-1}$. Let p and q be the maximum and the minimum values of $\alpha + \beta$ respectively. Then $p + q$ is equal to :

JEE Main 2025 (Online) 4th April Evening Shift

Options:

- A. 130
- B. 150
- C. 160
- D. 140

Answer: A

Solution:

$$\begin{aligned}\omega_1 &= (8 \sin \theta + 7 \cos \theta) + i(\sin \theta + 4 \cos \theta) \\ \omega_2 &= (\sin \theta + 4 \cos \theta) + i(8 \sin \theta + 7 \cos \theta) \\ \alpha &= (8 \sin \theta + 7 \cos \theta) + (\sin \theta + 4 \cos \theta) \\ &\quad - (\sin \theta + 4 \cos \theta) + (8 \sin \theta + 7 \cos \theta) = 0 \\ \beta &= (8 \sin \theta + 7 \cos \theta)^2 + (\sin \theta + 4 \cos \theta)^2 \\ &= 65 \sin^2 \theta + 65 \cos^2 \theta + 56 \sin 2\theta + 4 \sin 2\theta \\ &= 65 + 60 \sin 2\theta \\ (\alpha + \beta)_{\max} &= 125 = p \\ (\alpha + \beta)_{\min} &= 5 = q \\ p + q &= 130\end{aligned}$$

Question 36

Among the statements

(S1) : The set $\{z \in \mathbb{C} - \{-i\} : |z| = 1 \text{ and } \frac{z-i}{z+i} \text{ is purely real}\}$ contains exactly two elements, and

(S2) : The set $\{z \in \mathbb{C} - \{-1\} : |z| = 1 \text{ and } \frac{z-1}{z+1} \text{ is purely imaginary}\}$ contains infinitely many elements.

JEE Main 2025 (Online) 7th April Morning Shift

Options:

- A. both are incorrect
- B. both are correct
- C. only (S2) is correct
- D. only (S1) is correct

Answer: C

Solution:

$$\begin{aligned}\frac{z-i}{z+i} &= \frac{\bar{z}+i}{\bar{z}-i} \\ &= \frac{z\bar{z}-i\bar{z}-iz-1}{z\bar{z}+zi+i\bar{z}-1} \\ &= \frac{z+\bar{z}}{2} = 0 \\ &= 2x = 0 \\ &= x = 0 \quad (\text{y-axis})\end{aligned}$$

$$\begin{aligned}|z| &= 1 \\ \therefore z &= i \quad (z \neq -i \text{ is given})\end{aligned}$$

Statement 1 is incorrect

$$\begin{aligned}
 \frac{z-i}{z+i} + \frac{\bar{z}-1}{\bar{z}+1} &= 0 \\
 &= z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0 \\
 &= z\bar{z} = 1 \\
 &= |z| = 1
 \end{aligned}$$

Statement 2 is correct

Question 37

If the locus of $z \in \mathbb{C}$, such that $\operatorname{Re}\left(\frac{z-1}{2z+i}\right) + \operatorname{Re}\left(\frac{\bar{z}-1}{2\bar{z}-i}\right) = 2$, is a circle of radius r and center (a, b) , then $\frac{15ab}{r^2}$ is equal to :

JEE Main 2025 (Online) 7th April Evening Shift

Options:

A.

16

B.

24

C.

12

D.

18

Answer: D

Solution:

$$\operatorname{Re}\left(\frac{z-1}{2z+i}\right) + \operatorname{Re}\left(\frac{\bar{z}-1}{2\bar{z}-i}\right) = 2$$

$$\text{Here, } \frac{z-1}{2z+i} = \left(\frac{\bar{z}-1}{2\bar{z}-i}\right) = 2$$

$$= \operatorname{Re}\left(\frac{z-1}{2z+i}\right) + \operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 2$$

$$= 2 \operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 2 \Rightarrow \operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 1$$

$$\text{Let } z = x + iy$$

$$\operatorname{Re}\left(\frac{(x-1) + iy}{2x + i(2y+1)}\right) = 1 \Rightarrow \operatorname{Re}\left[\frac{((x-1) + iy)(2x - i(2y+1))}{(2x + i(2y+1))(2x - i(2y+1))}\right] = 1$$

$$\Rightarrow \frac{2x(x-1) + y(2y+1)}{4x^2 + (2y+1)^2} = 1$$

$$\Rightarrow 2x^2 - 2x + 2y^2 + y = 4x^2 + 4y^2 + 1 + 4y$$

$$\Rightarrow 2x^2 + 2y^2 + 3y + 2x + 1 = 0$$

$$\Rightarrow x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$$

$$\text{centre} = \left(-\frac{1}{2}, -\frac{3}{4}\right), r = \sqrt{\frac{1}{4} + \frac{9}{16} - \frac{1}{2}} = \frac{\sqrt{5}}{4}$$

$$a = -\frac{1}{2}, b = -\frac{3}{4}, r^2 = \frac{5}{16}$$

$$15 \frac{ab}{r^2} = 15 \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{4}\right) \times \frac{16}{5} = 18$$

Question 38

Let $A = \left\{ \theta \in [0, 2\pi] : 1 + 10 \operatorname{Re} \left(\frac{2 \cos \theta + i \sin \theta}{\cos \theta - 3i \sin \theta} \right) = 0 \right\}$. Then $\sum_{\theta \in A} \theta^2$ is equal to

JEE Main 2025 (Online) 8th April Evening Shift

Options:

A.

$$\frac{21}{4}\pi^2$$

B.

$$6\pi^2$$

C.

$$\frac{27}{4}\pi^2$$

D.

$$8\pi^2$$

Answer: A

Solution:

$$1 + 10 \operatorname{Re} \left(\frac{2 \cos \theta + i \sin \theta}{\cos \theta - 3i \sin \theta} \right) = 0$$

$$\therefore z + \bar{z} = 2 \operatorname{Re}(z)$$

$$\frac{2 \cos \theta + i \sin \theta}{\cos \theta - 3i \sin \theta} + \frac{2 \cos \theta - i \sin \theta}{\cos \theta + 3i \sin \theta} = 2 \times \left(\frac{-1}{10} \right)$$

$$\frac{(2 \cos^2 \theta - 3 \sin^2 \theta) + (2 \cos^2 \theta) - (3 \sin^2 \theta)}{\cos^2 \theta + 9 \sin^2 \theta} = \frac{-2}{10}$$

$$\Rightarrow \frac{2 \cos^2 \theta - 3 \sin^2 \theta}{\cos^2 \theta + 9 \sin^2 \theta} = \frac{-1}{10}$$

$$\Rightarrow 20 \cos^2 \theta - 30 \sin^2 \theta = -\cos^2 \theta - 9 \sin^2 \theta$$

$$21 \cos^2 \theta - 21 \sin^2 \theta = 0$$

$$\Rightarrow \cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow \sum \theta^2 = \frac{\pi^2}{16} + \frac{9\pi^2}{16} + \frac{25\pi^2}{16} + \frac{49\pi^2}{16} = \frac{84\pi^2}{16} = \frac{21\pi^2}{4}$$

Question 39

If $S = \{z \in \mathbb{C} : |z - i| = |z + i| = |z - 1|\}$, then, $n(S)$ is:

[27-Jan-2024 Shift 1]

Options:

A.

$$1$$

B.

$$0$$

C.

$$3$$

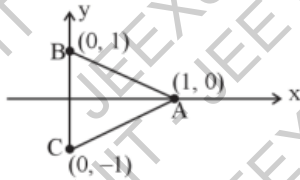
D.

$$2$$

Answer: A

Solution:

$$|z - i| = |z + i| = |z - 1|$$



ABC is a triangle. Hence its circum-centre will be the only point whose distance from A, B, C will be same.
So $n(S) = 1$

Question40

If α satisfies the equation $x^2 + x + 1 = 0$ and $(1 + \alpha)^7 = A + B\alpha + C\alpha^2$, $A, B, C \geq 0$, then $5(3A - 2B - C)$ is equal to

[27-Jan-2024 Shift 1]

Answer: 5

Solution:

$$x^2 + x + 1 = 0 \Rightarrow x = \omega, \omega^2 = \alpha$$

$$\text{Let } \alpha = \omega$$

$$\text{Now } (1 + \alpha)^7 = -\omega^{14} = -\omega^2 = 1 + \omega$$

$$A = 1, B = 1, C = 0$$

$$\therefore 5(3A - 2B - C) = 5(3 - 2 - 0) = 5$$

Question41

Let the complex numbers α and $1/\alpha$ lie on the circles $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$.

Then, the value of $100|\alpha|^2$ is. ____

[27-Jan-2024 Shift 2]

Answer: 20

Solution:

$$\begin{aligned}
 |z - z_0|^2 &= 4 \\
 \Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) &= 4 \\
 \Rightarrow \alpha\bar{\alpha} - \alpha\bar{z}_0 - z_0\bar{\alpha} + |z_0|^2 &= 4 \\
 \Rightarrow |\alpha|^2 - \alpha\bar{z}_0 - z_0\bar{\alpha} &= 2 \dots\dots\dots(1)
 \end{aligned}$$

$$\begin{aligned}
 |z - z_0|^2 &= 16 \\
 \Rightarrow \left(\frac{1}{\alpha} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) &= 16 \\
 \Rightarrow (1 - \alpha z_0)(1 - \alpha \bar{z}_0) &= 16 |\alpha|^2 \\
 \Rightarrow 1 - \alpha \bar{z}_0 - \alpha z_0 + |\alpha|^2 |z_0|^2 &= 16 |\alpha|^2 \\
 \Rightarrow 1 - \alpha \bar{z}_0 - \alpha z_0 &= 14 |\alpha|^2 \dots\dots\dots(2)
 \end{aligned}$$

From (1) and (2)

$$\begin{aligned}
 \Rightarrow 5 |\alpha|^2 &= 1 \\
 \Rightarrow 100 |\alpha|^2 &= 20
 \end{aligned}$$

Question 42

If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then

[27-Jan-2024 Shift 2]

Options:

A.

$$2S_{12} = S_{11} + S_{10}$$

B.

$$S_{12} = S_{11} + S_{10}$$

C.

$$2S_{11} = S_{12} + S_{10}$$

D.

$$S_{11} = S_{10} + S_{12}$$

Answer: B

Solution:

$$x^2 - x - 1 = 0$$

$$S_n = 2023\alpha^n + 2024\beta^n$$

$$\begin{aligned}
 S_{n-1} + S_{n-2} &= 2023\alpha^{n-1} + 2024\beta^{n-1} + 2023\alpha^{n-2} + 2024\beta^{n-2} \\
 &= 2023\alpha^{n-2}[1 + \alpha] + 2024\beta^{n-2}[1 + \beta] \\
 &= 2023\alpha^{n-2}[\alpha^2] + 2024\beta^{n-2}[\beta^2] \\
 &= 2023\alpha^n + 2024\beta^n
 \end{aligned}$$

$$S_{n-1} + S_{n-2} = S_n$$

Put $n = 12$

$$S_{11} + S_{10} = S_{12}$$

Question43

If $z = \frac{1}{2} - 2i$, is such that $|z + 1| = \alpha z + \beta(1 + i)$, $i = \sqrt{-1}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha + \beta$ is equal to

[29-Jan-2024 Shift 1]

Options:

- A. -4
- B. 3
- C. 2
- D. -1

Answer: B

Question44

Let α, β be the roots of the equation $x^2 - x + 2 = 0$ with $\text{Im}(\alpha) > \text{Im}(\beta)$. Then $\alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2$ is equal to

[29-Jan-2024 Shift 1]

Answer: 13

Solution:

$$\begin{aligned} & \alpha^6 + \alpha^4 + \beta^4 - 5\alpha^2 \\ &= \alpha^4(\alpha - 2) + \alpha^4 - 5\alpha^2 + (\beta - 2)^2 \\ &= \alpha^5 - \alpha^4 - 5\alpha^2 + \beta^2 - 4\beta + 4 \\ &= \alpha^3(\alpha - 2) - \alpha^4 - 5\alpha^2 + \beta - 2 - 4\beta + 4 \\ &= -2\alpha^3 - 5\alpha^2 - 3\beta + 2 \\ &= -2\alpha(\alpha - 2) - 5\alpha^2 - 3\beta + 2 \\ &= -7\alpha^2 + 4\alpha - 3\beta + 2 \\ &= -7(\alpha - 2) + 4\alpha - 3\beta + 2 \\ &= -3\alpha - 3\beta + 16 = -3(1) + 16 = 13 \end{aligned}$$

Question45

Let r and θ respectively be the modulus and amplitude of the complex number $z = 2 - i(2 \tan 5\pi/8)$, then (r, θ) is equal to

[29-Jan-2024 Shift 2]

Options:

A.

$$\left(2 \sec \frac{3\pi}{8}, \frac{3\pi}{8} \right)$$

B.

$$\left(2 \sec \frac{3\pi}{8}, \frac{5\pi}{8} \right)$$

C.

$$\left(2 \sec \frac{5\pi}{8}, \frac{3\pi}{8} \right)$$

D.

$$\left(2 \sec \frac{11\pi}{8}, \frac{11\pi}{8} \right)$$

Answer: A

Solution:

$$z = 2 - i \left(2 \tan \frac{5\pi}{8} \right) = x + iy \quad (\text{let})$$

$$r = \sqrt{x^2 + y^2} \quad \& \quad \theta = \tan^{-1} \frac{y}{x}$$

$$r = \sqrt{(2)^2 + \left(2 \tan \frac{5\pi}{8} \right)^2}$$

$$= \left| 2 \sec \frac{5\pi}{8} \right| = \left| 2 \sec \left(\pi - \frac{3\pi}{8} \right) \right|$$

$$= 2 \sec \frac{3\pi}{8}$$

$$\& \quad \theta = \tan^{-1} \left(\frac{-2 \tan \frac{5\pi}{8}}{2} \right)$$

$$= \tan^{-1} \left(\tan^2 \left(\pi - \frac{5\pi}{8} \right) \right)$$

$$= \frac{3\pi}{8}$$

Question 46

Let α, β be the roots of the equation $x^2 - \sqrt{6}x + 3 = 0$ such that $\text{Im}(\alpha) > \text{Im}(\beta)$. Let a, b be integers not divisible by 3 and n be a natural number such that $\frac{\alpha^{99}}{\beta} + \alpha^{98} = 3^n(a + ib)$, $i = \sqrt{-1}$. Then $n + a + b$ is equal to _____

[29-Jan-2024 Shift 2]

Answer: 49

Solution:

$$x^2 - \sqrt{6}x + 6 = 0$$

$$x = \frac{\sqrt{6} \pm i\sqrt{6}}{2} = \frac{\sqrt{6}}{2}(1 \pm i)$$

$$\alpha = \sqrt{3} \left(e^{i\frac{\pi}{4}} \right), \beta = \sqrt{3} \left(e^{-i\frac{\pi}{4}} \right)$$

$$\begin{aligned} \therefore \frac{\alpha^{99}}{\beta} + \alpha^{98} &= \alpha^{98} \left(\frac{\alpha}{\beta} + 1 \right) \\ &= \frac{\alpha^{98}(\alpha + \beta)}{\beta} = 3^{49} \left(e^{i99\frac{\pi}{4}} \right) \times \sqrt{2} \end{aligned}$$

$$= 3^{49}(-1 + i)$$

$$= 3^n(a + ib)$$

$$\therefore n = 49, a = -1, b = 1$$

$$\therefore n + a + b = 49 - 1 + 1 = 49$$

Question 47

Let the set $C = \{(x, y) \mid x^2 - 2^y = 2023, x, y \in \mathbb{N}\}$.

Then $\sum_{(x, y) \in C} (x + y)$ is equal to _____

[29-Jan-2024 Shift 2]

Answer: 46

Solution:

$$x^2 - 2^y = 2023$$

$$\Rightarrow x = 45, y = 1$$

$$\sum_{(x, y) \in C} (x + y) = 46$$

Question 48

If $z = x + iy$, $xy \neq 0$, satisfies the equation $z^2 + \bar{iz} = 0$, then $|z^2|$ is equal to :

[30-Jan-2024 Shift 1]

Options:

A.

9

B.

1

C.

4

D.

1/4

Answer: B

Solution:

$$z^2 = -i\bar{z}$$

$$|z^2| = |i\bar{z}|$$

$$|z^2| = |z|$$

$$|z|^2 - |z| = 0$$

$$|z|(|z| - 1) = 0$$

$$|z| = 0 \text{ (not acceptable)}$$

$$\therefore |z| = 1$$

$$\therefore |z|^2 = 1$$

Question 49

If z is a complex number, then the number of common roots of the equation $z^{1985} + z^{100} + 1 = 0$ and $z^3 + 2z^2 + 2z + 1 = 0$, is equal to :

[30-Jan-2024 Shift 2]

Options:

A.

1

B.

2

C.

0

D.

3

Answer: B

Solution:

$$z^{1985} + z^{100} + 1 = 0 \text{ \& } z^3 + 2z^2 + 2z + 1 = 0$$

$$(z+1)(z^2 - z + 1) + 2z(z+1) = 0$$

$$(z+1)(z^2 + z + 1) = 0$$

$$\Rightarrow z = -1, \quad z = w, w^2$$

Now putting $z = -1$ not satisfy

Now put $z = w$

$$\Rightarrow w^{1985} + w^{100} + 1$$

$$\Rightarrow w^2 + w + 1 = 0$$

$$\text{Also, } z = w^2$$

$$\Rightarrow w^{3970} + w^{200} + 1$$

$$\Rightarrow w + w^2 + 1 = 0$$

Two common root

Question 50

If α denotes the number of solutions of $|1 - i|^x = 2^x$ and $\beta =$

$\left(\frac{|z|}{\arg(z)} \right)$, where $z = \frac{\pi(1+i)^4}{4} \left(\frac{1 - \sqrt{\pi}i}{\sqrt{\pi} - i} + \frac{\sqrt{\pi} - i}{1 + \sqrt{\pi}i} \right)$, $i = \sqrt{-1}$, then

the distance of the point (α, β) from the line $4x - 3y =$

7 is _____

[31-Jan-2024 Shift 1]

Answer: 3

Solution:

$$(\sqrt{2})^x = 2^x \Rightarrow x = 0 \Rightarrow \alpha = 1$$

$$z = \frac{\pi(1+i)^4}{4} \left[\frac{\sqrt{\pi} - \pi i - i - \sqrt{\pi}}{\pi + 1} + \frac{\sqrt{\pi} - i - \pi i - \sqrt{\pi}}{1 + \pi} \right]$$

$$= -\frac{\pi i}{2}(1 + 4i + 6i^2 + 4i^3 + 1)$$

$$= 2\pi i$$

$$\beta = \frac{2\pi}{\frac{\pi}{2}} = 4$$

Distance from $(1, 4)$ to $4x - 3y = 7$

$$\text{Will be } \frac{15}{5} = 3$$

Question 51

Let z_1 and z_2 be two complex number such that $z_1 + z_2 = 5$ and $z_1^3 + z_2^3 = 20 + 15i$. Then $|z_1^4 + z_2^4|$ equals-

[31-Jan-2024 Shift 2]

Options:

A.

$$30\sqrt{3}$$

B.

$$75$$

C.

$$15\sqrt{15}$$

D.

$$25\sqrt{3}$$

Answer: B

Solution:

$$z_1 + z_2 = 5$$

$$z_1^3 + z_2^3 = 20 + 15i$$

$$z_1^3 + z_2^3 = (z_1 + z_2)^3 - 3z_1z_2(z_1 + z_2)$$

$$z_1^3 + z_2^3 = 125 - 3z_1z_2(5)$$

$$\Rightarrow 20 + 15i = 125 - 15z_1z_2$$

$$\Rightarrow 3z_1z_2 = 25 - 4 - 3i$$

$$\Rightarrow 3z_1z_2 = 21 - 3i$$

$$\Rightarrow z_1 \cdot z_2 = 7 - i$$

$$\Rightarrow (z_1 + z_2)^2 = 25$$

$$\Rightarrow z_1^2 + z_2^2 = 25 - 2(7 - i)$$

$$\Rightarrow 11 + 2i$$

$$(z_1^2 + z_2^2)^2 = 121 - 4 + 44i$$

$$\Rightarrow z_1^4 + z_2^4 + 2(7 - i)^2 = 117 + 44i$$

$$\Rightarrow z_1^4 + z_2^4 = 117 + 44i - 2(49 - 1 - 14i)$$

$$\Rightarrow |z_1^4 + z_2^4| = 75$$

Question52

Let $\alpha, \beta \in \mathbb{N}$ be roots of equation $x^2 - 70x + \lambda = 0$, where $\lambda/2, \lambda/3 \notin \mathbb{N}$. If λ assumes the minimum possible value, then

$\frac{(\sqrt{\alpha-1} + \sqrt{\beta-1})(\lambda+35)}{|\alpha-\beta|}$ is equal to :

[30-Jan-2024 Shift 1]

Answer: 60

Solution:

$$x^2 - 70x + \lambda = 0$$

$$\alpha + \beta = 70$$

$$\alpha\beta = \lambda$$

$$\therefore \alpha(70 - \alpha) = \lambda$$

Since, 2 and 3 does not divide λ

$$\therefore \alpha = 5, \beta = 65, \lambda = 325$$

By putting value of α, β, λ we get the required value 60 .

Question53

The number of real solutions of the equation $x(x^2 + 3x| + 5x - 1| + 6x - 2|) = 0$ is

[30-Jan-2024 Shift 2]

Answer: 1

Solution:

$$x=0 \text{ and } x^2+3x+5x-1+6x-2=0$$

Here all terms are +ve except at $x=0$

So there is no value of x

Satisfies this equation

Only solution $x=0$

No of solution 1

Question54

Let S be the set of positive integral values of a for which $\frac{ax^2+2(a+1)x+9a+4}{x^2-8x+32} > 0, \forall x \in \mathbb{R}$.

Then, the number of elements in S is :

[31-Jan-2024 Shift 1]

Options:

A.

1

B.

0

C.

∞

D.

3

Answer: B

Solution:

$$ax^2+2(a+1)x+9a+4 < 0 \quad \forall x \in \mathbb{R}$$

$$\therefore a < 0$$

Question55

For $0 < c < b < a$, let $(a+b-2c)x^2 + (b+c-2a)x + (c+a-2b) = 0$ and $\alpha \neq 1$ be one of its root. Then, among the two statements

(I) If $\alpha \in (-1, 0)$, then b cannot be the geometric mean of a and c

(II) If $\alpha \in (0, 1)$, then b may be the geometric mean of a and c

[31-Jan-2024 Shift 1]

Options:

- A.
Both (I) and (II) are true
B.
Neither (I) nor (II) is true
C.
Only (II) is true
D.
Only (I) is true

Answer: A

Solution:

$$f(x) = (a+b-2c)x^2 + (b+c-2a)x + (c+a-2b)$$

$$f(x) = a+b-2c + b+c-2a + c+a-2b = 0$$

$$f(1) = 0$$

$$\therefore \alpha \cdot 1 = \frac{c+a-2b}{a+b-2c}$$

$$\alpha = \frac{c+a-2b}{a+b-2c}$$

$$\text{If } -1 < \alpha < 0$$

$$-1 < \frac{c+a-2b}{a+b-2c} < 0$$

$$b+c < 2a \text{ and } b > \frac{a+c}{2}$$

therefore, b cannot be G.M. between a and c .

$$\text{If, } 0 < \alpha < 1$$

$$0 < \frac{c+a-2b}{a+b-2c} < 1$$

$$b > c \text{ and } b < \frac{a+c}{2}$$

Therefore, b may be the G.M. between a and c .

Question 56

The number of solutions, of the equation $e^{\sin x} - 2e^{-\sin x} = 2$ is

[31-Jan-2024 Shift 2]

Options:

- A.
2
- B.
more than 2
- C.
1
- D.
0

Answer: D

Solution:

Take $e^{\sin x} = t (t > 0)$

$$\Rightarrow t - \frac{2}{t} = 2$$

$$\Rightarrow \frac{t^2 - 2}{t} = 2$$

$$\Rightarrow t^2 - 2t - 2 = 0$$

$$\Rightarrow t^2 - 2t + 1 = 3$$

$$\Rightarrow (t-1)^2 = 3$$

$$\Rightarrow t = 1 \pm \sqrt{3}$$

$$\Rightarrow t = 1 \pm 1.73$$

$$\Rightarrow t = 2.73 \text{ or } -0.73 \text{ (rejected as } t > 0)$$

$$\Rightarrow e^{\sin x} = 2.73$$

$$\Rightarrow \log_e e^{\sin x} = \log_e 2.73$$

$$\Rightarrow \sin x = \log_e 2.73 > 1$$

So no solution.

Question 57

Let a, b, c be the length of three sides of a triangle satisfying the condition $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$. If the set of all possible values of x is the interval (α, β) , then $12(\alpha^2 + \beta^2)$ is equal to _____

[31-Jan-2024 Shift 2]

Answer: 36

Solution:

$$(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$$

$$\Rightarrow a^2x^2 - 2abx + b^2 + b^2x^2 - 2bcx + c^2 = 0$$

$$\Rightarrow (ax - b)^2 + (bx - c)^2 = 0$$

$$\Rightarrow ax - b = 0, \quad bx - c = 0$$

$$\Rightarrow a + b > c, \quad b + c > a, \quad c + a > b$$

$$a + ax > bx, \quad ax + bx > a, \quad ax^2 + a > ax$$

$$a + ax > ax^2, \quad ax + ax^2 > a, \quad x^2 - x + 1 > 0$$

$$x^2 - x - 1 < 0, \quad x^2 + x - 1 > 0, \quad \text{always true}$$

$$\frac{1 - \sqrt{5}}{2} < x < \frac{1 + \sqrt{5}}{2}$$

$$x < \frac{-1 - \sqrt{5}}{2}, \quad \text{or } x > \frac{-1 + \sqrt{5}}{2}$$

$$\Rightarrow \frac{\sqrt{5} - 1}{2} < x < \frac{\sqrt{5} + 1}{2}$$

$$\Rightarrow \alpha = \frac{\sqrt{5} - 1}{2}, \quad \beta = \frac{\sqrt{5} + 1}{2}$$

$$12(\alpha^2 + \beta^2) = 12 \left(\frac{(\sqrt{5} - 1)^2 + (\sqrt{5} + 1)^2}{4} \right) = 36$$

Question 58

Let $S = \{z \in \mathbb{C} : |z - 1| = 1 \text{ and } (\sqrt{2} - 1)(z + z) - i(z - z) = 2\sqrt{2}\}$. Let $z_1, z_2 \in S$ be such that

$|z_1| = \max_{z \in S} |z|$ and $z_2 = \min_{z \in S} |z|$. Then $|\sqrt{2}z_1 - z_2|^2$ equals :

[1-Feb-2024 Shift 1]

Options:

A.

1

B.

4

C.

3

D.

2

Answer: D

Solution:

$$\text{Let } Z = x + iy$$

$$\text{Then } (x-1)^2 + y^2 = 1 \rightarrow (1)$$

$$(\sqrt{2}-1)(2x) - i(2iy) = 2\sqrt{2}$$

$$\Rightarrow (\sqrt{2}-1)x + y = \sqrt{2} \rightarrow (2)$$

Solving (1) & (2) we get

$$\text{Either } x = 1 \text{ or } x = \frac{1}{2-\sqrt{2}} \rightarrow (3)$$

On solving (3) with (2) we get

$$\text{For } x = 1 \Rightarrow y = 1 \Rightarrow Z_2 = 1 + i$$

& for

$$x = \frac{1}{2-\sqrt{2}} \Rightarrow y = \sqrt{2} - \frac{1}{\sqrt{2}} \Rightarrow Z_1 = \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}$$

Now

$$|\sqrt{2}z_1 - z_2|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} + 1 \right) \sqrt{2} + i - (1 + i) \right|^2$$

$$= (\sqrt{2})^2$$

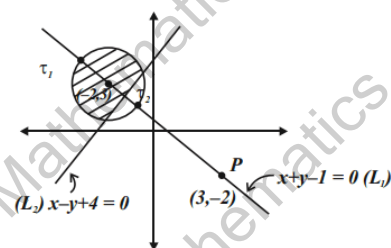
$$= 2$$

Question 59

Let $P = \{z \in \mathbb{C} : |z + 2 - 3i| \leq 1\}$ and $Q = \{z \in \mathbb{C} : z(1+i) + \bar{z}(1-i) \leq -8\}$. Let in $P \cap Q$, $|z - 3 + 2i|$ be maximum and minimum at z_1 and z_2 respectively. If $|z_1|^2 + 2|z_2|^2 = \alpha + \beta\sqrt{2}$, where α, β are integers, then $\alpha + \beta$ equals _____ [1-Feb-2024 Shift 1]

Answer: 8

Solution:



Clearly for the shaded region z_1 is the intersection of the circle and the line passing through P (L_1) and z_2 is intersection of line L_1 & L_2

$$\text{Circle : } (x+2)^2 + (y-3)^2 = 1$$

$$L_1 : x + y - 1 = 0$$

$$L_2 : x - y + 4 = 0$$

On solving circle & L_1 we get

$$z_1 : \left(-2 - \frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right)$$

On solving L_1 and z_2 is intersection of line L_1 & L_2 we get $z_2 : \left(\frac{-3}{2}, \frac{5}{2}\right)$

$$|z_1|^2 + 2|z_2|^2 = 14 + 5\sqrt{2} + 17$$

$$= 31 + 5\sqrt{2}$$

$$\text{So } \alpha = 31$$

$$\beta = 5$$

$$\alpha + \beta = 36$$

Question60

If z is a complex number such that $|z| \geq 1$, then the minimum value of $\left|z + \frac{1}{2}(3 + 4i)\right|$ is:

[1-Feb-2024 Shift 2]

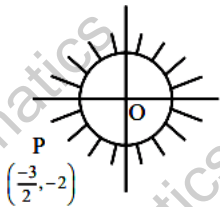
Options:

- A.
 $\frac{5}{2}$
- B.
 2
- C.
 $\frac{3}{2}$
- D.
None of above

Answer: D

Solution:

$$|z| \geq 1$$



Min. value of $\left|z + \frac{3}{2} + 2i\right|$ is actually zero.

Question61

$$\text{Let } S = \{x \in \mathbb{R} : (\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10\}$$

Then the number of elements in S is :

[1-Feb-2024 Shift 1]

Options:

- A.
 4
- B.
 0

C.

2

D.

1

Answer: C

Solution:

$$(\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10$$

$$\text{Let } (\sqrt{3} + \sqrt{2})^x = t$$

$$t + \frac{1}{t} = 10$$

$$t^2 - 10t + 1 = 0$$

$$t = \frac{10 \pm \sqrt{100 - 4}}{2} = 5 \pm 2\sqrt{6}$$

$$(\sqrt{3} + \sqrt{2})^x = (\sqrt{3} + \sqrt{2})^2$$

$$x = 2 \text{ or } x = -2$$

Number of solutions = 2

Question 62

Let α and β be the roots of the equation $px^2 + qx - r = 0$, where $p \neq 0$. If p, q and r be the consecutive terms of a non-constant G.P and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{4}$, then the value of $(\alpha - \beta)^2$ is :

[1-Feb-2024 Shift 2]

Options:

A.

80/9

B.

9

C.

20/3

D.

8

Answer: A

Solution:

$$px^2 + qx - r = 0$$

$$p = A, q = AR, r = AR^2$$

$$Ax^2 + ARx - AR^2 = 0$$

$$x^2 + Rx - R^2 = 0$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{4}$$

$$\therefore \frac{\alpha + \beta}{\alpha\beta} = \frac{3}{4} \Rightarrow \frac{-R}{-R^2} = \frac{3}{4} \Rightarrow R = \frac{4}{3}$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = R^2 - 4(-R^2) = 5 \left(\frac{16}{9} \right)$$

$$= 80/9$$

Question63

Let $p, q \in \mathbb{R}$ and $(1 - \sqrt{3}i)^{200} = 2^{199}(p + iq)$, $i = \sqrt{-1}$ Then $p + q + q^2$ and $p - q + q^2$ are roots of the equation.

[24-Jan-2023 Shift 1]

Options:

A. $x^2 + 4x - 1 = 0$

B. $x^2 - 4x + 1 = 0$

C. $x^2 + 4x + 1 = 0$

D. $x^2 - 4x - 1 = 0$

Answer: B

Solution:

$$(1 - \sqrt{3}i)^{200} = 2^{199}(p + iq)$$

$$2^{200} \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)^{200} = 2^{199}(p + iq)$$

$$2 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = p + iq$$

$$p = -1, q = -\sqrt{3}$$

$$\alpha = p + q + q^2 = 2 - \sqrt{3}$$

$$\beta = p - q + q^2 = 2 + \sqrt{3}$$

$$= 4$$

$$= 1$$

$$\text{equation } x^2 - 4x + 1 = 0$$

Question64

The value of $\left(\frac{1 + \sin \frac{2\pi}{9} + i \cos \frac{2\pi}{9}}{1 + \sin \frac{2\pi}{9} - i \cos \frac{2\pi}{9}} \right)^3$ is

[24-Jan-2023 Shift 2]

Options:

A. $\frac{-1}{2}(1 - i\sqrt{3})$

B. $\frac{1}{2}(1 - i\sqrt{3})$

C. $\frac{-1}{2}(\sqrt{3} - i)$

D. $\frac{1}{2}(\sqrt{3} + i)$

Answer: C

Solution:

Let $\sin \frac{2\pi}{9} + i \cos \frac{2\pi}{9} = z$

$$\left(\frac{1+z}{1+\bar{z}} \right)^3 = \left(\frac{1+z}{1+\frac{1}{z}} \right)^3 = z^3$$

$$\Rightarrow \left(i \left(\cos \frac{2\pi}{9} - i \sin \frac{2\pi}{9} \right) \right)^3 = z^3$$

$$= -i \left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right) = -i \left(\frac{-1}{2} - i \frac{\sqrt{3}}{2} \right)$$

$$= \frac{-1}{2}(\sqrt{3} - i).$$

Question65

Let $z_1 = 2 + 3i$ and $z_2 = 3 + 4i$. The set

$$S = \left\{ z \in \mathbb{C} : \left| z - z_1 \right|^2 - \left| z - z_2 \right|^2 = \left| z_1 - z_2 \right|^2 \right\}$$

represents a

[25-Jan-2023 Shift 1]

Options:

A. straight line with sum of its intercepts on the coordinate axes equals 14

B. hyperbola with the length of the transverse axis 7

C. straight line with the sum of its intercepts on the coordinate axes equals -18

D. hyperbola with eccentricity 2

Answer: A

Solution:

$$((x-2)^2 + (y-3)^2) - ((x-3)^2 + (y-4)^2) = 1 + 1$$

$$\Rightarrow x + y = 7$$

Question66

Let z be a complex number such that $\left| \frac{z-2i}{z+i} \right| = 2$, $z \neq -i$. Then z lies on the circle of radius 2 and centre

[25-Jan-2023 Shift 2]

Options:

A. (2, 0)

B. (0, 0)

C. (0, 2)

D. $(0, -2)$

Answer: D

Solution:

$$\begin{aligned}(z-2i)(\bar{z}+2i) &= 4(z+i)(\bar{z}-i) \\ zz + 4 + 2i(z-\bar{z}) &= 4(\bar{z}\bar{z} + 1 + i(z-\bar{z})) \\ 3zz - 6i(z-\bar{z}) &= 0 \\ x^2 + y^2 - 2i(2iy) &= 0 \\ x^2 + y^2 + 4y &= 0\end{aligned}$$

Question 67

For two non-zero complex number z_1 and z_2 , if $\operatorname{Re}(z_1 z_2) = 0$ and $\operatorname{Re}(z_1 + z_2) = 0$, then which of the following are possible ?

(A) $\operatorname{Im}(z_1) > 0$ and $\operatorname{Im}(z_2) > 0$

(B) $\operatorname{Im}(z_1) < 0$ and $\operatorname{Im}(z_2) > 0$

(C) $\operatorname{Im}(z_1) > 0$ and $\operatorname{Im}(z_2) < 0$

(D) $\operatorname{Im}(z_1) < 0$ and $\operatorname{Im}(z_2) < 0$

Choose the correct answer from the options given below :

[29-Jan-2023 Shift 1]

Options:

A. B and D

B. B and C

C. A and B

D. A and C

Answer: B

Solution:

$$\begin{aligned}z_1 &= x_1 + iy_1 \\ z_2 &= x_2 + iy_2 \\ \operatorname{Re}(z_1 z_2) &= x_1 x_2 - y_1 y_2 = 0 \\ \operatorname{Re}(z_1 + z_2) &= x_1 + x_2 = 0 \\ x_1 \& x_2 \text{ are of opposite sign} \\ y_1 \& y_2 \text{ are of opposite sign}\end{aligned}$$

Question 68

Let $\alpha = 8 - 14i$, $A = \left\{ z \in \mathbb{C} : \frac{\alpha z - \bar{\alpha} \bar{z}}{z^2 - (\bar{z})^2 - 112i} = 1 \right\}$ and $B = \{ z \in \mathbb{C} : |z + 3i| = 4 \}$

Then $\sum_{z \in A \cap B} (\operatorname{Re} z - \operatorname{Im} z)$ is equal to _____.

[29-Jan-2023 Shift 2]

Answer: 14

Solution:

$$\alpha = 8 - 14i$$

$$z = x + iy$$

$$az = (8x + 14y) + i(-14x + 8y)$$

$$z + z = 2x \quad z - z = 2iy$$

$$\text{Set A: } \frac{2i(-14x + 8y)}{i(4xy - 112)} = 1$$

$$(x - 4)(y + 7) = 0$$

$$x = 4 \quad \text{or} \quad y = -7$$

$$\text{Set B: } x^2 + (y + 3)^2 = 16$$

$$\text{when } x = 4 \quad y = -3$$

$$\text{when } y = -7 \quad x = 0$$

$$\therefore A \cap B = \{4 - 3i, 0 - 7i\}$$

$$\text{So, } \sum_{z \in A \cap B} (\operatorname{Re} z - \operatorname{Im} z) = 4 - (-3) + (0 - (-7)) = 14$$

Question 69

Let $z = 1 + i$ and $z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}$. Then $\frac{12}{\pi} \arg(z_1)$ is equal to _____.

[30-Jan-2023 Shift 1]

Answer: 9

Solution:

$$\begin{aligned} z_1 &= \frac{1 + i}{\bar{z}(1 - z) + \frac{1}{z}} \\ z_1 &= \frac{1 + i(1 - i)}{(1 - i)(1 - 1 - i) + \frac{1}{1 + i}} \\ &= \frac{1 + i - i^2}{(1 - i)(-i) + \frac{1 - i}{2}} \\ &= \frac{2 + i}{-3i - 1} = \frac{4 + 2i}{-3i - 1} \\ &= \frac{-(4 + 2i)(3i - 1)}{(3i)^2 - (1)^2} \\ \operatorname{Arg}(z_1) &= \frac{3\pi}{4} \\ \therefore \frac{12}{\pi} \arg(z_1) &= \frac{12}{\pi} \times \frac{3\pi}{4} = 9 \end{aligned}$$

Question 70

For all $z \in \mathbb{C}$ on the curve $C_1: |z| = 4$, let the locus of the point $z + \frac{1}{z}$ be the curve C_2 . Then

[31-Jan-2023 Shift 1]

Options:

- A. the curves C_1 and C_2 intersect at 4 points
- B. the curves C_1 lies inside C_2
- C. the curves C_1 and C_2 intersect at 2 points
- D. the curves C_2 lies inside C_1

Answer: A

Solution:

$$\text{Let } w = z + \frac{1}{z} = 4e^{i\theta} + \frac{1}{4}e^{-i\theta}$$

$$\Rightarrow w = \frac{17}{4} \cos \theta + i \frac{15}{4} \sin \theta$$

$$\text{So locus of } w \text{ is ellipse } \frac{x^2}{\left(\frac{17}{4}\right)^2} + \frac{y^2}{\left(\frac{15}{4}\right)^2} = 1$$

Locus of z is circle $x^2 + y^2 = 16$
So intersect at 4 points

Question 71

The complex number $z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$ is equal to:

[31-Jan-2023 Shift 2]

Options:

A. $\sqrt{2} \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$

B. $\cos \frac{\pi}{12} - i \sin \frac{\pi}{12}$

C. $\sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$

D. $\sqrt{2} i \left(\cos \frac{5\pi}{12} - i \sin \frac{5\pi}{12} \right)$

Answer: A

Solution:

$$\begin{aligned} Z &= \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i} \\ &= \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i} \times \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}i}{\frac{1}{2} - \frac{\sqrt{3}}{2}i} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i \end{aligned}$$

Apply polar form,

$$r \cos \theta = \frac{\sqrt{3}-1}{2}$$

$$r \sin \theta = \frac{\sqrt{3}+1}{2}$$

$$\text{Now, } \tan \theta = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\text{So, } \theta = \frac{5\pi}{12}$$

Question 72

Let α be a root of the equation

$(a-c)x^2 + (b-a)x + (c-b) = 0$ where a, b, c are distinct real numbers such that the matrix

$$\begin{bmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}$$

is singular. Then the value of $\frac{(a-c)^2}{(b-a)(c-b)} + \frac{(b-a)^2}{(a-c)(c-b)} + \frac{(c-b)^2}{(a-c)(b-a)}$ is

[24-Jan-2023 Shift 1]

Options:

- A. 6
- B. 3
- C. 9
- D. 12

Answer: B

Solution:

$$\Delta = 0 = \begin{vmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$\Rightarrow \alpha^2(c-b) - \alpha(c-a) + (b-a) = 0$$

It is singular when $\alpha = 1$

$$\begin{aligned} & \frac{(a-c)^2}{(b-a)(c-b)} + \frac{(b-a)^2}{(a-c)(c-b)} + \frac{(c-b)^2}{(a-c)(b-a)} \\ & \frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{(a-b)(b-c)(c-a)} \\ & = 3 \frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = 3 \end{aligned}$$

Question 73

Let $\lambda \in \mathbb{R}$ and let the equation E be $|x|^2 - 2|x| + |\lambda - 3| = 0$. Then the largest element in the set $S = \{x + \lambda : x \text{ is an integer solution of E}\}$ is
[24-Jan-2023 Shift 1]

Answer: 5

Solution:

$$|x|^2 - 2|x| + |\lambda - 3| = 0$$

$$|x|^2 - 2|x| + |\lambda - 3| - 1 = 0$$

$$(|x| - 1)^2 + |\lambda - 3| = 1$$

At $\lambda = 3$, $x = 0$ and 2 ,

at $\lambda = 4$ or 2 , then

$x = 1$ or -1

So maximum value of $x + \lambda = 5$

Question 74

The number of real solutions of the equation $3\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) + 5 = 0$, is

[24-Jan-2023 Shift 2]

Options:

- A. 4
- B. 0
- C. 3
- D. 2

Answer: B

Solution:

$$3\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) + 5 = 0$$

$$3\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 2\left(x + \frac{1}{x}\right) + 5 = 0$$

$$\text{Let } x + \frac{1}{x} = t$$

$$3t^2 - 2t - 1 = 0$$

$$3t^2 - 3t + t - 1 = 0$$

$$3t(t-1) + 1(t-1) = 0$$

$$(t-1)(3t+1) = 0$$

$$t = 1, -\frac{1}{3}$$

$$x + \frac{1}{x} = 1, -\frac{1}{3} \Rightarrow \text{No solution.}$$

Question 75

Let

$$S = \left\{ \alpha : \log_2(9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2 \right\}.$$

Then the maximum value of β for which the equation $x^2 - 2\left(\sum_{\alpha \in S} \alpha\right)x + \sum_{\alpha \in S} (\alpha+1)^2 \beta = 0$ has real roots, is

[25-Jan-2023 Shift 1]

Answer: 25

Solution:

$$\log_2(9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2$$

$$\Rightarrow \frac{9^{2\alpha-4} + 13}{\frac{5}{2} \cdot 3^{2\alpha-4} + 1} = 4$$

$$\sum_{\alpha \in S} \alpha = 5 \text{ and } \sum_{\alpha \in S} (\alpha+1)^2 = 25$$

$$\Rightarrow x^2 - 50x + 25\beta = 0 \text{ has real roots}$$

$$\Rightarrow \beta \leq 25$$

$$\Rightarrow \beta_{\max} = 25$$

Question 76

Let $a \in \mathbb{R}$ and let α, β be the roots of the equation $x^2 + 60^{\frac{1}{4}}x + a = 0$. If $\alpha^4 + \beta^4 = -30$, then the product of all possible values of a is _____.

[25-Jan-2023 Shift 2]

Answer: 45

Solution:

$$\alpha + \beta = -60 \times \frac{1}{4} \quad \& \quad \alpha\beta = a$$

Given $\alpha^4 + \beta^4 = -30$

$$\Rightarrow (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = -30$$

$$\Rightarrow \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2a^2 = -30$$

$$\Rightarrow \left\{60 \times \frac{1}{2} - 2a\right\}^2 - 2a^2 = -30$$

$$\Rightarrow 60 + 4a^2 - 4a \times 60 \times \frac{1}{2} - 2a^2 = -30$$

$$\Rightarrow 2a^2 - 4.60 \times \frac{1}{2}a + 90 = 0$$

Product $= \frac{90}{2} = 45$

Question 77

Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation :

[29-Jan-2023 Shift 1]

Options:

- A. $7x^2 + 245x - 250 = 0$
- B. $7x^2 - 245x + 250 = 0$
- C. $49x^2 - 245x + 250 = 0$
- D. $49x^2 + 245x + 250 = 0$

Answer: C

Solution:

$$14x^2 - 31x + 3\lambda = 0$$

$$\alpha + \beta = \frac{31}{14} \dots (1) \text{ and } \alpha\beta = \frac{3\lambda}{14}$$

$$35x^2 - 53x + 4\lambda = 0$$

$$\alpha + \gamma = \frac{53}{35} \dots (3) \text{ and } \alpha\gamma = \frac{4\lambda}{35} \dots$$

$$\frac{(2)}{(4)} \Rightarrow \frac{\beta}{\gamma} = \frac{3 \times 35}{4 \times 14} = \frac{15}{8} \Rightarrow \beta = \frac{15}{8}\gamma$$

$$(1) - (3) \Rightarrow \beta - \gamma = \frac{31}{14} - \frac{53}{35} = \frac{155 - 106}{70} = \frac{7}{10}$$

$$\frac{15}{8}\gamma - \gamma = \frac{7}{10} \Rightarrow \gamma = \frac{4}{5}$$

$$\Rightarrow \beta = \frac{15}{8} \times \frac{4}{5} = \frac{3}{2}$$

$$\Rightarrow \alpha = \frac{31}{14} - \beta = \frac{31}{14} - \frac{3}{2} = \frac{5}{7}$$

$$\Rightarrow \lambda = \frac{14}{3} \alpha\beta = \frac{14}{3} \times \frac{5}{7} \times \frac{3}{2} = 5$$

so, sum of roots $\frac{3\alpha}{\beta} + \frac{4\alpha}{\gamma} = \left(\frac{3\alpha\gamma + 4\alpha\beta}{\beta\gamma} \right)$

$$= \frac{\left(3 \times \frac{4\lambda}{35} + 4 \times \frac{3\lambda}{14} \right)}{\frac{\beta\gamma}{14 \times 35\beta\gamma}} = \frac{12\lambda(14 + 35)}{14 \times 35\beta\gamma}$$

$$= \frac{49 \times 12 \times 5}{490 \times \frac{3}{2} \times \frac{4}{5}} = 5$$

Product of roots

$$= \frac{3\alpha}{\beta} \times \frac{4\alpha}{\gamma} = \frac{12\alpha^2}{\beta\gamma} = \frac{12 \times \frac{25}{49}}{\frac{3}{2} \times \frac{4}{5}} = \frac{250}{49}$$

So, required equation is $x^2 - 5x + \frac{250}{49} = 0$

$$\Rightarrow 49x^2 - 245x + 250 = 0$$

Question 78

If the value of real number $a > 0$ for which $x^2 - 5ax + 1 = 0$ and $x^2 - ax - 5 = 0$ have a common real roots is $\frac{3}{\sqrt{2\beta}}$ then β is equal to _____.

[30-Jan-2023 Shift 2]

Answer: 13

Solution:

Two equations have common root

$$\begin{aligned}\therefore (4a)(26a) &= (-6)^2 = 36 \\ \Rightarrow a^2 &= \frac{9}{26} \therefore a = \frac{3}{\sqrt{26}} \Rightarrow \beta = 13\end{aligned}$$

Question 79

The number of real roots of the equation $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$, is:

[31-Jan-2023 Shift 1]

Options:

- A. 0
- B. 1
- C. 3
- D. 2

Answer: B

Solution:

$$\begin{aligned}&\sqrt{(x-1)(x-3)} + \sqrt{(x-3)(x+3)} \\&= \sqrt{4\left(x - \frac{12}{4}\right)\left(x - \frac{2}{4}\right)} \\&\Rightarrow \sqrt{x-3} = 0 \Rightarrow x = 3 \text{ which is in domain} \\&\sqrt{x-1} + \sqrt{x+3} = \sqrt{4x-2} \\&2\sqrt{(x-1)(x+3)} = 2x-4 \\&x^2 + 2x - 3 = x^2 - 4x + 4 \\&6x = 7 \\&x = 7/6 \text{ or}\end{aligned}$$

Question 80

The equation $e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0$, $x \in \mathbb{R}$ has:

[31-Jan-2023 Shift 2]

Options:

- A. two solutions and both are negative
- B. no solution
- C. four solutions two of which are negative

D. two solutions and only one of them is negative

Answer: A

Solution:

$$e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0$$

$$\text{Let } e^x = t$$

$$\text{Now, } t^4 + 8t^3 + 13t^2 - 8t + 1 = 0$$

Dividing equation by t^2 ,

$$t^2 + 8t + 13 - \frac{8}{t} + \frac{1}{t^2} = 0$$

$$t^2 + \frac{1}{t^2} + 8\left(t - \frac{1}{t}\right) + 13 = 0$$

$$\left(t - \frac{1}{t}\right)^2 + 2 + 8\left(t - \frac{1}{t}\right) + 13 = 0$$

$$\text{Let } t - \frac{1}{t} = z$$

$$z^2 + 8z + 15 = 0$$

$$(z+3)(z+5) = 0$$

$$z = -3 \text{ or } z = -5$$

$$\text{So, } t - \frac{1}{t} = -3 \text{ or } t - \frac{1}{t} = -5$$

$$t^2 + 3t - 1 = 0 \text{ or } t^2 + 5t - 1 = 0$$

$$t = \frac{-3 \pm \sqrt{13}}{2} \text{ or } t = \frac{-5 \pm \sqrt{29}}{2}$$

as $t = e^x$ so t must be positive,

$$t = \frac{\sqrt{13}-3}{2} \text{ or } \frac{\sqrt{29}-5}{2}$$

$$\text{So, } x = \ln\left(\frac{\sqrt{13}-3}{2}\right) \text{ or } x = \ln\left(\frac{\sqrt{29}-5}{2}\right)$$

Hence two solution and both are negative.

Question 81

If the center and radius of the circle $\left| \frac{z-2}{z-3} \right| = 2$ are respectively (α, β) and γ , then $3(\alpha + \beta + \gamma)$ is equal to

[1-Feb-2023 Shift 1]

Options:

A. 11

B. 9

C. 10

D. 12

Answer: D

Solution:

$$\sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$= x^2 + y^2 - 4x + 4 = 4x^2 + 4y^2 - 24x + 36$$

$$= 3x^2 + 3y^2 - 20x + 32 = 0$$

$$= x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$= (\alpha, \beta) = \left(\frac{10}{3}, 0\right)$$

$$\gamma = \sqrt{\frac{100}{9} - \frac{32}{3}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

$$3(\alpha, \beta, \gamma) = 3\left(\frac{10}{3} + \frac{2}{3}\right)$$

$$= 12$$

Question82

Let a, b be two real numbers such that $ab < 0$. If the complex number $\frac{1+ai}{b+i}$ is of unit modulus and $a+ib$ lies on the circle $|z-1| = |2z|$, then a possible value of $\frac{1+[a]}{4b}$, where $[t]$ is greatest integer function, is :
[1-Feb-2023 Shift 2]

Options:

- A. $-\frac{1}{2}$
- B. -1
- C. 1
- D. $\frac{1}{2}$
- E. 0

Answer: E

Solution:

$$\begin{aligned} \left| \frac{1+ai}{b+i} \right| &= 1 \\ |1+ia| &= |b+i| \\ a^2 + 1 &= b^2 + 1 \Rightarrow a = \pm b \Rightarrow b = -a \text{ as } ab < 0 \\ (a+ib) \text{ lies on } |z-1| &= |2z| \\ |a+ib-1| &= 2|a+ib| \\ (a-1)^2 + b^2 &= 4(a^2 + b^2) \\ (a-1)^2 &= a^2 = 4(2a^2) \\ 1-2a &= 6a^2 \Rightarrow 6a^2 + 2a - 1 = 0 \\ a &= \frac{-2 \pm \sqrt{28}}{12} = \frac{-1 \pm \sqrt{7}}{6} \\ a &= \frac{\sqrt{7}-1}{6} \text{ and } b = \frac{1-\sqrt{7}}{6} \\ [a] &= 0 \\ \therefore \frac{1+[a]}{4b} &= \frac{6}{4(1-\sqrt{7})} = -\left(\frac{1+\sqrt{7}}{4}\right) \\ \text{Similarly when } a &= \frac{-1-\sqrt{7}}{6} \text{ and } b = \frac{1+\sqrt{7}}{6} \text{ then } [a] = -1 \\ \therefore \frac{1+[a]}{4b} &= \frac{1-1}{4 \times \frac{1+\sqrt{7}}{6}} = 0 \end{aligned}$$

Question83

Two dice are thrown independently. Let A be the event that the number appeared on the 1st die is less than the number appeared on the 2nd die, B be the event that the number appeared on the 1st die is even and that on the second die is odd, and C be the event that the number appeared on the 1st die is odd and that on the 2nd is even. Then
[1-Feb-2023 Shift 2]

Options:

- A. the number of favourable cases of the event $(A \cup B) \cap C$ is 6
- B. A and B are mutually exclusive
- C. The number of favourable cases of the events A, B and C are 15, 6 and 6 respectively
- D. B and C are independent

Answer: A

Solution:

A : no. on 1st die < no. on 2nd die
A : no. on 1st die = even & no. on 2nd die = odd
C : no. on 1st die = odd & no. on 2nd die = even
 $n(A) = 5 + 4 + 3 + 2 + 1 = 15$
 $n(B) = 9$
 $n(C) = 9$
 $n((A \cup B) \cap C) = (A \cap C) \cup (B \cap C)$
 $= (3 + 2 + 1) + 0 = 6.$

Question84

Let

$$S = \{x : x \in \mathbb{R} \text{ and } (\sqrt{3} + \sqrt{2})^{x^2-4} + (\sqrt{3} - \sqrt{2})^{x^2-4} = 10\}.$$

Then $n(S)$ is equal to
[1-Feb-2023 Shift 1]

Options:

- A. 2
- B. 4
- C. 6
- D. 0

Answer: B

Solution:

Sol. Let $(\sqrt{3} + \sqrt{2})^{x^2-4} = t$
 $t + \frac{1}{t} = 10$
 $\Rightarrow t = 5 + 2\sqrt{6}, 5 - 2\sqrt{6}$
 $\Rightarrow (\sqrt{3} + \sqrt{2})^{x^2-4} = 5 + 2\sqrt{6}, 5 - 2\sqrt{6}$
 $\Rightarrow x^2 - 4 = 2, -2 \text{ or } x^2 = 6, 2$
 $\Rightarrow x = \pm\sqrt{2}, \pm\sqrt{6}$

Question85

Let $a \neq b$ be two-zero real numbers. Then the number of elements in the set
 $X = \{z \in \mathbb{C} : \operatorname{Re}(az^2 + bz) = a \text{ and } \operatorname{Re}(bz^2 + az) = b\}$ is equal to :
[6-Apr-2023 shift 2]

Options:

- A. 0
- B. 2
- C. 1
- D. 3

Answer: A

Solution:

(1) Bonus
 $\because z + \bar{z} = 2 \operatorname{Re}(z)$ If $z = x + iy$

$$\Rightarrow z + \bar{z} = 2x$$

$$z^2 + (\bar{z})^2 = 2(x^2 - y^2)$$

$$(az^2 + bz) + (a\bar{z}^2 + b\bar{z}) = 2a \dots (1)$$

$$(bz^2 + az) + (b\bar{z}^2 + a\bar{z}) = 2b \dots (2)$$

add (1) and (2)

$$(a+b)z^2 + (a+b)z + (a+b)\bar{z}^2 + (a+b)\bar{z} = 2(a+b)$$

$$(a+b)[z^2 + z + (\bar{z})^2 + \bar{z}] = 2(a+b)$$

sub. (1) and (2)

$$(a-b)[z^2 - z + \bar{z}^2 - \bar{z}] = 2(a-b) \dots (3)$$

$$z^2 + \bar{z}^2 - z - \bar{z} = 2 \dots (4)$$

Case I: If $a+b \neq 0$

From (3) & (4)

$$2x + 2(x^2 - y^2) = 2 \Rightarrow x^2 - y^2 + x = 1 \dots (5)$$

$$2(x^2 - y^2) - 2x = 2 \Rightarrow x^2 - y^2 - x = 1 \dots (6)$$

$$(5) - (6)$$

$$2x = 0 \Rightarrow x = 0$$

$$\text{from (5) } y^2 = -1 \Rightarrow \text{not possible}$$

$$\therefore \text{Ans} = 0$$

Case II: If $a+b = 0$ then infinite number of solution.

So, the set X have infinite number of elements.

Question 86

For $\alpha, \beta, z \in \mathbb{C}$ and $\lambda > 1$, if $\sqrt{\lambda-1}$ is the radius of the circle $|z-\alpha|^2 + |z-\beta|^2 = 2\lambda$, then $|\alpha-\beta|$ is equal to

[6-Apr-2023 shift 2]

Answer: 2

Solution:

$$|z-z_1|^2 + |z-z_2|^2 = |z_1-z_2|^2$$

$$z_1 = \alpha, z_2 = \beta$$

$$|\alpha-\beta|^2 = 2\lambda$$

$$|\alpha-\beta| = \sqrt{2\lambda}$$

$$2r = \sqrt{2\lambda}$$

$$2\sqrt{\lambda-1} = \sqrt{2\lambda}$$

$$\Rightarrow 4(\lambda-1) = 2\lambda$$

$$\lambda = 2$$

$$|\alpha-\beta| = 2$$

Question 87

If for $z = \alpha + i\beta$, $|z+2| = z+4(1+i)$, then $\alpha+\beta$ and $\alpha\beta$ are the roots of the equation

[8-Apr-2023 shift 1]

Options:

A. $x^2 + 3x - 4 = 0$

B. $x^2 + 7x + 12 = 0$

C. $x^2 + x - 12 = 0$

D. $x^2 + 2x - 3 = 0$

Answer: B

Solution:

$$\begin{aligned}
 |z+2| &= |\alpha+i\beta+2| \\
 &= \alpha+i\beta+4+4i \\
 \sqrt{(\alpha+2)^2+\beta^2} &= (\alpha+4)+i(\beta+4) \quad \beta+4=0 \\
 (\alpha+2)^2+16 &= (\alpha+4)^2 \\
 \alpha^2+4+4\alpha+16 &= \alpha^2+16+8\alpha \\
 4 &= 4\alpha \\
 \alpha &= 1 \\
 \alpha &= 1, \beta = -4 \\
 \alpha+\beta &= -3, \alpha\beta = -4 \\
 \text{Sum of roots} &= -7 \\
 \text{Product of roots} &= 12 \\
 x^2+7x+12 &= 0
 \end{aligned}$$

Question88

Let $A = \left\{ \theta \in (0, 2\pi) : \frac{1+2i\sin\theta}{1-i\sin\theta} \text{ is purely imaginary} \right\}$. Then the sum of the elements in A is.

[8-Apr-2023 shift 2]

Options:

- A. π
- B. 3π
- C. 4π
- D. 2π

Answer: C

Solution:

$$\begin{aligned}
 z &= \frac{1+2i\sin\theta}{1-i\sin\theta} \times \frac{1+i\sin\theta}{1+i\sin\theta} \\
 z &= \frac{1-2\sin^2\theta+i(3\sin\theta)}{1+\sin^2\theta} \\
 \operatorname{Re}(z) &= 0 \\
 \frac{1-2\sin^2\theta}{1+\sin^2\theta} &= 0 \\
 \sin\theta &= \frac{\pm 1}{\sqrt{2}} \\
 A &= \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\} \\
 \text{sum} &= 4\pi \quad (\text{Option 3})
 \end{aligned}$$

Question89

Let the complex number $z = x + iy$ be such that $\frac{2z-3i}{2z+i}$ is purely imaginary. If $x + y^2 = 0$, then $y^4 + y^2 - y$ is equal to :

[10-Apr-2023 shift 1]

Options:

- A. $\frac{3}{2}$
- B. $\frac{2}{3}$
- C. $\frac{4}{3}$
- D. $\frac{3}{4}$

Answer: D

Solution:

$$z = x + iy$$
$$\frac{(2z - 3i)}{2z + i} = \text{purely imaginary}$$

$$\text{Means } \operatorname{Re} \left(\frac{2z - 3i}{2z + i} \right) = 0$$

$$\Rightarrow \frac{(2z - 3i)}{(2z + i)} = \frac{2(x + iy) - 3i}{2(x + iy) + i}$$

$$= \frac{2x + 2yi - 3i}{2x + i2y + i}$$

$$= \frac{2x + i(2y - 3)}{2x + i(2y + 1)} \times \frac{2x - i(2y + 1)}{2x - i(2y + 1)}$$

$$\operatorname{Re} \left[\frac{2z - 3i}{2z + i} \right] = \frac{4x^2 + (2y - 3)(2y + 1)}{4x^2 + (2y + 1)^2} = 0$$

$$\Rightarrow 4x^2 + (2y - 3)(2y + 1) = 0$$

$$\Rightarrow 4x^2 + [4y^2 + 2y - 6y - 3] = 0$$

$$\because x + y^2 = 0 \Rightarrow x = -y^2$$

$$\Rightarrow 4(-y^2)^2 + 4y^2 - 4y - 3 = 0$$

$$\Rightarrow 4y^4 + 4y^2 - 4y - 3 = 0$$

$$\Rightarrow y^4 + y^2 - y = \frac{3}{4}$$

Therefore, correct answer is option (4).

Question90

Let $S = \left\{ z = x + iy : \frac{2z - 3i}{4z + 2i} \text{ is a real number} \right\}$. Then which of the following is NOT correct?

[10-Apr-2023 shift 2]

Options:

A. $y \in \left(-\infty, -\frac{1}{2} \right) \cup \left(-\frac{1}{2}, \infty \right)$

B. $(x, y) = \left(0, -\frac{1}{2} \right)$

C. $x = 0$

D. $y + x^2 + y^2 \neq -\frac{1}{4}$

Answer: B

Solution:

$$\frac{2z - 3i}{4z + 2i} \in \mathbb{R}$$

$$\frac{2(x + iy) - 3i}{4(x + iy) + 2i} = \frac{2x + (2y - 3)i}{4x + (4y + 2)i} \times \frac{4x - (4y + 2)i}{4x - (4y + 2)i}$$

$$4x(2y - 3) - 2x(4y + 2) = 0$$

$$x = 0 \quad y \neq -\frac{1}{2}$$

$$\text{Ans.} = 2$$

Question91

Let w_1 be the point obtained by the rotation of $z_1 = 5 + 4i$ about the origin through a right angle in the anticlockwise direction, and w_2 be the point obtained by the rotation of $z_2 = 3 + 5i$ about the origin through a right angle in the clockwise direction. Then the principal argument of $w_1 - w_2$ is equal to :

[11-Apr-2023 shift 1]

Options:

A. $\pi - \tan^{-1} \frac{8}{9}$

B. $-\pi + \tan^{-1} \frac{8}{9}$

C. $\pi - \tan^{-1} \frac{33}{5}$

D. $-\pi + \tan^{-1} \frac{33}{5}$

Answer: A

Solution:

$$W_1 = z_1 i = (5 + 4i)i = -4 + 5i \dots (i)$$

$$W_1 = z_2(-i) = (3 + 5i)(-i) = 5 - 3i \dots (2)$$

$$W_1 - W_2 = -9 + 8i$$

$$\text{Principal argument} = \pi - \tan^{-1} \left(\frac{8}{9} \right)$$

Question 92

For $a \in \mathbb{C}$, let $A = \{z \in \mathbb{C} : \operatorname{Re}(a + \bar{z}) > \operatorname{Im}(\bar{a} + z)\}$ and $B = \{z \in \mathbb{C} : \operatorname{Re}(a + \bar{z}) < \operatorname{Im}(\bar{a} + z)\}$. The among the two statements:

(S1) : If $\operatorname{Re}(a), \operatorname{Im}(a) > 0$, then the set A contains all the real numbers

(S2) : If $\operatorname{Re}(a), \operatorname{Im}(a) < 0$, then the set B contains all the real numbers,

[11-Apr-2023 shift 2]

Options:

A. only (S1) is true

B. both are false

C. only (S2) is true

D. both are true

Answer: B

Solution:

$$\text{Let } a = x_1 + iy_1, z = x + iy$$

$$\text{Now } \operatorname{Re}(a + \bar{z}) > \operatorname{Im}(\bar{a} + z)$$

$$\therefore x_1 + x > -y_1 + y$$

$$x_1 = 2, y_1 = 10, x = -12, y = 0$$

Given inequality is not valid for these values.

S1 is false.

$$\text{Now } \operatorname{Re}(a + \bar{z}) < \operatorname{Im}(\bar{a} + z)$$

$$x_1 + x < -y_1 + y$$

$$x_1 = -2, y_1 = -10, x = 12, y = 0$$

Given inequality is not valid for these values. S2 is false.

Question 93

Let $S = \left\{ z \in \mathbb{C} - \{i, 2i\} : \frac{z^2 + 8iz - 15}{z^2 + 3iz - 2} \in \mathbb{R} \right\}$. If $\alpha - \frac{13}{11}i \in S$, $a \in \mathbb{R} - \{0\}$, then $242\alpha^2$ is equal to _____.

[11-Apr-2023 shift 2]

Answer: 1680

Solution:

$$\left(\frac{z^2 + 8iz - 15}{z^2 - 3iz - 2} \right) \in \mathbb{R}$$

$$\Rightarrow 1 + \frac{(11iz - 13)}{(z^2 - 3iz - 2)} \in \mathbb{R}$$

$$\text{Put } z = \alpha - \frac{13}{11}i$$

$$\Rightarrow (z^2 - 3iz - 2) \text{ is imaginary}$$

$$\text{Put } z = x + iy$$

$$\Rightarrow (x^2 - y^2 + 2xyi - 3ix + 3y - 2) \in \text{Imaginary}$$

$$\Rightarrow \text{Re}(x^2 - y^2 + 3y - 2 + (2xy - 3x)i) = 0$$

$$\Rightarrow x^2 - y^2 + 3y - 2 = 0$$

$$x^2 = y^2 - 3y + 2$$

$$x^2 = (y-1)(y-2) \therefore z = \alpha - \frac{13}{11}i$$

$$\text{Put } x = \alpha, y = -\frac{13}{11}$$

$$\alpha^2 = \left(-\frac{13}{11} - 11 \right) \left(-\frac{13}{11} - 2 \right)$$

$$\alpha^2 = \frac{(24 \times 35)}{121}$$

$$242\alpha^2 = 48 \times 35 = 1680$$

Question 94

Let C be the circle in the complex plane with centre $z_0 = \frac{1}{2}(1 + 3i)$ and radius $r = 1$. Let $z_1 = 1 + i$ and the complex number z_2 be outside the circle C such that $|z_1 - z_0|z_2 - z_0| = 1$. If $z_0 \cdot z_1$ and z_2 are collinear, then the smaller value of $|z_2|^2$ is equal to
[12-Apr-2023 shift 1]

Options:

A. $\frac{7}{2}$

B. $\frac{13}{2}$

C. $\frac{5}{2}$

D. $\frac{3}{2}$

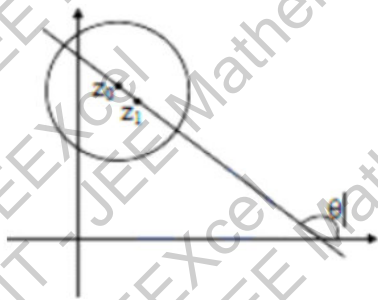
Answer: C

Solution:

$$|z_1 - z_0| = \left| \frac{1-i}{2} \right| = \frac{1}{2}$$

$$\Rightarrow |z_2 - z_0| = \sqrt{2} : \text{centre } \left(\frac{1}{2}, \frac{3}{2} \right)$$

$$z_0 \left(\frac{1}{2}, \frac{3}{2} \right) \text{ and } z_1(1, 1)$$



$$\tan \theta = -1 \Rightarrow \theta = 135^\circ$$

$$z_2 \left(\frac{1}{2} + \sqrt{2} \cos 135^\circ, \frac{3}{2} + \sqrt{2} \sin 135^\circ \right)$$

or

$$\left(\frac{1}{2} - \sqrt{2} \cos 135^\circ, \frac{3}{2} - \sqrt{2} \sin 135^\circ \right)$$

$$\Rightarrow z_2 \left(-\frac{1}{2}, \frac{5}{2} \right) \text{ or } z_2 \left(\frac{3}{2}, \frac{1}{2} \right)$$

$$\Rightarrow |z_3|^2 = \frac{26}{4}, \frac{5}{2}$$

$$\Rightarrow |z_2|_{\min}^2 = \frac{5}{2}$$

Question95

Let $S = \{z \in \mathbb{C} : \bar{z} = i(z^2 + \operatorname{Re}(\bar{z}))\}$. Then $\sum_{z \in S} |z|^2$ is equal to

[13-Apr-2023 shift 2]

Options:

A. 4

B. $\frac{7}{2}$

C. 3

D. $\frac{5}{2}$

Answer: A

Solution:

$$\text{Let } z = x + iy$$

$$\bar{z} = i(z^2 + \operatorname{Re}(\bar{z}))$$

$$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy + x)$$

$$\Rightarrow x - iy = -2xy + i(x^2 - y^2 + x)$$

$$x + 2xy = 0 \text{ and } x^2 - y^2 + x + y = 0$$

$$x(1 + 2y) = 0 \text{ and } x^2 - y^2 + x + y = 0$$

$$\text{If } x = 0 \text{ then } -y^2 + y = 0$$

$$\Rightarrow y = 1, 0$$

$$\text{If } y = \frac{-1}{2} \text{ then } x^2 - \frac{1}{4} + x - \frac{1}{2} = 0$$

$$\Rightarrow x = -\frac{3}{2}, \frac{1}{2}$$

$$= \left\{ 0 + i0, 0 + i, -\frac{3}{2} - \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i \right\}$$

$$\sum_{z \in S} |z|^2 = 0 + 1 + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4$$

Question 96

If the set $\left\{ \operatorname{Re} \left(\frac{z - \bar{z} + z\bar{z}}{2 - 3z + 5z} \right) : z \in \mathbb{C}, \operatorname{Re}(z) = 3 \right\}$ is equal to the interval $(\alpha, \beta]$, then $24(\beta - \alpha)$ is equal to
[15-Apr-2023 shift 1]

Options:

- A. 36
- B. 27
- C. 30
- D. 42

Answer: C

Solution:

$$\text{Let } z_1 = \left(\frac{z - \bar{z} + z\bar{z}}{2 - 3z + 5z} \right)$$

$$\text{Let } z = 3 + iy$$

$$\bar{z} = 3 - iy$$

$$z_1 = \frac{2iy + (9 + y^2)}{2 - 3(3 + iy) + 5(3 - iy)}$$

$$= \frac{9 + y^2 + i(2y)}{8 - 8iy}$$

$$= \frac{(9 + y^2) + i(2y)}{8(1 - iy)}$$

$$\operatorname{Re}(z_1) = \frac{(9 + y^2) - 2y^2}{8(1 + y^2)}$$

$$= \frac{9 - y^2}{8(1 + y^2)}$$

$$= \frac{1}{8} \left[\frac{10 - (1 + y^2)}{(1 + y^2)} \right]$$

$$= \frac{1}{8} \left[\frac{10}{(1 + y^2)} - 1 \right]$$

$$1 + y^2 \in [1, \infty]$$

$$\frac{1}{1 + y^2} \in (0, 1]$$

$$\frac{10}{1 + y^2} \in (0, 10]$$

$$\frac{10}{1 + y^2} - 1 \in (-1, 9]$$

$$\operatorname{Re}(z_1) \in \left(\frac{-1}{8}, \frac{9}{8} \right]$$

$$\alpha = \frac{-1}{8}, \beta = \frac{9}{8}$$

$$24(\beta - \alpha) = 24 \left(\frac{9}{8} - \frac{-1}{8} \right) = 30$$

Question 97

The sum of all the roots of the equation $|x^2 - 8x + 15| - 2x + 7 = 0$ is :
[6-Apr-2023 shift 1]

Options:

- A. $11 - \sqrt{3}$
- B. $9 - \sqrt{3}$
- C. $9 + \sqrt{3}$
- D. $11 + \sqrt{3}$

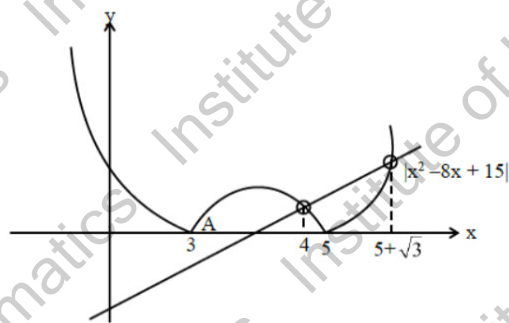
Answer: C

Solution:

$$\begin{aligned} |x^2 - 8x + 15| &= 2x - 7 \\ x^2 - 8x + 15 &= 2x - 7 \quad \& \quad x^2 - 8x + 15 = 7 - 2x \\ x^2 - 10x + 22 &= 0 \quad \& \quad x^2 - 6x + 8 = 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 5 + \sqrt{3} & x_2 &= 5 - \sqrt{3} \text{ (reject)} & x_3 &= 4 & x_4 &= 2 \text{ (reject)} \end{aligned}$$

$$\text{Sum of roots is } = 5 + \sqrt{3} + 4 = 9 + \sqrt{3}$$



Question 98

Let α, β, γ , be the three roots of the equation $x^3 + bx + c = 0$. If $\beta\gamma = 1 = -\alpha$, then $b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$ is equal to
[8-Apr-2023 shift 1]

Options:

- A. $\frac{155}{8}$
- B. 21
- C. 19
- D. $\frac{169}{8}$

Answer: C

Solution:

$$x^3 + bx + c = 0 \begin{cases} \rightarrow \alpha \\ \rightarrow \beta \\ \rightarrow \gamma \end{cases}$$

$$\begin{aligned} \beta\gamma &= 1 \\ \alpha &= -1 \\ \text{Put } \alpha &= -1 \\ -1 - b + c &= 0 \\ c - b &= 1 \\ \text{also} \\ \alpha \cdot \beta \cdot \gamma &= -c \\ -1 &= -c \Rightarrow c = 1 \\ \therefore b &= 0 \\ x^3 + 1 &= 0 \\ \alpha &= -1, \beta = -\omega, \gamma = -\omega^2 \\ \therefore b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3 \\ 0 + 2 + 3 + 6 + 8 &= 19 \end{aligned}$$

Question 99

Let m and n be the numbers of real roots of the quadratic equations $x^2 - 12x + [x] + 31 = 0$ and $x^2 - 5|x+2| - 4 = 0$ respectively, where $[x]$ denotes the greatest integer less than or equal to x . Then $m^2 + mn + n^2$ is equal to

[8-Apr-2023 shift 2]

Answer: 9

Solution:

$$\begin{aligned} x^2 - 12x + [x] + 31 &= 0 \\ \{x\} &= x^2 - 11x + 31 \\ 0 &\leq x^2 - 11x + 31 < 1 \\ x^2 - 11x + 30 &< 1 \\ x &\in (5, 6) \\ \text{so } [x] &= 5 \\ x^2 - 12x + 5 + 31 &= 0 \\ x^2 - 12x + 36 &= 0 \\ x &= 6 \text{ but } x \in (5, 6) \\ \text{so } x &\in \emptyset \\ m &= 0 \end{aligned}$$

Now

$$x^2 - 5|x+2| - 4 = 0$$

$\begin{aligned} x &\geq -2 \\ x^2 - 5x - 14 &= 0 \\ (x-7)(x+2) &= 0 \\ x &= 7, -2 \end{aligned}$	$\begin{aligned} x &< -2 \\ x^2 + 5x + 6 &= 0 \\ (x+3)(x+2) &= 0 \\ x &= -3, -2 \end{aligned}$
---	--

$$\begin{aligned} x &= \{7, -2, -3\} \\ n &= 3 \\ m^2 + mn + n^2 &= n^2 = 9 \end{aligned}$$

Question 100

If a and b are the roots of the equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$ is equal to

[11-Apr-2023 shift 1]

Answer: 51

Solution:

By newton's theorem

$$S_{n+2} - 7S_{n+1} - S_n = 0$$

$$S_{21} - 7S_{20} - S_{19} = 0$$

$$S_{20} - 7S_{19} - S_{18} = 0$$

$$S_{19} - 7S_{18} - S_{17} = 0$$

$$S_{21} + S_{17} = S_{21} + (S_{19} - 7S_{18})$$

$$= \frac{S_{19}}{50S_{19} + (S_{21} - 7S_{20})} \cdot \frac{S_{19}}{S_{19}}$$

$$= 51 \cdot \frac{S_{19}}{S_{19}} = 51$$

Question101

The number of points where the curve $f(x) = e^{8x} - e^{6x} - 3e^{4x} - e^{2x} + 1$, $x \in \mathbb{R}$ cuts x-axis, is equal to

[11-Apr-2023 shift 2]

Answer: 2

Solution:

$$\text{Let } e^{2x} = t$$

$$\Rightarrow t^4 - t^3 - 3t^2 - t + 1 = 0$$

$$\Rightarrow t_2 + \frac{1}{t_2} - \left(t + \frac{1}{t}\right) - 3 = 0$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right) - 5 = 0$$

$$\Rightarrow t + \frac{1}{t} = \frac{1 + \sqrt{21}}{2}$$

Two real values of t.

Question102

Let α, β be the roots of the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$. Then $\frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}}$ is equal to

[12-Apr-2023 shift 1]

Options:

A. 9

B. 729

C. 72

D. 81

Answer: D

Solution:

$$\alpha, \beta = \frac{-\sqrt{6} \pm \sqrt{6-12}}{2} = \frac{-\sqrt{6} \pm \sqrt{6}i}{2}$$
$$= \sqrt[3]{e^{\pm \frac{3\pi i}{4}}}$$

Required expression

$$\frac{(\sqrt{3})^{23} \left(2 \cos \frac{69\pi}{4} \right) + (\sqrt{3})^{14} \left(2 \cos \frac{42\pi}{4} \right)}{(\sqrt{3})^{15} \left(2 \cos \frac{45\pi}{4} \right) + (\sqrt{3})^{10} \left(2 \cos \frac{30\pi}{4} \right)}$$
$$(\sqrt{3})^8 = 81$$

Question103

Let α, β be the roots of the equation $x^2 - \sqrt{2}x + 2 = 0$, Then $\alpha^{14} + \beta^{14}$ is equal to
[13-Apr-2023 shift 2]

Options:

A. $-128\sqrt{2}$

B. $-64\sqrt{2}$

C. -128

D. -64

Answer: C

Solution:

$$x^2 - \sqrt{2}x + 2 = 0$$
$$x = \frac{\sqrt{2} \pm \sqrt{-6}}{2}$$
$$= \sqrt{2} \left(\frac{1 \pm i\sqrt{3}}{2} \right)$$
$$= -\sqrt{2}\omega, -\sqrt{2}\omega^2$$
$$\Rightarrow \alpha = -\sqrt{2}\omega, \beta = -\sqrt{2}\omega^2$$
$$\alpha^{14} + \beta^{14} = 2^7(\omega^{14} + \omega^{28}) = 2^7(\omega^2 + \omega) = -128$$

Question104

The number of real roots of the equation $x |x| - 5 |x + 2| + 6 = 0$, is
[15-Apr-2023 shift 1]

Options:

A. 5

B. 6

C. 4

D. 3

Answer: D

Solution:

$$\begin{aligned}
 & x|x+5|+x+2|+6=0 \\
 & C-1: -x \in [0, \infty] \\
 & x^2-5x-4=0 \\
 & x = \frac{5 \pm \sqrt{25+16}}{2} = \frac{5 \pm \sqrt{41}}{2} \\
 & x = \frac{5 + \sqrt{41}}{2} \\
 & C-2: -x \in [-2, 0) \\
 & -x^2-5x-4=0 \\
 & x^2+5x+4=0 \\
 & x = -1, -4 \\
 & x = -1 \\
 & C-3: x \in [-\infty, -2) \\
 & -x^2+5x+16=0 \\
 & x^2-5x-16=0 \\
 & x = \frac{5 \pm \sqrt{25+64}}{2} \\
 & x = \frac{5 \pm \sqrt{89}}{2} \\
 & x = \frac{5 - \sqrt{89}}{2}
 \end{aligned}$$

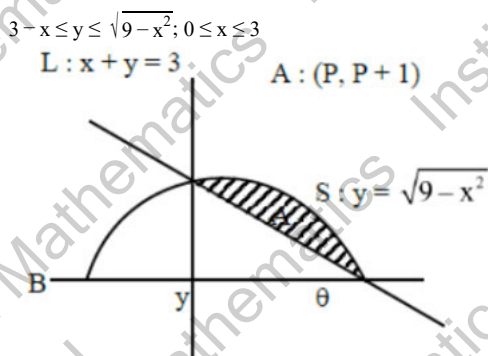
Question 105

Let the point $(p, p+1)$ lie inside the region $E = \{(x, y) : 3-x \leq y \leq \sqrt{9-x^2}, 0 \leq x \leq 3\}$. If the set of all values of p is the interval (a, b) , then $b^2 + b - a^2$ is equal to _____.

[6-Apr-2023 shift 1]

Answer: 3

Solution:



$$L(A) > 0 \Rightarrow P+P+1-3 > 0 \Rightarrow P > 1 \dots (1)$$

$$S(A) < 0 \Rightarrow P+1-\sqrt{9-P^2} < 0$$

$$\Rightarrow P+1 < \sqrt{9-P^2}$$

$$\Rightarrow P+2P+1 < 9-P^2$$

$$\Rightarrow 2P^2+2P-8 < 0$$

$$\Rightarrow P^2+P-4 < 0$$

$$\Rightarrow P \in \left(\frac{-1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2} \right) \dots (2)$$

$$(1) \cap (2) P \in \left(1, \frac{\sqrt{17}-1}{2} \right) \equiv (a, b)$$

$$b^2+b-a^2=4-1=3$$

Question 106

Let a, b, c be three distinct positive real numbers such that $(2a)^{\log_c a} = (bc)^{\log_c b}$ and $b^{\log_c 2} = a^{\log_c c}$. Then

$6a + 5bc$ is equal to _____.

[10-Apr-2023 shift 1]

Answer: 8

Solution:

$$\begin{aligned}(2a)^{\ln a} &= (bc)^{\ln b} \quad 2a > 0, bc > 0 \\ \ln a(\ln 2 + \ln a) &= \ln b(\ln b + \ln c) \\ \ln 2 \cdot \ln a &= \ln b \cdot \ln c \\ \ln 2 = \alpha, \ln a = x, \ln b = y, \ln c = z \\ \alpha y &= xz \\ x(\alpha + x) &= y(y + z) \\ \alpha &= \frac{xz}{y} \\ x\left(\frac{xz}{y} + x\right) &= y(y + z) \\ x^2(z + y) &= y^2(y + z) \\ y + z = 0 \text{ or } x^2 = y^2 &\Rightarrow x = -y \\ bc = 1 \text{ or } ab &= 1 \\ bc = 1 \text{ or } ab &= 1\end{aligned}$$

$$(1) \text{ if } bc = 1 \Rightarrow (2a)^{\ln a} = 1 \begin{cases} a = 1 \\ a = 1/2 \end{cases}$$

$$(a, b, c) = \left(\frac{1}{2}, \lambda, \frac{1}{\lambda}\right), \lambda \neq 1, 2, \frac{1}{2}$$

then

$$6a + 5bc = 3 + 5 = 8$$

$$(II) (a, b, c) = \left(\lambda, \frac{1}{\lambda}, \frac{1}{2}\right), \lambda \neq 1, 2, \frac{1}{2}$$

In this situation infinite answer are possible
So, Bonus.

Question 107

The number of integral solutions x of $\log_{\left(x + \frac{7}{2}\right)} \left(\frac{x-7}{2x-3}\right)^2 \geq 0$ is :

[11-Apr-2023 shift 1]

Options:

- A. 5
- B. 7
- C. 8
- D. 6

Answer: D

Solution:

$$\log_{x+\frac{7}{2}} \left(\frac{x-7}{2x-3} \right)^2 \geq 0$$

$$\text{Feasible region: } x + \frac{7}{2} > 0 \Rightarrow x > -\frac{7}{2}$$

$$\text{And } x + \frac{7}{2} \neq 1 \Rightarrow x \neq \frac{-5}{2}$$

$$\text{Taking intersection: } x \in \left(\frac{-7}{2}, \infty \right) - \left\{ \frac{-5}{2}, \frac{3}{2}, 7 \right\}$$

Now $\log_a b \geq 0$ if $a > 1$ and $b \geq 1$

$$a \in (0, 1) \text{ and } b \in (0, 1)$$

$$C-I: x + \frac{7}{2} > 1 \text{ and } \left(\frac{x-7}{2x-3} \right)^2 \geq 1$$

$$x > -\frac{5}{2}; (2x-3)^2 - (x-7)^2 \leq 0$$

$$(2x-3+x-7)(2x-3-x+7) \leq 0$$

$$(3x-10)(x+4) \leq 0$$

$$x \in \left[-4, \frac{10}{3} \right]$$

$$\text{Intersection: } x \in \left(\frac{-5}{2}, \frac{10}{3} \right]$$

$$C-II: x + \frac{7}{2} \in (0, 1) \text{ and } \left(\frac{x-7}{2x-3} \right)^2 \in (0, 1)$$

$$0 < x + \frac{7}{2} < 1; \left(\frac{x-7}{2x-3} \right)^2 < 1$$

$$-\frac{7}{2} < x < \frac{-5}{2}; (x-7)^2 < (2x-3)^2$$

$$x \in (-\infty, -4) \cup \left(\frac{10}{3}, \infty \right)$$

No common values of x .

Hence intersection with feasible region

$$\text{We get } x \in \left(\frac{-5}{2}, \frac{10}{3} \right] - \left\{ \frac{3}{2} \right\}$$

Integral value of x are $\{-2, -1, 0, 1, 2, 3\}$

No. of integral values = 6

Question 108

If the sum of the squares of the reciprocals of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)^2$ is equal to:
[24-Jun-2022-Shift-1]

Options:

A. 18

B. 24

C. 36

D. 96

Answer: B

Solution:

$$3x^2 + \lambda x - 1 = 0$$

Given, two roots are α and β .

$$\therefore \text{Sum of roots} = \alpha + \beta = \frac{-\lambda}{3}$$

$$\text{And product of roots} = \alpha\beta = \frac{-1}{3}$$

Given that,

Sum of square of reciprocal of roots α and β is 15 .

$$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} = 15$$

$$\Rightarrow \frac{\alpha^2 + \beta^2}{\alpha^2 \beta^2} = 15$$

$$\Rightarrow \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = 15$$

$$\Rightarrow \frac{\frac{\lambda^2}{9} + 2 \times \frac{1}{3}}{\frac{1}{9}} = 15$$

$$\Rightarrow \frac{\lambda^2 + 6}{\frac{1}{9}} = 15$$

$$\Rightarrow \lambda^2 + 6 = 15$$

$$\Rightarrow \lambda^2 = 9$$

$$\text{Now, } 6(\alpha^2 + \beta^2)^2$$

$$= 6\{(\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)\}^2$$

$$= 6(\alpha + \beta)^2[(\alpha + \beta)^2 - 2\alpha\beta - \alpha\beta]^2$$

$$= 6\left(\frac{-\lambda}{3}\right)^2 \left[\left(\frac{-\lambda}{3}\right)^2 - 3 \cdot \frac{-1}{3}\right]^2$$

$$= 6 \times \frac{\lambda^2}{9} \times \left[\frac{\lambda^2}{9} + 1\right]$$

$$= 6 \times \frac{9}{9} \times \left[\frac{9}{9} + 1\right]^2$$

$$= 6 \times (2)^2$$

$$= 6 \times 4 = 24$$

Question 109

Let $S = \{z \in \mathbb{C} : |z - 3| \leq 1 \text{ and } z(4 + 3i) + z(4 - 3i) \leq 24\}$. If $\alpha + i\beta$ is the point in S which is closest to $4i$, then $25(\alpha + \beta)$ is equal to _____
[24-Jun-2022-Shift-2]

Answer: 80

Solution:

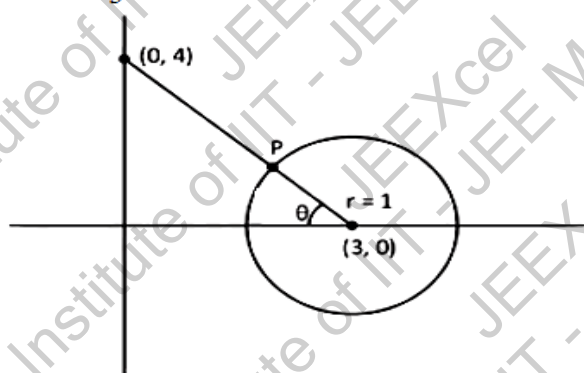
Here $|z - 3| < 1$

$$\Rightarrow (x - 3)^2 + y^2 < 1$$

$$\text{and } z = (4 + 3i) + \bar{z}(4 - 3i) \leq 24$$

$$\Rightarrow 4x - 3y \leq 12$$

$$\tan \theta = \frac{4}{3}$$



$$\therefore \text{Coordinate of } P = (3 - \cos \theta, \sin \theta)$$

$$= \left(3 - \frac{3}{5}, \frac{4}{5} \right)$$

$$\therefore \alpha + i\beta = \frac{12}{5} + \frac{4}{5}i$$

$$\therefore 25(\alpha + \beta) = 80$$

Question 110

Let a circle C in complex plane pass through the points $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z (\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3 , then $\arg(z)$ is equal to:

[25-Jun-2022-Shift-1]

Options:

A. $\tan^{-1} \left(\frac{2}{\sqrt{5}} \right) - \pi$

B. $\tan^{-1} \left(\frac{24}{7} \right) - \pi$

C. $\tan^{-1}(3) - \pi$

D. $\tan^{-1}\left(\frac{3}{4}\right) - \pi$

Answer: B

Solution:

$$z_1 = 3 + 4i, z_2 = 4 + 3i \text{ and } z_3 = 5i$$

$$\text{Clearly, } C \equiv x^2 + y^2 = 25$$

Let $z(x, y)$

$$\Rightarrow \left(\frac{y-4}{x-3}\right) \left(\frac{2}{-4}\right) = -1$$

$$\Rightarrow y = 2x - 2 \equiv L$$

$\therefore z$ is intersection of C & L

$$\Rightarrow z \equiv \left(\frac{-7}{5}, \frac{-24}{5}\right)$$

$$\therefore \text{Arg}(z) = -\pi + \tan^{-1}\left(\frac{24}{7}\right)$$

Question 111

Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 = iz_2$ and $\arg\left(\frac{z_1}{z_2}\right) = \pi$. Then

[25-Jun-2022-Shift-2]

Options:

A. $\arg z_2 = \frac{\pi}{4}$

B. $\arg z_2 = -\frac{3\pi}{4}$

C. $\arg z_1 = \frac{\pi}{4}$

D. $\arg z_1 = -\frac{3\pi}{4}$

Answer: C

Solution:

$$\because \frac{z_1}{z_2} = -i \Rightarrow z_1 = -iz_2$$

$$\Rightarrow \arg(z_1) = -\frac{\pi}{2} + \arg(z_2) \dots \dots (i)$$

$$\text{Also } \arg(z_1) - \arg(z_2) = \pi$$

$$\Rightarrow \arg(z_1) + \arg(z_2) = \pi \dots \dots (ii)$$

$$\text{From (i) and (ii), we get } \arg(z_1) = \frac{\pi}{4} \text{ and } \arg(z_2) = \frac{3\pi}{4}$$

Question 112

Let $A = \left\{ z \in \mathbb{C} : \left| \frac{z+1}{z-1} \right| < 1 \right\}$ and $B = \left\{ z \in \mathbb{C} : \arg\left(\frac{z-1}{z+1}\right) = \frac{2\pi}{3} \right\}$. Then $A \cap B$ is :

[26-Jun-2022-Shift-1]

Options:

A. a portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second and third quadrants only

B. a portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second quadrant only

C. an empty

D. a portion of a circle of radius $\frac{2}{\sqrt{3}}$ that lies in the third quadrant only

Answer: B

Solution:

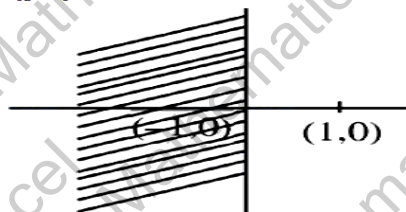
Set A

$$\Rightarrow \left| \frac{z+1}{z-1} \right| < 1$$

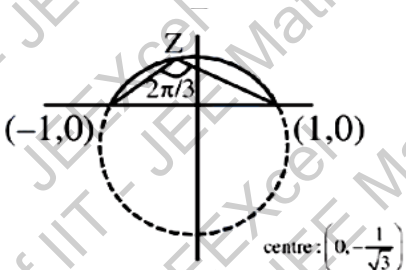
$$\Rightarrow |z+1| < |z-1|$$

$$\Rightarrow (x+1)^2 + y^2 < (x-1)^2 + y^2$$

$$\Rightarrow x < 0$$



Set B



$$\Rightarrow \arg\left(\frac{z-1}{z+1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow x^2 + y^2 + \frac{2y}{\sqrt{3}} - 1 = 0$$

$$A \cap B$$

$$\Rightarrow \text{Centre} \left(0, -\frac{1}{\sqrt{3}}\right)$$

Question 113

If $z^2 + z + 1 = 0$, $z \in \mathbb{C}$, then

$$\left| \sum_{n=1}^{15} \left(z^n + (-1)^n \frac{1}{z^n} \right)^2 \right| \text{ is equal to } \underline{\hspace{2cm}}$$

[26-Jun-2022-Shift-2]

Answer: 2

Solution:

$$\because z^2 + z + 1 = 0$$

$$\Rightarrow \omega \text{ or } \omega^2$$

$$\therefore \left| \sum_{n=1}^{15} \left(z^n + (-1)^n \frac{1}{z^n} \right)^2 \right|$$

$$= \left| \sum_{n=1}^{15} z^{2n} + \sum_{n=1}^{15} z^{-2n} + 2 \cdot \sum_{n=1}^{15} (-1)^n \right|$$

$$= |0 + 0 - 2|$$

$$= 2$$

Question 114

The area of the polygon, whose vertices are the non-real roots of the equation $\bar{z} = iz^2$ is :

[27-Jun-2022-Shift-1]

Options:

A. $\frac{3\sqrt{3}}{4}$

B. $\frac{3\sqrt{3}}{2}$

C. $\frac{3}{2}$

D. $\frac{3}{4}$

Answer: A

Solution:

$$\bar{z} = iz^2$$

$$\text{Let } z = x + iy$$

$$x - iy = i(x^2 - y^2 + 2ixy)$$

$$x - iy = i(x^2 - y^2) - 2xy$$

$$\therefore x = -2yx \text{ or } x^2 - y^2 = -y$$

$$x = 0 \text{ or } y = -\frac{1}{2}$$

Case - I

$$x = 0$$

$$-y^2 = -y$$

$$y = 0, 1$$

Case - II

$$y = -\frac{1}{2}$$

$$\Rightarrow x^2 - \frac{1}{4} = \frac{1}{2} \Rightarrow x = \pm \frac{\sqrt{3}}{2}$$

$$x = \left\{ 0, i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2} \right\}$$

$$\text{Area of polygon} = \frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left| -\sqrt{3} - \frac{\sqrt{3}}{2} \right| = \frac{3\sqrt{3}}{4}$$

Question 115

The number of points of intersection of $|z - (4 + 3i)| = 2$ and $|z| + |z - 4| = 6$, $z \in \mathbb{C}$, is
[27-Jun-2022-Shift-2]

Options:

A. 0

B. 1

C. 2

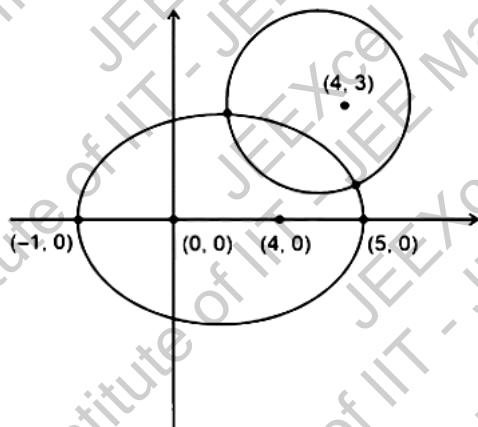
D. 3

Answer: C

Solution:

$$C_1: |z - (4 + 3i)| = 2 \text{ and } C_2: |z| + |z - 4| = 6, z \in \mathbb{C}$$

C_1 represents a circle with centre $(4, 3)$ and radius 2 and C_2 represents an ellipse with foci at $(0, 0)$ and $(4, 0)$ and length of major axis = 6, and length of semi-major axis $2\sqrt{5}$ and $(4, 2)$ lies inside the both C_1 and C_2 and $(4, 3)$ lies outside the C_2



$\therefore$ number of intersection points = 2

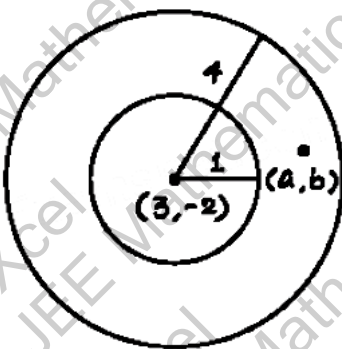
Question 116

The number of elements in the set $\{z = a + ib \in \mathbb{C} : a, b \in \mathbb{Z} \text{ and } 1 < |z - 3 + 2i| < 4\}$ is ____
[28-Jun-2022-Shift-1]

Answer: 40

Solution:

$$1 < |z - 3 + 2i| < 4$$



$$1 < (a - 3)^2 + (b + 2)^2 < 16$$

$$(0, \pm 2), (\pm 2, 0), (\pm 1, \pm 2), (\pm 2, \pm 1)$$

$$(\pm 2, \pm 3), (3 \pm, \pm 2), (\pm 1, \pm 1), (2 \pm, \pm 2)$$

$$(\pm 3, 0), (0, \pm 3), (\pm 3 \pm 1), (\pm 1, \pm 3)$$

Total 40 points

Question 117

Sum of squares of modulus of all the complex numbers z satisfying $\bar{z} = iz^2 + z^2 - z$ is equal to _____
[28-Jun-2022-Shift-2]

Answer: 2

Solution:

Let $z = x + iy$

$$\text{So } 2x = (1+i)(x^2 - y^2 + 2xyi)$$

$$\Rightarrow 2x = x^2 - y^2 - 2xy$$

(i) and

$$x^2 - y^2 + 2xy = 0$$

From (i) and (ii) we get

$$x = 0 \text{ or } y = -\frac{1}{2}$$

When $x = 0$ we get $y = 0$

$$\text{When } y = -\frac{1}{2} \text{ we get } x^2 - x - \frac{1}{4} = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{2}}{2}$$

So there will be total 3 possible values of z , which are 0 , $\left(\frac{-1+\sqrt{2}}{2}\right) - \frac{1}{2}i$ and $\left(\frac{-1-\sqrt{2}}{2}\right) - \frac{1}{2}i$

Sum of squares of modulus

$$= 0 + \left(\frac{\sqrt{2}-1}{2}\right)^2 + \frac{1}{4} + \left(\frac{\sqrt{2}+1}{2}\right)^2 = +\frac{1}{4}$$

$$= 2$$

Question 118

Let α and β be the roots of the equation $x^2 + (2i - 1) = 0$. Then, the value of $|\alpha^8 + \beta^8|$ is equal to:
[29-Jun-2022-Shift-1]

Options:

A. 50

B. 250

C. 1250

D. 1500

Answer: A

Solution:

Given equation,

$$x^2 + (2i - 1) = 0$$

$$\Rightarrow x^2 = 1 - 2i$$

Let α and β are the two roots of the equation.

As, we know roots of a equation satisfy the equation so

$$\alpha^2 = 1 - 2i$$

$$\text{and } \beta^2 = 1 - 2i$$

$$\therefore \alpha^2 = \beta^2 = 1 - 2i$$

$$\therefore |\alpha^2| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Now, $|\alpha^8 + \beta^8|$

$$|\alpha^8 + \alpha^8|$$

$$= 2|\alpha^8|$$

$$= 2|\alpha^2|^4$$

$$= 2(\sqrt{5})^4$$

$$= 2 \times 25$$

$$= 50$$

Question 119

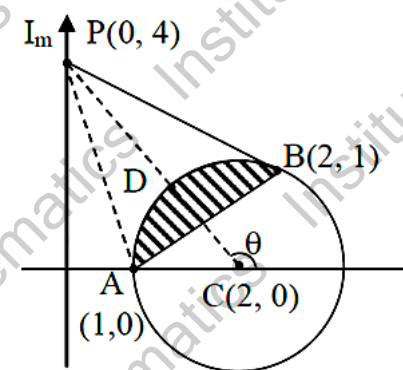
Let $S = \{z \in \mathbb{C} : |z - 2| \leq 1, z(1+i) + \bar{z}(1-i) \leq 2\}$. Let $|z - 4i|$ attains minimum and maximum values, respectively, at $z_1 \in S$ and $z_2 \in S$. If $5(|z_1|^2 + |z_2|^2) = \alpha + \beta\sqrt{5}$, where α and β are integers, then the value of $\alpha + \beta$ is equal to

[29-Jun-2022-Shift-1]

Answer: 26

Solution:

$$|z - 2| \leq 1$$



$$(x - 2)^2 + y^2 \leq 1$$

&

$$z(1+i) + \bar{z}(1-i) \leq 2$$

Put $z = x + iy$

$$\therefore x - y \leq 1 \dots (2)$$

$$PA = \sqrt{17}, PB = \sqrt{13}$$

Maximum is PA & Minimum is PD

Let $D(2 + \cos \theta, 0 + \sin \theta)$

$$\therefore m_{cp} = \tan \theta = -2$$

$$\cos \theta = -\frac{1}{\sqrt{5}}, \sin \theta = \frac{2}{\sqrt{5}}$$

$$\therefore D\left(2 - \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow z_1 = \left(2 - \frac{1}{\sqrt{5}}\right) + \frac{2i}{\sqrt{5}}$$

$$|z_1| = \frac{25 - 4\sqrt{5}}{5} \text{ \& } z_2 = 1$$

$$\therefore |z_2|^2 = 1$$

$$\therefore 5(|z_1|^2 + |z_2|^2) = 30 - 4\sqrt{5}$$

$$\therefore \alpha = 30$$

$$\beta = -4$$

$$\therefore \alpha + \beta = 26$$

Question 120

Let $\arg(z)$ represent the principal argument of the complex number z . Then, $|z| = 3$ and $\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$ intersect [29-Jun-2022-Shift-2]

Options:

- A. exactly at one point.
- B. exactly at two points.
- C. nowhere.
- D. at infinitely many points.

Answer: C

Solution:

Let $z = x + iy$

$$\therefore |z| = \sqrt{x^2 + y^2}$$

Given, $|z| = 3$

$$\therefore \sqrt{x^2 + y^2} = 3$$

$$\Rightarrow x^2 + y^2 = 9 = 3^2$$

This represents a circle with center at $(0, 0)$ and radius $= 3$

Now, given

$$\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x + iy - 1) - \arg(x + iy + 1) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x - 1 + iy) - \arg(x + 1 + iy) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1} \times \frac{y}{x+1}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{xy+y-xy+y}{x^2-1}}{\frac{x^2-1+y^2}{x^2-1}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{xy+y-xy+y}{x^2-1+y^2}\right) = \frac{\pi}{4}$$

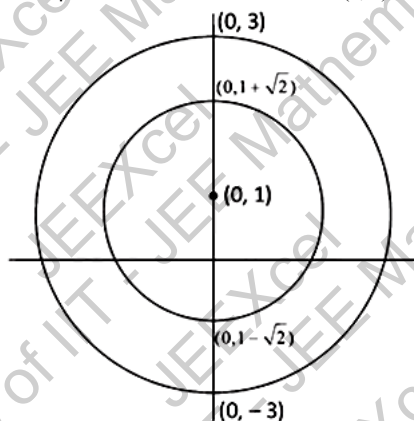
$$\Rightarrow \frac{2y}{x^2-1+y^2} = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow 2y = x^2 + y^2 - 1$$

$$\Rightarrow x^2 + y^2 - 2y - 1 = 0$$

$$\Rightarrow x^2 + (y-1)^2 = (\sqrt{2})^2$$

This represents a circle with center at $(0, 1)$ and radius $\sqrt{2}$.



From diagram you can see both the circles do not cut anywhere.

Question 121

The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is
[24-Jun-2022-Shift-2]

Options:

- A. $\log_e 3$
- B. $-\log_e 3$
- C. $\log_e 6$
- D. $-\log_e 6$

Answer: B

Solution:

$$(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$$

$$\text{Let } e^x = t$$

$$\therefore (t^2 - 4)(6t^2 - 5t + 1) = 0$$

$$\Rightarrow (t^2 - 4)(2t - 1)(3t - 1) = 0$$

$$\therefore t = 2, -2, \frac{1}{2}, \frac{1}{3}$$

$$\therefore e^x = 2 \Rightarrow x = \ln 2$$

$$e^x = -2 \text{ (not possible)}$$

$$e^x = \frac{1}{2} \Rightarrow x = -\ln 2$$

$$e^x = \frac{1}{3} \Rightarrow x = -\ln 3$$

$$\therefore \text{Sum of all real roots}$$

$$= \ln 2 - \ln 2 - \ln 3$$

$$= -\ln 3$$

Question 122

For a natural number n , let $\alpha_n = 19^n - 12^n$. Then, the value of $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$ is
[25-Jun-2022-Shift-1]

Answer: 4

Solution:

$$\alpha_n = 19^n - 12^n$$

Let equation of roots 12&19 i.e.

$$x^2 - 31x + 228 = 0$$

$$\Rightarrow (31 - x) = \frac{228}{x} \text{ (where x can be 19 or 12)}$$

$$\begin{aligned} \therefore \frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} &= \frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57(19^8 - 12^8)} \\ &= \frac{19^9(31 - 19) - 12^9(31 - 12)}{57(19^8 - 12^8)} \\ &= \frac{228(19^8 - 12^8)}{57(19^8 - 12^8)} = 4 \end{aligned}$$

Question123

Let $a, b \in \mathbb{R}$ be such that the equation $ax^2 - 2bx + 15 = 0$ has a repeated root α . If α and β are the roots of the equation $x^2 - 2bx + 21 = 0$, then $\alpha^2 + \beta^2$ is equal to :
[25-Jun-2022-Shift-2]

Options:

- A. 37
- B. 58
- C. 68
- D. 92

Answer: B

Solution:

$ax^2 - 2bx + 15 = 0$ has repeated root so $b^2 = 15a$ and $\alpha = \frac{15}{b}$

$\therefore \alpha$ is a root of $x^2 - 2bx + 21 = 0$

So $\frac{225}{b^2} = 9 \Rightarrow b^2 = 25$

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4b^2 - 42 = 100 - 42 = 58$

Question124

The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 0$ is _____
[26-Jun-2022-Shift-1]

Answer: 36

Solution:

$$x^4 - 3x^3 - x^2 - x^2 + 3x + 1 = 0$$

$$(x^2 - 1)(x^2 - 3x - 1) = 0$$

Let the root of $x^2 - 3x - 1 = 0$ be α and β and other two roots of given equation are 1 and -1

So sum of cubes of roots

$$= 1^3 + (-1)^3 + \alpha^3 + \beta^3$$

$$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= (3)^3 - 3(-1)(3)$$

$$= 36$$

Question 125

If the sum of all the roots of the equation

$$e^{2x} - 11e^x - 45e^{-x} + \frac{81}{2} = 0 \text{ is } \log_e p, \text{ then } p \text{ is equal to } \underline{\hspace{2cm}}$$

[27-Jun-2022-Shift-1]

Answer: 45

Solution:

Let $e^x = t$ then equation reduces to

$$t^2 - 11t - \frac{45}{t} + \frac{81}{2} = 0$$

$$\Rightarrow 2t^3 - 22t^2 + 81t - 45 = 0 \dots (i)$$

If roots of $e^{2x} - 11e^x - 45e^{-x} + \frac{81}{2} = 0$ are α, β, γ then roots of (i) will be $e^{\alpha}, e^{\beta}, e^{\gamma}$ using product of roots

$$e^{\alpha_1 + \alpha_2 + \alpha_3} = 45$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = \ln 45 \Rightarrow p = 45$$

Question 126

Let α, β be the roots of the equation $x^2 - 4\lambda x + 5 = 0$ and α, γ be the roots of the equation

$$x^2 - (3\sqrt{2} + 2\sqrt{3})x + 7 + 3\lambda\sqrt{3} = 0, \lambda > 0. \text{ If } \beta + \gamma = 3\sqrt{2}, \text{ then } (\alpha + 2\beta + \gamma)^2 \text{ is equal to } \underline{\hspace{2cm}}$$

[27-Jun-2022-Shift-2]

Answer: 98

Solution:

$$\because \alpha, \beta \text{ are roots of } x^2 - 4\lambda x + 5 = 0$$

$$\therefore \alpha + \beta = 4\lambda \text{ and } \alpha\beta = 5$$

Also, α, γ are roots of

$$x^2 - (3\sqrt{2} + 2\sqrt{3})x + 7 + 3\lambda\sqrt{3} = 0, \lambda > 0$$

$$\therefore \alpha + \gamma = 3\sqrt{2} + 2\sqrt{3}, \alpha\gamma = 7 + 3\sqrt{3}\lambda$$

$\therefore \alpha$ is common root

$$\therefore \alpha^2 - 4\lambda\alpha + 5 = 0$$

$$\text{and } \alpha^2 - (3\sqrt{2} + 2\sqrt{3})\alpha + 7 + 3\sqrt{3}\lambda = 0$$

$$\text{From (i) - (ii) : we get } \alpha = \frac{2 + 3\sqrt{3}\lambda}{3\sqrt{2} + 2\sqrt{3} - 4\lambda}$$

$$\therefore \beta + \gamma = 3\sqrt{2}$$

$$\therefore 4\lambda + 3\sqrt{2} + 2\sqrt{3} - 2\alpha = 3\sqrt{2}$$

$$\Rightarrow 3\sqrt{2} = 4\lambda + 3\sqrt{2} + 2\sqrt{3} - \frac{4 + 6\sqrt{3}\lambda}{3\sqrt{2} + 2\sqrt{3} - 4\lambda}$$

$$\Rightarrow 8\lambda^2 + 3(\sqrt{3} + 2\sqrt{2})\lambda - 4 - 3\sqrt{6} = 0$$

$$\therefore \lambda = \frac{6\sqrt{2} - 3\sqrt{2} \pm \sqrt{9(11 - 4\sqrt{6}) + 32(4 + 3\sqrt{6})}}{16}$$

$$\therefore \lambda = \sqrt{2}$$

$$\therefore (\alpha + 2\beta + \gamma)^2 = (\alpha + \beta + \beta + \gamma)^2$$

$$= (4\sqrt{2} + 3\sqrt{2})^2$$

$$= (7\sqrt{2})^2 = 98$$

Question 127

The number of real solutions of the equation $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$ is _____
[28-Jun-2022-Shift-1]

Answer: 2

Solution:

Dividing by e^{2x}

$$e^{2x} + 4e^x - 58 + 4e^{-x} + e^{-2x} = 0$$

$$\Rightarrow (e^x + e^{-x})^2 + 4(e^x + e^{-x}) - 60 = 0$$

$$\text{Let } e^x + e^{-x} = t \in [2, \infty)$$

$$\Rightarrow t^2 + 4t - 60 = 0$$

$$\Rightarrow t = 6 \text{ is only possible solution}$$

$$e^x + e^{-x} = 6 \Rightarrow e^{2x} - 6e^x + 1 = 0$$

$$\text{Let } e^x = p$$

$$p^2 - 6p + 1 = 0$$

$$\Rightarrow p = \frac{3 + \sqrt{5}}{2} \text{ or } \frac{3 - \sqrt{5}}{2}$$

$$\text{So } x = \ln\left(\frac{3 + \sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3 - \sqrt{5}}{2}\right)$$

Question 128

Let $f(x)$ be a quadratic polynomial such that $f(-2) + f(3) = 0$. If one of the roots of $f(x) = 0$ is -1 , then the sum of the roots of $f(x) = 0$ is equal to :
[28-Jun-2022-Shift-2]

Options:

A. $\frac{11}{3}$

B. $\frac{7}{3}$

C. $\frac{13}{3}$

D. $\frac{14}{3}$

Answer: A

Solution:

$\therefore x = -1$ be the roots of $f(x) = 0$
 $\therefore$ Let $f(x) = A(x+1)(x-1)\dots\dots (i)$
Now, $f(-2) + f(3) = 0$
 $\Rightarrow A[-1(-2-b) + 4(3-b)] = 0$
 $b = \frac{14}{3}$

$\therefore$ Second root of $f(x) = 0$ will be $\frac{14}{3}$

$\therefore$ Sum of roots $= \frac{14}{3} - 1 = \frac{11}{3}$

Question 129

Let α be a root of the equation $1 + x^2 + x^4 = 0$. Then, the value of $\alpha^{1011} + \alpha^{2022} - \alpha^{3033}$ is equal to :
[29-Jun-2022-Shift-2]

Options:

- A. 1
- B. α
- C. $1 + \alpha$
- D. $1 + 2\alpha$

Answer: A

Solution:

Given, α is a root of the equation $1 + x^2 + x^4 = 0$

$\therefore \alpha$ will satisfy the equation.

$$\therefore 1 + \alpha^2 + \alpha^4 = 0$$

$$\alpha^2 = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore \alpha^2 = \omega \text{ or } \omega^2$$

Now,

$$\alpha^{1011} + \alpha^{2022} - \alpha^{3033}$$

$$= \alpha \cdot (\alpha^2)^{505} + (\alpha^2)^{1011} - \alpha \cdot (\alpha^2)^{1516}$$

$$= \alpha(\omega)^{505} + (\omega)^{1011} - \alpha \cdot (\omega)^{1516}$$

$$= \alpha \cdot (\omega^3)^{168} \cdot \omega + (\omega^3)^{337} - \alpha \cdot (\omega^3)^{505} \cdot \omega$$

$$= \alpha\omega + 1 - \alpha\omega$$

$$= 1$$

Question 130

Let $x, y > 0$. If $x^3 y^2 = 2^{15}$, then the least value of $3x + 2y$ is
[24-Jun-2022-Shift-2]

Options:

- A. 30
- B. 32
- C. 36
- D. 40

Answer: D

Solution:

$$x, y > 0 \text{ and } x^3 y^2 = 2^{15}$$

$$\text{Now, } 3x + 2y = (x + x + x) + (y + y)$$

So, by $A \cdot M \geq G.M$ inequality

$$\frac{3x + 2y}{5} \geq \sqrt[5]{x^3 \cdot y^2}$$

$$\therefore 3x + 2y \geq 5 \sqrt[5]{2^{15}} \geq 40$$

$$\therefore \text{Least value of } 3x + 4y = 40$$

Question 131

Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\frac{1}{p} + \frac{1}{q}\right)^{-2}$ is equal to _____
[26-Jun-2022-Shift-2]

Answer: 4

Solution:

$$\therefore p + q = 3 \dots\dots (i)$$

$$\text{and } p^4 + q^4 = 369 \dots\dots (ii)$$

$$\{(p + q)^2 - 2pq\}^2 - 2p^2q^2 = 369$$

$$\text{or } (9 - 2pq)^2 - 2(pq)^2 = 369$$

$$\text{or } (pq)^2 - 18pq - 144 = 0$$

$$\therefore pq = -6 \text{ or } 24$$

But $pq = 24$ is not possible

$$\therefore pq = -6$$

$$\text{Hence, } \left(\frac{1}{p} + \frac{1}{q}\right)^{-2} = \left(\frac{pq}{p+q}\right)^2 = (-2)^2 = 4$$

Question 132

If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, then $\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$ is equal to _____
[25-Jul-2022-Shift-1]

Options:

A. -4

B. -1

C. 1

D. 4

Answer: B

Solution:

When, $x^5 = 1$

then $x^5 - 1 = 0$

$$\Rightarrow (x-1)(x^4 + x^3 + x^2 + x + 1) = 0$$

Given, $x^4 + x^3 + x^2 + x + 1 = 0$ has roots α, β, γ and 8.

$\therefore$ Roots of $x^5 - 1 = 0$ are 1, α, β, γ and 8.

We know, Sum of p^{th} power of n^{th} roots of unity = 0. (If p is not multiple of n) or n (If p is multiple of n)

$\therefore$ Here, Sum of p^{th} power of n^{th} roots of unity

Here, $p = 2021$, which is not multiple of 5.

$$\therefore 1^{2021} + \alpha^{2021} + \beta^{2021} + \gamma^{2021} + 8^{2021} = 0$$

$$\Rightarrow \alpha^{2021} + \beta^{2021} + \gamma^{2021} + 8^{2021} = -1$$

Question 133

For $n \in \mathbb{N}$, let $S_n = \left\{ z \in \mathbb{C} : |z - 3 + 2i| = \frac{n}{4} \right\}$ and $T_n = \left\{ z \in \mathbb{C} : |z - 2 + 3i| = \frac{1}{n} \right\}$.

Then the number of elements in the set $\{n \in \mathbb{N} : S_n \cap T_n = \emptyset\}$ is :

[25-Jul-2022-Shift-1]

Options:

A. 0

B. 2

C. 3

D. 4

Answer: D

Solution:

$S_n = \left\{ z \in \mathbb{C} : |z - 3 + 2i| = \frac{n}{4} \right\}$ represents a circle with centre $C_1(3, -2)$ and radius $r_1 = \frac{n}{4}$

Similarly T_n represents circle with centre $C_2(2, -3)$ and radius $r_2 = \frac{1}{n}$

As $S_n \cap T_n = \emptyset$

$$C_1 C_2 > r_1 + r_2 \quad \text{OR} \quad C_1 C_2 < |r_1 - r_2|$$

$$\sqrt{2} > \frac{n}{4} + \frac{1}{n} \quad \text{OR} \quad \sqrt{2} < \left| \frac{n}{4} - \frac{1}{n} \right|$$

$n = 1, 2, 3, 4$ n may take infinite values

Question 134

For $z \in \mathbb{C}$ if the minimum value of $(|z - 3\sqrt{2}| + |z - p\sqrt{2}i|)$ is $5\sqrt{2}$, then a value Question: of p is

[25-Jul-2022-Shift-2]

Options:

A. 3

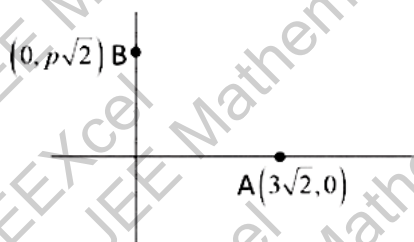
B. $\frac{7}{2}$

C. 4

D. $\frac{9}{2}$

Answer: C

Solution:



It is sum of distance of z from $(3\sqrt{2}, 0)$ and $(0, p\sqrt{2})$. For minimising, z should lie on AB and $AB = 5\sqrt{2}$

$$(AB)^2 = 18 + 2p^2$$

$$p = \pm 4$$

Question 135

Let O be the origin and A be the point $z_1 = 1 + 2i$. If B is the point z_2 , $\text{Re}(z_2) < 0$, such that OAB is a right angled isosceles triangle with OB as hypotenuse, then which of the following is NOT true?

[26-Jul-2022-Shift-1]

Options:

A. $\arg z_2 = \pi - \tan^{-1} 3$

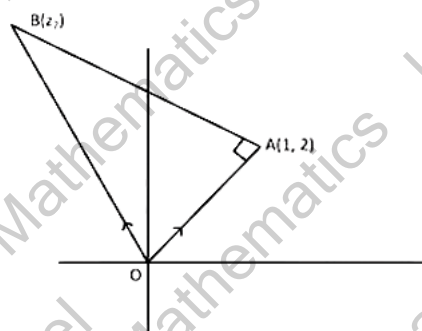
B. $\arg(z_1 - 2z_2) = -\tan^{-1} \frac{4}{3}$

C. $|z_2| = \sqrt{10}$

D. $|2z_1 - z_2| = 5$

Answer: D

Solution:



$$\frac{z_2 - 0}{(1 + 2i) - 0} = \frac{|OB|}{|OA|} e^{i\frac{\pi}{4}}$$

$$\Rightarrow \frac{z_2}{1 + 2i} = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$\text{OR } z_2 = (1 + 2i)(1 + i)$$

$$= -1 + 3i$$

$$\arg z_2 = \pi - \tan^{-1} 3$$

$$|z_2| = \sqrt{10}$$

$$z_1 - 2z_2 = (1 + 2i) + 2 - 6i = 3 - 4i$$

$$\arg(z_1 - 2z_2) = -\tan^{-1} \frac{4}{3}$$

$$|2z_1 - z_2| = |2 + 4i + 1 - 3i| = |3 + i| = \sqrt{10}$$

Question 136

If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then
[26-Jul-2022-Shift-2]

Options:

A. $x + 2y - 4 = 0$

B. $x^2 + y - 4 = 0$

C. $x + 2y + 4 = 0$

D. $x^2 - y + 3 = 0$

Answer: C

Solution:

$$|z - i| = |z + 5i|$$

So, z lies on $\perp^r$ bisector of $(0, 1)$ and $(0, -5)$

i.e., line $y = -2$

as $|z| = 2$

$$\Rightarrow z = -2i$$

$$x = 0 \text{ and } y = -2$$

$$\text{so, } x + 2y + 4 = 0$$

Question 137

Let the minimum value v_0 of $v = |z|^2 + |z - 3|^2 + |z - 6i|^2$, $z \in \mathbb{C}$ is attained at $z = z_0$. Then

$\left| 2z_0^2 - \bar{z}_0^3 + 3 \right|^2 + v_0^2$ is equal to

[27-Jul-2022-Shift-1]

Options:

A. 1000

B. 1024

C. 1105

D. 1196

Answer: A

Solution:

$$\text{Let } z = x + iy$$

$$v = x^2 + y^2 + (x - 3)^2 + y^2 + x^2 + (y - 6)^2$$

$$= (3x^2 - 6x + 9) + (3y^2 - 12y + 36)$$

$$= 3(x^2 + y^2 - 2x - 4y + 15)$$

$$= 3[(x - 1)^2 + (y - 2)^2 + 10]$$

$$v_{\min} \text{ at } z = 1 + 2i = z_0 \text{ and } v_0 = 30$$

$$\text{so } |2(1 + 2i)^2 - (1 - 2i)^3 + 3|^2 + 900$$

$$= |2(-3 + 4i) - (1 - 8i) - 6i(1 - 2i) + 3|^2 + 900.$$

$$= |-6 + 8i - (1 + 8i - 6i - 12) + 3|^2 + 900$$

$$= |8 + 6i|^2 + 900$$

$$= |8 + 6i|^2 + 900$$

$$= 1000$$

Question 138

Let $S = \{z \in \mathbb{C} : z^2 + \bar{z} = 0\}$. Then $\sum_{z \in S} (\operatorname{Re}(z) + \operatorname{Im}(z))$ is equal to _____.

[27-Jul-2022-Shift-1]

Answer: 0

Solution:

$$\because z^2 + \bar{z} = 0$$

$$\text{Let } z = x + iy$$

$$\because x^2 - y^2 + 2ixy + x - iy = 0$$

$$(x^2 - y^2 + x) + i(2xy - y) = 0$$

$$\because x^2 + y^2 = 0 \text{ and } (2x - 1)y = 0$$

$$\text{if } x = +\frac{1}{2} \text{ then } y = \pm \frac{\sqrt{3}}{2}$$

$$\text{And if } y = 0 \text{ then } x = 0, -1$$

$$\therefore z = 0 + 0i, -1 + 0i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\therefore \sum (\operatorname{Re}(z) + \operatorname{Im}(z)) = 0$$

Question 139

Let S be the set of all (α, β) , $\pi < \alpha, \beta < 2\pi$, for which the complex number $\frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$ is purely imaginary

and $\frac{1 + i \cos \beta}{1 - 2i \cos \beta}$ is purely real. Let $Z_{\alpha\beta} = \sin 2\alpha + i \cos 2\beta$, $(\alpha, \beta) \in S$. Then $\sum_{(\alpha, \beta) \in S} \left(iZ_{\alpha\beta} + \frac{1}{iZ_{\alpha\beta}} \right)$ is equal to :

[27-Jul-2022-Shift-2]

Options:

A. 3

B. $3i$

C. 1

D. $2 - i$

Answer: C

Solution:

$$\because \frac{1 - i \sin \alpha}{1 + 2i \sin \alpha} \text{ is purely imaginary}$$

$$\therefore \frac{1 - i \sin \alpha}{1 + 2i \sin \alpha} + \frac{1 + i \sin \alpha}{1 - 2i \sin \alpha} = 0$$

$$\Rightarrow 1 - 2 \sin^2 \alpha = 0$$

$$\therefore \alpha = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{and } \frac{1 + i \cos \beta}{1 - 2i \cos \beta} \text{ is purely real}$$

$$\frac{1 + i \cos \beta}{1 - 2i \cos \beta} - \frac{1 - i \cos \beta}{1 + 2i \cos \beta} = 0$$

$$\Rightarrow \cos \beta = 0$$

$$\therefore \beta = \frac{3\pi}{2}$$

$$\therefore S = \left\{ \left(\frac{5\pi}{4}, \frac{3\pi}{2} \right), \left(\frac{7\pi}{4}, \frac{3\pi}{2} \right) \right\}$$

$$Z_{\alpha\beta} = 1 - i \text{ and } Z_{\alpha\beta} = -1 - i$$

$$\therefore \sum_{(\alpha, \beta) \in S} \left(iZ_{\alpha\beta} + \frac{1}{iZ_{\alpha\beta}} \right) = i(-2i) + \frac{1}{i} \left[\frac{1}{1+i} + \frac{1}{-1+i} \right]$$

$$= 2 + \frac{1}{i} \cdot \frac{2i}{-2} = 1$$

Question 140

Let $S_1 = \left\{ z_1 \in \mathbb{C} : |z_1 - 3| = \frac{1}{2} \right\}$ and $S_2 = \{ z_2 \in \mathbb{C} : z_2 - |z_2 + 1| = z_2 + |z_2 - 1| \}$. Then, for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$ is :
[28-Jul-2022-Shift-1]

Options:

- A. 0
- B. $\frac{1}{2}$
- C. $\frac{3}{2}$
- D. $\frac{5}{2}$

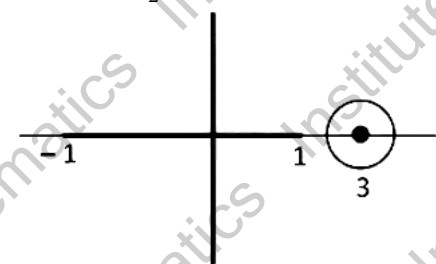
Answer: C

Solution:

$$\begin{aligned} \because |z_2 + |z_2 - 1||^2 &= |z_2 - |z_2 + 1||^2 \\ \Rightarrow (z_2 + |z_2 - 1|)(\overline{z_2} + |z_2 - 1|) &= (z_2 - |z_2 + 1|)(\overline{z_2} - |z_2 + 1|) \\ \Rightarrow z_2(\overline{z_2} - 1 + |z_2 + 1|) + \overline{z_2}(\overline{z_2} - 1 + |z_2 + 1|) &= |z_2 + 1|^2 - |z_2 - 1|^2 \\ \Rightarrow (z_2 + \overline{z_2})(|z_2 + 1| + |z_2 - 1|) &= 2(z_2 + \overline{z_2}) \\ \Rightarrow \text{Either } z_2 + \overline{z_2} = 0 \text{ or } |z_2 + 1| + |z_2 - 1| &= 2 \end{aligned}$$

So, z_2 lies on imaginary axis or on real axis within $[-1, 1]$

Also $|z_1 - 3| = \frac{1}{2} \Rightarrow z_1$ lies on the circle having center 3 and radius $\frac{1}{2}$.



Clearly $|z_1 - z_2|_{\min} = \frac{3}{2}$

Question 141

Let $z = a + ib$, $b \neq 0$ be complex numbers satisfying $z^2 = \overline{z} \cdot 2^{1-|z|}$. Then the least value of $n \in \mathbb{N}$, such that $z^n = (z+1)^n$, is equal to _____.
[28-Jul-2022-Shift-2]

Answer: 6

Solution:

$$\begin{aligned} \because z^2 &= \overline{z} \cdot 2^{1-|z|} \dots (1) \\ \Rightarrow |z|^2 &= |\overline{z}| \cdot 2^{1-|z|} \\ \Rightarrow |z| &= 2^{1-|z|} \\ \because b \neq 0 \Rightarrow |z| &\neq 0 \\ \therefore |z| &= 1 \dots (2) \\ \because z &= a + ib \text{ then } \sqrt{a^2 + b^2} = 1 \dots (3) \end{aligned}$$

Now again from equation (1), equation (2), equation (3) we get :

$$a^2 - b^2 + i2ab = (a - ib)^2$$

$$\therefore a^2 - b^2 = a \text{ and } 2ab = -b$$

$$\therefore a = -\frac{1}{2} \text{ and } b = \pm \frac{\sqrt{3}}{2}$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ or } z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z^n = (z + 1)^n \Rightarrow \left(\frac{z+1}{z}\right)^n = 1$$

$$\left(1 + \frac{1}{z}\right)^n = 1$$

$$\left(\frac{1 + \sqrt{3}i}{2}\right)^n = 1, \text{ then minimum value of } n \text{ is } 6.$$

Question 142

If $z = 2 + 3i$, then $z^5 + \left(\frac{1}{z}\right)^5$ is equal to :

[29-Jul-2022-Shift-1]

Options:

A. 244

B. 224

C. 245

D. 265

Answer: A

Solution:

$$z = (2 + 3i)$$

$$\Rightarrow z^5 = (2 + 3i)((2 + 3i)^2)^2$$

$$= (2 + 3i)(-5 + 12i)^2$$

$$= (2 + 3i)(-119 - 120i)$$

$$= -238 - 240i - 357i + 360$$

$$= 122 - 597i$$

$$\overline{z^5} = 122 + 597i$$

$$z^5 + \overline{z^5} = 244$$

Question 143

If $z \neq 0$ be a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum value of $|z|$ is:

[29-Jul-2022-Shift-2]

Options:

A. $\sqrt{2}$

B. 1

C. $\sqrt{2} - 1$

D. $\sqrt{2} + 1$

Answer: D

Solution:

We know,

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\therefore \left| |z| - \frac{1}{|z|} \right| \leq \left| z - \frac{1}{z} \right|$$

$$\Rightarrow \left| |z| - \frac{1}{|z|} \right| \leq 2 \quad \left[\text{Given } \left| z - \frac{1}{z} \right| = 2 \right]$$

$$\Rightarrow \left| \frac{|z|^2 - 1}{|z|} \right| \leq 2$$

$$\Rightarrow -2 \leq \frac{|z|^2 - 1}{|z|} \leq 2$$

$$\therefore \frac{|z|^2 - 1}{|z|} \leq 2$$

$$\Rightarrow |z|^2 - 1 \leq 2|z|$$

$$\Rightarrow |z|^2 - 2|z| - 1 \leq 0$$

$$\Rightarrow |z|^2 - 2|z| + 1 - 2 \leq 0$$

$$\Rightarrow (|z| - 1)^2 - 2 \leq 0$$

$$\Rightarrow -\sqrt{2} \leq |z| - 1 \leq \sqrt{2}$$

$$\Rightarrow 1 - \sqrt{2} \leq |z| \leq 1 + \sqrt{2} \dots (1)$$

or

$$-2 \leq \frac{|z|^2 - 1}{|z|}$$

$$\Rightarrow |z|^2 - 1 \leq -2|z|$$

$$\Rightarrow |z|^2 + 2|z| - 1 \leq 0$$

$$\Rightarrow |z|^2 + 2|z| + 1 - 2 \leq 0$$

$$\Rightarrow (|z| + 1)^2 - 2 \leq 0$$

$$\Rightarrow -\sqrt{2} \leq |z| + 1 \leq \sqrt{2}$$

$$\Rightarrow -\sqrt{2} - 1 \leq |z| \leq \sqrt{2} - 1 \dots (2)$$

From (1) and (2) we get,

Maximum value of $|z| = \sqrt{2} + 1$ and minimum value of $|z| = -\sqrt{2} - 1$

Question 144

Let $S = \{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z + i| = |z - 1|\}$. Then the set of all values of x , for which $u = 2x + iy \in S$ for some $y \in \mathbb{R}$, is
[29-Jul-2022-Shift-2]

Options:

A. $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$

B. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

C. $\left(-\sqrt{2}, \frac{1}{2}\right]$

D. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

Answer: B

Solution:

Question 145

If the numbers appeared on the two throws of a fair six faced die are α and β , then the probability that $x^2 + \alpha x + \beta > 0$, for all $x \in \mathbb{R}$, is :
[25-Jul-2022-Shift-1]

Options:

A. $\frac{17}{36}$

B. $\frac{4}{9}$

C. $\frac{1}{2}$

D. $\frac{19}{36}$

Answer: A

Solution:

For $x^2 + \alpha x + \beta > 0 \forall x \in \mathbb{R}$ to hold, we should have $\alpha^2 - 4\beta < 0$

If $\alpha = 1$, β can be 1, 2, 3, 4, 5, 6 i.e., 6 choices

If $\alpha = 2$, β can be 2, 3, 4, 5, 6 i.e., 5 choices

If $\alpha = 3$, β can be 3, 4, 5, 6 i.e., 4 choices

If $\alpha = 4$, β can be 5 or 6 i.e., 2 choices

If $\alpha = 6$, No possible value for β i.e., 0 choices

Hence total favourable outcomes

$$= 6 + 5 + 4 + 2 + 0 + 0$$

$$= 17$$

Total possible choices for α and $\beta = 6 \times 6 = 36$

$$\text{Required probability} = \frac{17}{36}$$

Question146

Let a, b be two non-zero real numbers. If p and r are the roots of the equation $x^2 - 8ax + 2a = 0$ and q and s are the roots of the equation $x^2 + 12bx + 6b = 0$, such that $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ are in A.P., then $a^{-1} - b^{-1}$ is equal to
[25-Jul-2022-Shift-1]

Answer: 38

Solution:

$\therefore$ Roots of $2ax^2 - 8ax + 1 = 0$ are $\frac{1}{p}$ and $\frac{1}{r}$ and roots of $6bx^2 + 12bx + 1 = 0$ are $\frac{1}{q}$ and $\frac{1}{s}$

Let $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ as $\alpha - 3\beta, \alpha - \beta, \alpha + \beta, \alpha + 3\beta$

So sum of roots $2\alpha - 2\beta = 4$ and $2\alpha + 2\beta = -2$

Clearly $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{2}$

Now product of roots, $\frac{1}{p} \cdot \frac{1}{r} = \frac{1}{2a} = -5 \Rightarrow \frac{1}{a} = -10$

and $\frac{1}{q} \cdot \frac{1}{s} = \frac{1}{6b} = -8 \Rightarrow \frac{1}{b} = -48$

So, $\frac{1}{a} - \frac{1}{b} = 38$

Question147

If for some p, q, r $\in \mathbb{R}$, not all have same sign, one of the roots of the equation $(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$ is also a root of the equation $x^2 + 2x - 8 = 0$, then $\frac{q^2 + r^2}{p^2}$ is equal to
[26-Jul-2022-Shift-1]

Answer: 272

Solution:

$$(p^2 + q^2)x^2 - 2q(p+r)x + q^2 + r^2 = 0 \quad \begin{matrix} \alpha \\ \beta \end{matrix}$$

$\therefore \alpha + \beta > 0$ and $\alpha\beta > 0$

Also, it has a common root with $x^2 + 2x - 8 = 0$

$\therefore$ The common root between above two equations is 4.

$$\Rightarrow 16(p^2 + q^2) - 8q(p+r) + q^2 + r^2 = 0$$

$$\Rightarrow (16p^2 - 8pq + q^2) + (16q^2 - 8qr + r^2) = 0$$

$$\Rightarrow (4p - q)^2 + (4q - r)^2 = 0$$

$$\Rightarrow q = 4p \text{ and } r = 16p$$

$$\therefore \frac{q^2 + r^2}{p^2} = \frac{16p^2 + 256p^2}{p^2} = 272$$

Question 148

The number of distinct real roots of the equation $x^5(x^3 - x^2 - x + 1) + x(3x^3 - 4x^2 - 2x + 4) - 1 = 0$ is

[26-Jul-2022-Shift-1]

Answer: 3

Solution:

$$x^8 + x^7 - x^6 + x^5 + 3x^4 - 4x^3 - 2x^2 + 4x - 1 = 0$$

$$\Rightarrow x^7(x-1) - x^5(x-1) + 3x^3(x-1) - x(x^2-1) + 2x(1-x) + (x-1) = 0$$

$$\Rightarrow (x-1)(x^7 - x^5 + 3x^3 - x(x^2-1) - 2x + 1) = 0$$

$$\Rightarrow (x-1)(x^7 - x^5 + 3x^3 - x^2 - 3x + 1) = 0$$

$$\Rightarrow (x-1)(x^5(x^2-1) + 3x(x^2-1) - 1(x^2-1)) = 0$$

$$\Rightarrow (x-1)(x^2-1)(x^5 + 3x - 1) = 0$$

$\therefore x = \pm 1$ are roots of above equation and $x^5 + 3x - 1$ is a monotonic term hence vanishes at exactly one value of x other than 1 or -1.

$\therefore 3$ real roots.

Question 149

The minimum value of the sum of the squares of the roots of $x^2 + (3-a)x + 1 = 2a$ is:

[26-Jul-2022-Shift-2]

Options:

A. 4

B. 5

C. 6

D. 8

Answer: C

Solution:

$$x^2 + (3-a)x + 1 = 2a \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$$

$$\alpha + \beta = a - 3, \alpha\beta = 1 - 2a$$

$$\Rightarrow \alpha^2 + \beta^2 = (a-3)^2 - 2(1-2a)$$

$$= a^2 - 6a + 9 - 2 + 4a$$

$$= a^2 - 2a + 7$$

$$= (a-1)^2 + 6$$

$$\text{So, } \alpha^2 + \beta^2 \geq 6$$

Question150

Let the abscissae of the two points P and Q on a circle be the roots of $x^2 - 4x - 6 = 0$ and the ordinates of P and Q be the roots of $y^2 + 2y - 7 = 0$. If PQ is a diameter of the circle $x^2 + y^2 + 2ax + 2by + c = 0$, then the value of $(a + b - c)$ is _____.

[26-Jul-2022-Shift-2]

Options:

- A. 12
- B. 13
- C. 14
- D. 16

Answer: A

Solution:

Abscissae of PQ are roots of $x^2 - 4x - 6 = 0$

Ordinates of PQ are roots of $y^2 + 2y - 7 = 0$

and PQ is diameter

$\Rightarrow$ Equation of circle is

$$x^2 + y^2 - 4x + 2y - 13 = 0$$

But, given $x^2 + y^2 + 2ax + 2by + c = 0$

By comparison $a = -2, b = 1, c = -13$

$$\Rightarrow a + b - c = -2 + 1 + 13 = 12$$

Question151

If α, β are the roots of the equation $x^2 - \left(5 + 3^{\sqrt{\log_3 5}} - 5^{\sqrt{\log_5 3}}\right) + 3 \left(3^{(\log_3 5)^{\frac{1}{3}}} - 5^{(\log_5 3)^{\frac{2}{3}}} - 1\right) = 0$ then the equation, whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is :

[27-Jul-2022-Shift-2]

Options:

- A. $3x^2 - 20x - 12 = 0$
- B. $3x^2 - 10x - 4 = 0$
- C. $3x^2 - 10x + 2 = 0$
- D. $3x^2 - 20x + 16 = 0$

Answer: B

Solution:

$$3^{\sqrt{\log_3 5}} - 5^{\sqrt{\log_3 3}} = 3^{\sqrt{\log_3 5}} - (3^{\log_3 5})^{\sqrt{\log_3 3}} \\ 3^{(\log_3 5) \frac{1}{3}} - 5^{(\log_3 3) \frac{2}{3}} = 5^{\frac{2}{3} \log_3 3} - 5^{(\log_3 3) \frac{2}{3}} = 0$$

Note: In the given equation 'x' is missing.

So

$$x^2 - 5x + 3(-1) = 0$$

$$\alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta}$$

$$= 5 - \frac{5}{3} = \frac{10}{3}$$

$$\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = 2 + \alpha\beta + \frac{1}{\alpha\beta} = 2 - 3 - \frac{1}{3} = -\frac{4}{3}$$

So Equation must be option (B).

Question 152

The sum of all real values of x for which $\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} - \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$ is equal to _____.

[28-Jul-2022-Shift-1]

Answer: 6

Solution:

$$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$$

$$\Rightarrow \frac{3x^2 - 9x + 17}{5x^2 - 7x + 19} = \frac{x^2 + 3x + 10}{3x^2 + 5x + 12}$$

$$\frac{-2x^2 - 2x - 2}{5x^2 - 7x + 19} = \frac{-2x^2 - 2x - 2}{3x^2 + 5x + 12}$$

Either $x^2 + x + 1 = 0$ or No real roots $\Rightarrow 5x^2 - 7x + 19 = 3x^2 + 5x + 12$

$$2x^2 - 12x + 7 = 0$$

$$\text{sum of roots} = 6$$

Question 153

Let α, β be the roots of the equation $x^2 - \sqrt{2}x + \sqrt{6} = 0$ and $\frac{1}{\alpha^2} + 1, \frac{1}{\beta^2} + 1$ be the roots of the equation

$x^2 + ax + b = 0$. Then the roots of the equation $x^2 - (a + b - 2)x + (a + b + 2) = 0$ are :

[28-Jul-2022-Shift-2]

Options:

A. non-real complex numbers

B. real and both negative

C. real and both positive

D. real and exactly one of them is positive

Answer: B

Solution:

$$\alpha + \beta = \sqrt{2}, \alpha\beta = \sqrt{6}$$

$$\frac{1}{\alpha^2} + 1 + \frac{1}{\beta^2} + 1 = 2 + \frac{\alpha^2 + \beta^2}{6}$$

$$= 2 + \frac{2 - 2\sqrt{6}}{6} = -a$$

$$\left(\frac{1}{\alpha^2} + 1\right) \left(\frac{1}{\beta^2} + 1\right) = 1 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\alpha^2\beta^2}$$

$$= \frac{7}{6} + \frac{2 - 2\sqrt{6}}{6} = b$$

$$\Rightarrow a + b = \frac{-5}{6}$$

$$\text{So, equation is } x^2 + \frac{17x}{6} + \frac{7}{6} = 0$$

$$\text{OR } 6x^2 + 17x + 7 = 0$$

Both roots of equation are -ve and distinct

Question 154

Let $f(x) = ax^2 + bx + c$ be such that $f(1) = 3$, $f(-2) = \lambda$ and $f(3) = 4$. If $f(0) + f(1) + f(-2) + f(3) = 14$, then λ is equal to :

[28-Jul-2022-Shift-2]

Options:

A. -4

B. $\frac{13}{2}$

C. $\frac{23}{2}$

D. 4

Answer: D

Solution:

$$f(1) = a + b + c = 3 \dots (i)$$

$$f(3) = 9a + 3b + c = 4 \dots (ii)$$

$$f(0) + f(1) + f(-2) + f(3) = 14$$

$$\text{OR } c + 3 + (4a - 2b + c) + 4 = 14$$

$$\text{OR } 4a - 2b + 2c = 7 \dots (iii)$$

$$\text{From (i) and (ii) } 8a + 2b = 1 \dots (iv)$$

$$\text{From (iii) } -(2) \times (i)$$

$$\Rightarrow 2a - 4b = 1 \dots (v)$$

$$\text{From (iv) and (v) } a = \frac{1}{6}, b = \frac{-1}{6} \text{ and } c = 3$$

$$f(-2) = 4a - 2b + c$$

$$= \frac{4}{6} + \frac{2}{6} + 3 = 4$$

Question 155

Let $\alpha, \beta (\alpha > \beta)$ be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n$, $n \in \mathbb{N}$, then

$$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$$
 is equal to

[29-Jul-2022-Shift-2]

Answer: 16

Solution:

α and β are the roots of the quadratic equation $x^2 - x - 4 = 0$.

$\therefore \alpha$ and β are satisfy the given equation.

$$\alpha^2 - \alpha - 4 = 0$$

$$\Rightarrow \alpha^{n+1} - \alpha^n - 4\alpha^{n-1} = 0 \dots\dots (i)$$

$$\text{and } \beta^2 - \beta - 4 = 0$$

$$\Rightarrow \beta^{n+1} - \beta^n - 4\beta^{n-1} = 0 \dots\dots (2) \text{Substituting (2) from (1), we get,}$$

$$(\alpha^{n+1} - \beta^{n+1}) - (\alpha^n - \beta^n) - 4(\alpha^{n-1} - \beta^{n-1}) = 0$$

$$\Rightarrow P_{n+1} - P_n - 4P_{n-1} = 0$$

$$\Rightarrow P_{n+1} = P_n + 4P_{n-1}$$

$$\Rightarrow P_{n+1} - P_n = 4P_{n-1}$$

$$\text{For } n = 14, P_{15} - P_{14} = 4P_{13}$$

$$\text{For } n = 15, P_{16} - P_{15} = 4P_{14}$$

$$\text{Now, } \frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$$

$$= \frac{P_{16}(P_{15} - P_{14}) - P_{15}(P_{15} - P_{14})}{P_{13}P_{14}}$$

$$= \frac{(P_{15} - P_{14})(P_{16} - P_{15})}{P_{13}P_{14}}$$

$$= \frac{(4P_{13})(4P_{14})}{P_{13}P_{14}}$$

$$= 16$$

Question156

Let $S = \{x \in [-6, 3] - \{-2, 2\} : \frac{|x+3|-1}{|x|-2} \geq 0\}$ and $T = \{x \in \mathbb{Z} : x^2 - 7|x| + 9 \leq 0\}$

Then the number of elements in $S \cap T$ is :

[28-Jul-2022-Shift-2]

Options:

A. 7

B. 5

C. 4

D. 3

Answer: D

Solution:

$$|x^2 - 7|x| + 9 \leq 0$$

$$\Rightarrow |x| \in \left[\frac{7 - \sqrt{13}}{2}, \frac{7 + \sqrt{13}}{2} \right]$$

As $x \in \mathbb{Z}$

So, x can be $\pm 2, \pm 3, \pm 4, \pm 5$

Out of these values of x ,

$x = 3, -4, -5$

satisfy S as well

$$n(S \cap T) = 3$$

Question157

Let $i = \sqrt{-1}$. If $\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}} + \frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}} = k$ and $n = \lfloor |k| \rfloor$ be the greatest integral part of $|k|$. Then,

$$\sum_{j=0}^{n+5} (j+5)^2 - \sum_{j=0}^{n+5} (j+5) \text{ is equal to}$$

[2021, 24 Feb. Shift-II]

Answer: 310

Solution:

$$\text{Given, } \frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}} + \frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}} = k$$

$$\therefore -1+i\sqrt{3} = 2e^{i2\pi/3}$$

$$1+i\sqrt{3} = 2e^{i\pi/3}$$

$$1-i = \sqrt{2}e^{-i\pi/4}$$

$$1+i = \sqrt{2}e^{i\pi/4}$$

$$\text{Now, } \frac{\left(2e^{i\frac{2\pi}{3}}\right)^{21}}{(\sqrt{2}e^{-i\pi/4})^{24}} + \frac{(2e^{i\pi/3})^{21}}{(\sqrt{2}e^{i\pi/4})^{24}}$$

$$= \frac{2^{21} \cdot e^{i14\pi}}{2^{12} \cdot e^{-i6\pi}} + \frac{2^{21} \cdot e^{i7\pi}}{2^{12} \cdot e^{i6\pi}}$$

$$= 2^9 \cdot e^{i20\pi} + 2^9 \cdot e^{i\pi}$$

$$= 2^9(1) + 2^9(-1)$$

$$\Rightarrow 2^9 - 2^9 = 0 = k \text{ (given)}$$

$$\therefore n = [k] = [101] = 0$$

$$\text{Now, } \sum_{j=0}^5 (j+5)^2 - \sum_{j=0}^5 (j+5) \quad [\because n=0]$$

$$= [5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2]$$

$$- [5 + 6 + 7 + 8 + 9 + 10]$$

$$= [(1^2 + 2^2 + 3^2 + \dots + 10^2) -$$

$$(1^2 + 2^2 + \dots + 4^2)] - [(1 + 2 + 3 + \dots + 10)$$

$$- (1 + 2 + 3 + 4)]$$

$$= \left[\frac{10 \times 11 \times 21}{6} - \frac{4 \times 5 \times 9}{6} \right] - \left[\frac{10 \times 11}{2} - \frac{4 \times 5}{2} \right]$$

$$= (385 - 30) - (55 - 10)$$

$$= 385 - 45 = 310$$

Question 158

Let z be those complex numbers which satisfy $|z+5| \leq 4$ and $z(1+i) + \bar{z}(1-i) \geq -10$, $i = \sqrt{-1}$. If the maximum value of $|z+1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is [2021, 26 Feb. Shift-II]

Answer: 48

Solution:

Given, $|z+5| \leq 4$, which is equation of circle.

$$|z+5| \leq 4$$

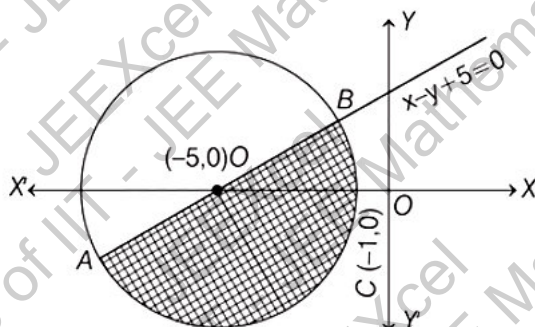
$$\Rightarrow (x+5)^2 + y^2 \leq 16$$

$$\text{and } z(1+i) + \bar{z}(1-i) \geq -10$$

$$\Rightarrow (z+z) + i(z-\bar{z}) \geq -10$$

$$\Rightarrow x-y+5 \geq 0$$

From Eqs. (i) and (iii), region bounded by inequalities are



Now, $|z + 1|^2 = |z - (-1)|^2$

Maximum value of $|z + 1|^2$ will be equal to $(AC)^2$.

Now, $(x + 5)^2 + y^2 = 16$

and $x - y + 5 = 0$

Given, $y = \pm 2\sqrt{2}$

and $x = \pm 2\sqrt{2} - 5$

$\therefore$ Coordinates are

$A(-2\sqrt{2} - 5, -2\sqrt{2})$

$B(2\sqrt{2} - 5, 2\sqrt{2})$

$C(-1, 0)$

Then,

$AC^2 = (2\sqrt{2} + 4)^2 + (2\sqrt{2})^2$

$= 32 + 16\sqrt{2}$

Given, that maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$

$\Rightarrow \alpha + \beta\sqrt{2} = 32 + 16\sqrt{2}$

$\Rightarrow \alpha = 32, \beta = 16$

$\therefore \alpha + \beta = 32 + 16 = 48$

Question 159

Let the lines $(2 - i)z = (2 + i)\bar{z}$ and $(2 + i)z + (i - 2)\bar{z} - 4i = 0$, (here $i^2 = -1$) be normal to a circle C. If the line $iz + \bar{z} + 1 + i = 0$ is tangent to this circle C, then its radius is [2021, 25 Feb. Shift-1]

Options:

A. $\frac{3}{\sqrt{2}}$

B. $\frac{1}{2\sqrt{2}}$

C. $3\sqrt{2}$

D. $\frac{3}{2\sqrt{2}}$

Answer: D

Solution:

Given, $(2 - i)z = (2 + i)\bar{z}$

Let $z = x + iy$, then $\bar{z} = x - iy$

$\Rightarrow (2 - i)(x + iy) = (2 + i)(x - iy)$

$\Rightarrow 2x - ix + 2iy + y = 2x + ix - 2iy + y$

$\Rightarrow 2ix - 4iy = 0$

$\therefore$ Equation of line $L_1 \Rightarrow x - 2y = 0 \dots\dots\dots (i)$

Also, $(2 + i)z + (i - 2)\bar{z} - 4i = 0$

$\Rightarrow (2 + i)(x + iy) + (i - 2)(x - iy) - 4i = 0$

$\Rightarrow 2x + ix + 2iy - y + ix - 2x + y$

$+ 2iy - 4i = 0$

$\Rightarrow 2ix + 4iy - 4i = 0$

$\therefore$ Equation of line $L_2 \Rightarrow x + 2y - 2 = 0 \dots\dots (ii)$

From Eqs. (i) and (ii),

$4y = 2$ or $y = 1/2$ and $x = 1$

Hence, centre $= (1, 1/2)$

Equation of third line

$L_3 \Rightarrow iz + \bar{z} + 1 + i = 0$

$\Rightarrow i(x + iy) + (x - iy) + 1 + i = 0$

$\Rightarrow ix - y + x - iy + 1 + i = 0$

$\Rightarrow (x - y + 1) + i(x - y + 1) = 0$

$\therefore$ Radius = Distance of point $(1, 1/2)$ to the line $x - y + 1 = 0$

$\therefore r = \frac{\left|1 - \frac{1}{2} + 1\right|}{\sqrt{1^2 + 1^2}} = \frac{3}{2\sqrt{2}}$

Question 160

Let α and β be two real numbers, such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$, for some integer $n \geq 1$. Then, the value of p_n^2 is
[2021, 26 Feb. Shift-III]

Answer: 324

Solution:

Given that, $\alpha + \beta = 1$, $\alpha\beta = -1$

Let α, β be roots of quadratic equation, then the quadratic equation be

$$x^2 - x - 1 = 0$$

$$\text{Now, } \alpha^2 - \alpha - 1 = 0$$

$$\Rightarrow \alpha^2 = \alpha + 1 \quad \dots\dots (i)$$

$$\text{Similarly, } \beta^2 = \beta + 1 \quad \dots\dots (ii)$$

Multiply α^{n-1} in Eq. (i), we get

$$\alpha^{n+1} = \alpha^n + \alpha^{n-1} \quad \dots\dots (iii)$$

Multiply β^{n-1} in Eq. (ii), we get

$$\beta^{n+1} = \beta^n + \beta^{n-1} \quad \dots\dots (iv)$$

Add Eqs. (iii) and (iv), we get

$$\alpha^{n+1} + \beta^{n+1} = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})$$

$$p_{n+1} = p_n + p_{n-1}$$

$$29 = p_n + 11$$

$$\Rightarrow p_n = 18$$

$$p_n^2 = (18)^2 = 324$$

Question 161

The number of solutions of the equation $\log_4(x-1) = \log_2(x-3)$ is
[2021, 26 Feb. Shift-1]

Answer: 1

Solution:

$$\log_4(x-1) = \log_2(x-3) \text{ (given)}$$

$$\Rightarrow \log_2 2(x-1) = \log_2(x-3)$$

Using property of logarithm,

$$\log_b c^a = \frac{1}{c} \log_b a$$

$$\Rightarrow \frac{1}{2} \log_2(x-1) = \log_2(x-3)$$

$$\Rightarrow \log_2(x-1) = 2\log_2(x-3)$$

$$\Rightarrow \log_2(x-1) = \log_2(x-3)^2$$

$$\text{On comparing, } x-1 = (x-3)^2$$

$$\text{or } x-1 = x^2 + 9 - 6x$$

$$\Rightarrow x^2 - 7x + 10 = 0$$

$$\Rightarrow x^2 - 5x - 2x + 10 = 0$$

$$\Rightarrow (x-5)(x-2) = 0$$

$$\Rightarrow x = 2, 5$$

$$x = 2 \text{ (rejected) as } x > 1$$

$$\therefore x = 5 \text{ is only solution i.e. number of solution is 1.}$$

Question 162

Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is
[2021, 25 Feb. Shift-II]

Options:

- A. 4
- B. 3
- C. 2
- D. 1

Answer: C

Solution:

We have, $x^2 - 6x - 2 = 0$

Given, α and β are roots of above quadratic equation, then

$$\alpha^2 - 6\alpha - 2 = 0$$

$$\beta^2 - 6\beta - 2 = 0$$

Also, given $a_n = \alpha^n - \beta^n$, then

$$\begin{aligned} \frac{a_{10} - 2a_8}{3a_9} &= \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{3(\alpha^9 - \beta^9)} \\ &= \frac{\alpha^{10} - 2\alpha^8 - \beta^{10} + 2\beta^8}{3(\alpha^9 - \beta^9)} = \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{3(\alpha^9 - \beta^9)} \end{aligned}$$

[from Eqs. (i) and (ii) $\alpha^2 - 2 = 6\alpha$, $\beta^2 - 2 = 6\beta$]

$$\begin{aligned} &= \frac{\alpha^8(6\alpha) - \beta^8(6\beta)}{3(\alpha^9 - \beta^9)} \\ &= \frac{6\alpha^9 - 6\beta^9}{3(\alpha^9 - \beta^9)} = \frac{6(\alpha^9 - \beta^9)}{3(\alpha^9 - \beta^9)} \\ &= 2 \end{aligned}$$

Question 163

If $\alpha, \beta \in \mathbb{R}$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to
[2021, 25 Feb. Shift-II]

Options:

- A. 3
- B. -3
- C. 7
- D. -7

Answer: D

Solution:

Given, root of $z^2 + \alpha z + \beta = 0$ is $1 - 2i$.

Since, it is quadratic equation and one root is complex in nature, its another root is complex conjugate.

$\therefore$ Two roots are $1 - 2i$ and $1 + 2i$.

Now, sum of roots $= -\frac{\alpha}{1} = -\alpha$

$$= (1 - 2i) + (1 + 2i) = 2$$

Gives, $\alpha = -2$

$$\text{Product of roots} = \frac{\beta}{1} = \beta$$

$$= (1 - 2i)(1 + 2i) = 1 + 4 = 5$$

$$\text{Gives, } \beta = 5$$

$$\therefore \alpha - \beta = -2 - 5 = -7$$

Question 164

The integer 'k', for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R, is [2021, 25 Feb. Shift-1]

Options:

A. 3

B. 2

C. 0

D. 4

Answer: A

Solution:

$$\text{Given, } x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$$

$$\text{Here, } a > 0$$

$$\therefore D < 0$$

$$\Rightarrow [2(3k - 1)]^2 - 4(8k^2 - 7) < 0$$

$$\Rightarrow 4(9k^2 + 1 - 6k) - 4(8k^2 - 7) < 0$$

$$\Rightarrow k^2 - 6k + 8 < 0$$

$$\Rightarrow (k - 4)(k - 2) < 0$$



$$k \in (2, 4)$$

$$\therefore \text{Required integer, } k = 3$$

Question 165

The sum of 162th power of the roots of the equation $x^3 - 2x^2 + 2x - 1 = 0$ is [2021, 26 Feb. Shift-1]

Answer: 3

Solution:

$$\text{Given, } x^3 - 2x^2 + 2x - 1 = 0$$

$$\text{i.e. } (x^3 - 1) - (2x^2 - 2x) = 0$$

$$\Rightarrow (x - 1)(x^2 + x + 1) - 2x(x - 1) = 0$$

$$\Rightarrow (x - 1)(x^2 + x + 1 - 2x) = 0$$

$$\Rightarrow (x - 1)(x^2 - x + 1) = 0$$

$$\therefore x = 1 \text{ and } x = \frac{-(-1) \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \text{Roots are } 1, -\omega_1, -\omega_2.$$

$$\text{Then, sum of } 162^{\text{th}} \text{ power of the roots}$$

$$= (1)^{162} + (-\omega_1)^{162} + (-\omega_2)^{162}$$

$$= 1 + \omega_1^{162} + \omega_2^{324}$$

$$= 1 + (\omega_1^3)^{54} + (\omega_2^3)^{108}$$

$$= 1 + (1)^{54} + (1)^{108} \quad [\because \omega^3 = 1]$$

$$= 1 + 1 + 1 = 3$$

Question 166

Let a, b, c be in an arithmetic progression. Let the centroid of the triangle with vertices $(a, c), (2, b)$ and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is [2021, 24 Feb. Shift-II]

Options:

A. $\frac{71}{256}$

B. $\frac{69}{256}$

C. $-\frac{69}{256}$

D. $-\frac{71}{256}$

Answer: D

Solution:

Given, a, b, c are in AP.
 $(a, c), (2, b), (a, b)$ are vertices of triangle.

$$\text{Centroid} = \left(\frac{10}{3}, \frac{7}{3}\right)$$

α and β are the roots of equation $ax^2 + bx + 1 = 0$

$\therefore a, b, c$ are in AP.

$$\therefore 2b = a + c$$

$$\text{Centroid} = \left(\frac{a+2+a}{3}, \frac{c+b+b}{3}\right)$$

$$= \left(\frac{2a+2}{3}, \frac{c+2b}{3}\right) = \left(\frac{10}{3}, \frac{7}{3}\right)$$

$$\Rightarrow \frac{2a+2}{3} = \frac{10}{3} \text{ and } \frac{c+2b}{3} = \frac{7}{3}$$

$$\Rightarrow a = 4$$

$$\Rightarrow c + a + c = 7 \quad [\because 2b = a + c]$$

$$\Rightarrow 2c = 7 - 4 \quad [\because a = 4]$$

$$c = 3/2$$

$$\text{Also, } 2b = a + c = 4 + \frac{3}{2}$$

$$\Rightarrow b = 11/4$$

Now, α and β are roots of $ax^2 + bx + 1 = 0$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-11/4}{4}$$

$$\Rightarrow \alpha + \beta = \frac{-11}{16}$$

$$\Rightarrow \alpha\beta = \frac{1}{a} = \frac{1}{4}$$

$$\Rightarrow \alpha\beta = \frac{1}{4}$$

Now, $\alpha^2 + \beta^2 - \alpha\beta$

$$= (\alpha + \beta)^2 - 3\alpha\beta$$

$$= \left(\frac{-11}{16}\right)^2 - 3 \times \frac{1}{4}$$

$$= \frac{121 - 192}{256} = \frac{-71}{256}$$

Question 167

The number of the real roots of the equation $(x+1)^2 + |x-5| = \frac{27}{4}$ is
[2021,24 Feb. Shift-II]

Answer: 2

Solution:

Given, equation $(x+1)^2 + |x-5| = \frac{27}{4}$

Case I For $x \geq 5$

$$11 \Rightarrow (x+1)^2 + (x-5) = \frac{27}{4}$$

$$\Rightarrow x^2 + 3x - 4 = \frac{27}{4}$$

$$\Rightarrow 4x^2 + 12x - 43 = 0$$

$$\therefore x = \frac{-12 \pm \sqrt{144 + 688}}{8}$$

$$= \frac{-12 \pm \sqrt{832}}{8} = \frac{-12 \pm 28.8}{8}$$

$$x = \frac{-3 \pm 7.2}{8}$$

$$x = \frac{-3 + 7.2}{8}, \frac{-3 - 7.2}{8}$$

Both the values are less than 5.

$\therefore$ No solution from here.

Case II $x < 5$

$$\Rightarrow (x+1)^2 + (x-5) = \frac{27}{4}$$

$$\Rightarrow x^2 + x - 6 = \frac{27}{4} = 0$$

$$\Rightarrow 4x^2 + 4x - 3 = 0$$

$$\Rightarrow x = \frac{-4 \pm \sqrt{16 + 48}}{8}$$

$$= \frac{-4 \pm 8}{8}$$

$$\Rightarrow x = \frac{-12}{8}, \frac{4}{8}, \text{ both are less than 5.}$$

$\therefore$ These values must be the solution. Hence, here 2 real roots are possible.

Question 168

If the least and the largest real values of α , for which the equation $z + \alpha |z-1| + 2i = 0$ ($z \in \mathbb{C}$ and $i = \sqrt{-1}$) has a solution, are p and q respectively, then $4(p^2 + q^2)$ is equal to
[2021,24 Feb. Shift-I]

Answer: 10

Solution:

Given, $\alpha_{\text{least}} = p$

$\alpha_{\text{max}} = q$

Equation given is $z + \alpha |z-1| + 2i = 0$;

$z \in \mathbb{C}$ and $i = \sqrt{-1}$

Let $z = x + iy$

Then, $z + \alpha |z-1| + 2i = 0$

$$\begin{aligned}
 &\Rightarrow x + iy + \alpha \sqrt{(x-1)^2 + y^2} + 2i = 0 \\
 &\Rightarrow (x + \alpha \sqrt{(x-1)^2 + y^2}) + i(y+2) = 0 \\
 &\therefore y+2 = 0 \text{ and } x + \alpha \sqrt{(x-1)^2 + y^2} = 0 \\
 &y = -2 \text{ and } x^2 = \alpha^2 (x^2 + 1 - 2x + y^2) \\
 &x^2 = \alpha^2 (x^2 - 2x + 5) \quad (\because y = -2) \\
 &\Rightarrow \alpha^2 = \frac{x^2}{x^2 - 2x + 5} \\
 &\therefore \alpha^2 \in \left[0, \frac{5}{4}\right] \\
 &\therefore \alpha \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right] \\
 &\text{Now, } 4(p^2 + q^2) = 4[(\alpha_{\text{least}})^2 + (\alpha_{\text{max}})^2] \\
 &= 4\left[\left(-\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right] \\
 &= 4 \times \left[\frac{5}{4} + \frac{5}{4}\right] = 10
 \end{aligned}$$

Question 169

Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation:
[24-Feb-2021 Shift 1]

Options:

- A. $x^2 - 2x + 2 = 0$
- B. $x^2 - 2x + 8 = 0$
- C. $x^2 - 2x + 136 = 0$
- D. $x^2 - 2x + 16 = 0$

Answer: D

Solution:

We have

$$\begin{aligned}
 (p^2 + q^2)^2 - 2p^2q^2 &= 272 \\
 ((p+q)^2 - 2pq)^2 - 2p^2q^2 &= 272 \\
 (4 - 2pq)^2 - 2p^2q^2 &= 272 \\
 16 - 16pq + 2p^2q^2 &= 272 \\
 (pq)^2 - 8pq - 128 &= 0 \\
 pq &= \frac{8 \pm 24}{2} = 16, -8 \\
 \therefore pq &= 16 \quad (\because p, q > 0)
 \end{aligned}$$

$\therefore$ Required equation :

$$x^2 - (2)x + 16 = 0$$

Question 170

If the equation $a|z|^2 + \overline{az} + az + d = 0$ represents a circle, where a, d are real constants, then which of the following condition is correct?

[2021, 18 March Shift-I]

Options:

A. $|\alpha|^2 - ad \neq 0$

B. $|\alpha|^2 - ad > 0$ and $a \in \mathbb{R} - \{0\}$

C. $|\alpha|^2 - ad \geq 0$ and $a \in \mathbb{R}$

D. $\alpha = 0, a, d \in \mathbb{R}^+$

Answer: B

Solution:

Given, $a|z|^2 + \overline{\alpha z} + \alpha \overline{z} + d = 0$

$\Rightarrow a|z|^2 + \alpha \overline{z} + \overline{\alpha} z + d = 0 \dots (i)$

Putting $z = x + iy$ and $\alpha = p + iq$ in Eq. (i), we get

$a(x^2 + y^2) + (p + iq)(x - iy) + (p - iq)(x + iy) + d = 0$

$\Rightarrow (x + iy) + d = 0$

$a(x^2 + y^2) + px + qy - ipy + iqx + px + qy - iqx + ipy + d = 0$

$\Rightarrow a(x^2 + y^2) + 2px + 2qy + d = 0$

$\Rightarrow x^2 + y^2 + \left(\frac{2p}{a}\right)x + \left(\frac{2q}{a}\right)y + \frac{d}{a} = 0$ be a circle

If $a \neq 0$ and $r^2 = \left(\frac{p^2}{a^2} + \frac{q^2}{a^2} - \frac{d}{a}\right) > 0$ If $a \neq 0$ and $r^2 = \left(\frac{p^2}{a^2} + \frac{q^2}{a^2} - \frac{d}{a}\right) > 0$

$\Rightarrow p^2 + q^2 - ad > 0$

$\Rightarrow |\alpha|^2 - ad > 0$

and $a \in \mathbb{R} - \{0\}$

Question 171

Let z_1, z_2 be the roots of the equation $z^2 + az + 12 = 0$ and z_1, z_2 form an equilateral triangle with origin.

Then, the value of $|a|$ is

[2021, 18 March Shift-I]

Answer: 6

Solution:

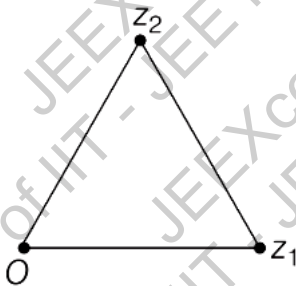
Given, z_1, z_2 are the roots of

$z^2 + az + 12 = 0$

$\therefore z_1 + z_2 = \frac{-a}{1} = -a$

and $z_1 z_2 = \frac{12}{1} = 12$

Now, z_1, z_2 and origin forms an equilateral triangle.



$$\begin{aligned}
 1 &\because z_1^2 + z_2^2 + 0^2 = z_1 z_2 + 0 + 0 \\
 &\Rightarrow z_1^2 + z_2^2 = z_1 z_2 \\
 &\Rightarrow z_1^2 + z_2^2 + 2z_1 z_2 = z_1 z_2 + 2z_1 z_2 \\
 &\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2 \\
 &\Rightarrow (-a)^2 = 3 \times (12) \\
 &\Rightarrow a^2 = 36 \Rightarrow |a|^2 = 36 \\
 &\Rightarrow |a| = \pm 6 \\
 &\text{But } |a| \geq 0 \\
 \therefore |a| &= 6
 \end{aligned}$$

Question 172

Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w , is equal to [2021, 18 March Shift-III]

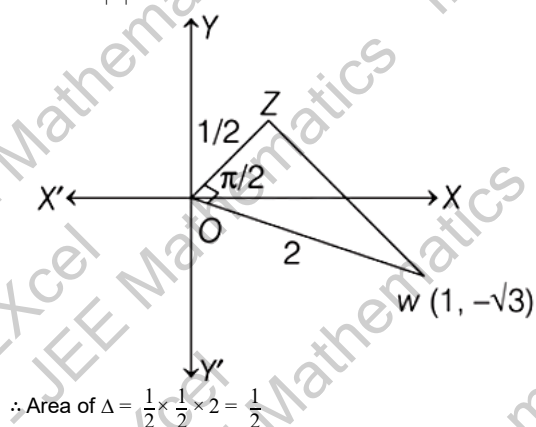
Options:

- A. 4
- B. $\frac{1}{2}$
- C. $\frac{1}{4}$
- D. 2

Answer: B

Solution:

$$\begin{aligned}
 \text{Given, } w &= 1 - \sqrt{3}i \\
 \Rightarrow |w| &= \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2 \\
 \text{and } |zw| &= 1 \Rightarrow |z||w| = 1 \\
 \Rightarrow |z| &= \frac{1}{|w|} = \frac{1}{2}
 \end{aligned}$$



Question 173

Let S_1 , S_2 and S_3 be three sets defined as

$$S_1 = \{z \in \mathbb{C} : |z - 1| \leq \sqrt{2}\}$$

$$S_2 = \{z \in \mathbb{C} : \operatorname{Re}[(1 - i)z] \geq 1\}$$

$$S_3 = \{z \in \mathbb{C} : \operatorname{Im}(z) \leq 1\}$$

Then, the set $S_1 \cap S_2 \cap S_3$

[2021, 17 March Shift-II]

Options:

- A. is a singleton
- B. has exactly two elements
- C. has infinitely many elements
- D. has exactly three elements

Answer: C

Solution:

For $|z-1| \leq \sqrt{2}, \dots$ (i)

z lies on and inside the circle of radius $\sqrt{2}$ units and centre $(1, 0)$.

For S_2 , let $z = x + iy$

Now $(1-i)(z) = (1-i)(x+iy)$

$= x + iy - ix + y = (x+y) + i(y-x)$

$\therefore \operatorname{Re}[(1-i)z] = (x+y)$, which is greater than or equal to one.

i.e., $x+y \geq 1 \dots\dots$ (ii)

Also, for S_3

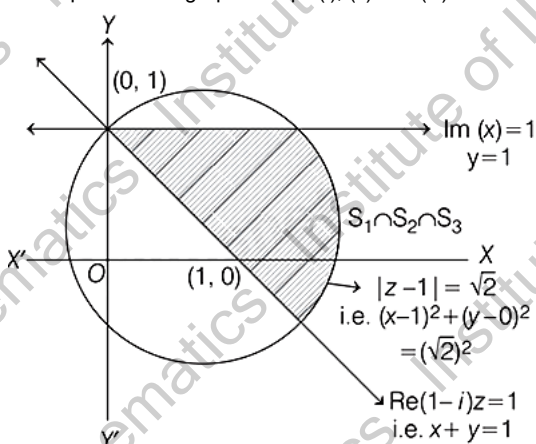
Let $z = x + iy$

$\therefore \operatorname{Im}(z) = y$, which is less than or equal to

one.

i.e., $y \leq 1 \dots\dots$ (iii)

Concept Draw the graph of Eqs. (i), (ii) and (iii) and then select the common region bounded by Eqs. (i), (ii) and (iii) for $S_1 \cap S_2 \cap S_3$.



$\therefore S_1 \cap S_2 \cap S_3$ has infinitely many elements.

Question 174

The area of the triangle with vertices $A(z)$, $B(iz)$ and $C(z+iz)$ is

[2021, 17 March Shift-I]

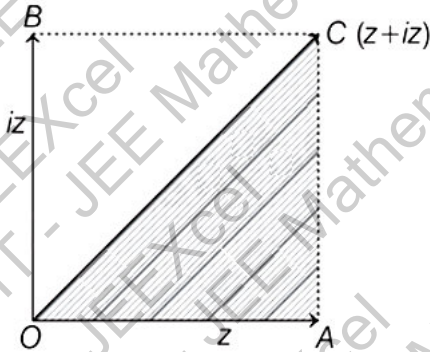
Options:

- A. 1
- B. $\frac{1}{2}|z|^2$
- C. $\frac{1}{2}$
- D. $\frac{1}{2}|z+iz|^2$

Answer: B

Solution:

Area of triangle whose vertices are A(z), B(iz), C(z + iz)



Area of the triangle

$$= \frac{1}{2} |z| |iz| = \frac{1}{2} |z|^2$$

Question 175

The value of $4 + \frac{1}{5 + \frac{1}{4 + \frac{1}{5 + \frac{1}{4 + \dots \infty}}}}$ is

[2021, 17 March Shift-I]

Options:

A. $2 + \frac{2}{5}\sqrt{30}$

B. $2 + \frac{4}{5}\sqrt{30}$

C. $4 + \frac{4}{5}\sqrt{30}$

D. $5 + \frac{2}{5}\sqrt{30}$

Answer: A

Solution:

$$\text{Let } x = 4 + \frac{1}{5 + \frac{1}{4 + \frac{1}{5 + \frac{1}{4 + \dots \infty}}}}$$

$$x = 4 + \frac{1}{5 + \frac{1}{x}}$$

$$\Rightarrow (x-4)(5x+1) = x$$

$$\Rightarrow 5x^2 - 19x - 4 = x$$

$$\Rightarrow 5x^2 - 20x - 4 = 0$$

$$\Rightarrow x = \frac{20 \pm \sqrt{400 + 80}}{10}$$

$$\Rightarrow x = \frac{20 \pm \sqrt{480}}{10}$$

$$\Rightarrow x = 2 \pm \sqrt{\frac{480}{100}}$$

$$= 2 \pm \frac{2}{5}\sqrt{30}$$

$$\because x \neq 0$$

$$\text{So, } x = 2 + \frac{2}{5}\sqrt{30}$$

Question 176

The number of elements in the set $\{x \in \mathbb{R} : (|x| - 3) |x + 4| = 6\}$ is equal to [2021, 16 March Shift-1]

Options:

- A. 3
- B. 2
- C. 4
- D. 1

Answer: B

Solution:

Given, set = $\{x \in \mathbb{R} : (|x| - 3) |x + 4| = 6\}$

As, we already know

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \text{ and}$$

$$|x + 4| = \begin{cases} x + 4 & x \geq -4 \\ -(x + 4) & x < -4 \end{cases}$$

Case I

$$x < -4$$

$$r(-x - 3)(-x - 4) = 6$$

$$(x + 3)(x + 4) = 6$$

$$x^2 + 7x + 12 = 6$$

$$x^2 + 7x + 6 = 0$$

$$(x + 6)(x + 1) = 0$$

$$x = -6 \text{ or } x = -1$$

We will reject $x = -1$ as, $-1 > -4$

$\therefore$ When $x < -4$, $x = -6$ is the solution.

Case II

$$-4 \leq x < 0$$

$$(-x - 3)(x + 4) = 6$$

$$\Rightarrow -(x + 3)(x + 4) = 6$$

$$\Rightarrow -(x^2 + 7x + 12) = 6$$

$$\Rightarrow x^2 + 7x + 18 = 0$$

As, the discriminant of this quadratic

$$\text{equation is } D = 7^2 - 4 \cdot 18 = 49 - 72 = -23$$

$$\therefore D = -23 \text{ and } D < 0$$

So, no real roots and as per the question,

$x \in \mathbb{R}$.

No solution when $-4 \leq x < 0$.

Case III

$$x \geq 0$$

$$(|x| - 3) |x + 4| = 6$$

$$\Rightarrow (x - 3)(x + 4) = 6$$

$$\Rightarrow x^2 + x - 12 = 6$$

$$\Rightarrow x^2 + x - 18 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 72}}{2} = \frac{-1 \pm \sqrt{73}}{2}$$

We will reject $x = \frac{-1 - \sqrt{73}}{2}$ as $\frac{-1 - \sqrt{73}}{2} < 0$ and here, $x \geq 0$.

So, $x = \frac{-1 + \sqrt{73}}{2}$, when $x \geq 0$.

$$\therefore x = -6 \text{ and } x = \frac{-1 + \sqrt{73}}{2}$$

are the two solutions which belong to the set.

Hence, number of solutions = 2

Question 177

Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients, such that $\int_0^1 P(x) dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then, the value of $g(b + c)$ is
[2021, 16 March Shift-II]

Options:

- A. 9
- B. 15
- C. 7
- D. 11

Answer: C

Solution:

$$\begin{aligned} P(x) &= x^2 + bx + c \\ \Rightarrow \int_0^1 (x^2 + bx + c) dx &= 1 \\ \Rightarrow \left[\frac{x^3}{3} + \frac{bx^2}{2} + cx \right]_0^1 &= 1 \\ \Rightarrow \frac{1}{3} + \frac{b}{2} + c &= 1 \\ \Rightarrow b + 2c &= 4/3 \quad \dots\dots (i) \\ \text{And, } P(x) &= (x - 2) \cdot Q(x) + 5 \\ \text{When, } x &= 2 \\ P(2) &= 5 \\ 4 + 2b + c &= 5 \\ c &= 1 - 2b \quad \dots\dots (ii) \\ \text{Putting } c &= 1 - 2b \text{ in Eq. (i),} \\ b + 2(1 - 2b) &= 4/3 \\ \Rightarrow -3b + 2 &= 4/3 \\ \Rightarrow b &= 2/9 \\ \therefore c &= 1 - 4/9 = 5/9 \\ 9(b + c) &= 9 \left(\frac{2}{9} + \frac{5}{9} \right) = 7 \end{aligned}$$

Question 178

Let z and w be two complex numbers, such that $w = \bar{z}z - 2z + 2$, $\left| \frac{z+i}{z-3i} \right| = 1$ and $\text{Re}(w)$ has minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which w^n is real, is equal to
[2021, 16 March Shift-1]

Answer: 4

Solution:

$$\begin{aligned} \text{Given, } w &= \bar{z}z - 2z + 2 \\ \left| \frac{z+i}{z-3i} \right| &= 1 \\ \Rightarrow |z+i| &= |z-3i| \\ \text{Let } z &= x + iy \\ \Rightarrow |x + i(y+1)| &= |x + i(y-3)| \\ \Rightarrow x^2 + (y+1)^2 &= x^2 + (y-3)^2 \\ \Rightarrow 2y + 1 &= -6y + 9 \\ \therefore y &= 1 \\ \text{Now, } w &= \bar{z}z - 2z + 2 \\ w &= |z|^2 - 2z + 2 \\ \Rightarrow w &= x^2 + y^2 - 2(x + iy) + 2 \end{aligned}$$

$$\Rightarrow w = (x^2 + y^2 - 2x + 2) + i(-2y)$$

$$\Rightarrow w = (x^2 + 1 - 2x + 2) + i(-2)$$

$$w = (x - 1)^2 + 2 - 2i$$

Re(w) has minimum value.

So, $(x - 1)^2 + 2$ is minimum when $x = 1$

$$\therefore w = 2 - 2i$$

$$= 2(1 - i)$$

$$= 2\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)$$

$$w = 2\sqrt{2} e^{-i\pi/4}$$

$$\text{Now, } w^n = (2\sqrt{2})^n e^{-\frac{in\pi}{4}}$$

$$= (2\sqrt{2})^n \left[\cos\left(\frac{n\pi}{4}\right) - i \sin\left(\frac{n\pi}{4}\right) \right]$$

This has to be zero for w^n to be real.

$$\text{So, } \sin\left(\frac{n\pi}{4}\right) = 0$$

$$\Rightarrow \frac{n\pi}{4} = 0, \pi, 2\pi, 3\pi \dots$$

$$\Rightarrow n = 0, 4, 8, 12 \dots$$

The minimum value of n is $4 (n \in \mathbb{N})$.

Question 179

The least value of $|z|$, where z is a complex number which satisfies the inequality

$$\exp\left(\frac{(|z|+3)(|z|-1)}{|z|+1} \log_e 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|,$$

$i = \sqrt{-1}$, is equal to:

[2021, 16 March Shift-II]

Options:

A. 3

B. $\sqrt{5}$

C. 2

D. 8

Answer: A

Solution:

$$\exp\left[\frac{(|z|+3)(|z|-1)}{(|z|+1)} \times \log_e 2\right] \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|$$

$$\exp\left[\frac{(|z|+3)(|z|-1)}{(|z|+1)} \times \log_e 2\right] \geq \log_{\sqrt{2}} 16$$

$$\Rightarrow \log_{\sqrt{2}} |5\sqrt{7} + 9i|$$

$$\Rightarrow \frac{(|z|+3)(|z|-1)}{(|z|+1)} \geq 3$$

$$\Rightarrow |z|^2$$

$$\Rightarrow |z| + 1$$

$$\Rightarrow (|z| - 3)(|z| + 2) \geq 0$$

$$\Rightarrow |z| = 3$$

Question 180

If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to

[2021, 18 March Shift-II]

Answer: 0

Solution:

Method (1)

Given, $P(x) = f(x^3) + xg(x^3) \dots\dots\dots (i)$

$\therefore P(1) = f(1) + g(1) \dots\dots\dots (ii)$

Given, $P(x)$ is divisible by $(x^2 + x + 1)$.

$\therefore P(x) = Q(x) \cdot (x^2 + x + 1)$

As, we know that ω and ω^2 are non-real cube roots of unity and this is also root

of $x^2 + x + 1 = 0$

$\therefore P(\omega) = P(\omega^2) = 0$

As, we know that ω and ω^2 are non-real cube roots of unity and this is also root of $x^2 + x + 1 = 0$

$\therefore P(\omega) = P(\omega^2) = 0 \dots\dots\dots (iii)$

From Eq. (i),

$P(\omega) = f(\omega^3) + \omega[g(\omega^3)] = 0$ [from Eq. (iii)]

$\Rightarrow f(1) + \omega g(1) = 0 \dots\dots\dots (iv)$

and $P(\omega^2) = 0$ [from Eq. (iii)]

$\Rightarrow f(\omega^6) + \omega^2 \cdot g(\omega^6) = 0$

$\Rightarrow f(1) + \omega^2 g(1) = 0 \dots\dots\dots (v)$

Now, adding Eqs. (iv) and (v), we get

$2f(1) + (\omega + \omega^2)g(1) = 0$

$\Rightarrow 2f(1) - g(1) = 0$ ($\because 1 + \omega + \omega^2 = 0$)

$\Rightarrow 2f(1) = g(1) \dots\dots\dots (vi)$

Subtracting Eq. (iv) from Eq. (v), we get

$0 + (\omega - \omega^2)g(1) = 0$

$\Rightarrow g(1) = 0$

$f(1) = \frac{g(1)}{2} = \frac{0}{2}$ [from Eq. (vi)]

$\Rightarrow f(1) = 0$

From Eq. (ii), $P(1) = f(1) + g(1) = 0 + 0 = 0$

Method (2)

$\therefore P(\omega) = 0$

$\Rightarrow f(1) + \omega g(1) = 0$

$\Rightarrow f(1) + \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)g(1) = 0$

$\Rightarrow \left(f(1) - \frac{g(1)}{2}\right) + i\left(\frac{\sqrt{3}}{2}g(1)\right) = 0$

On comparing real and imaginary parts from both sides, we have

$11 \ f(1) - \frac{g(1)}{2} = 0, \quad \frac{\sqrt{3}}{2}g(1) = 0$

$\Rightarrow f(1) = \frac{g(1)}{2}, \quad \Rightarrow g(1) = 0$

$\therefore f(1) = \frac{0}{2} = 0$

$\therefore P(1) = f(1) + g(1) = 0 + 0 = 0$

Question 181

The value of $3 +$

$$4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \dots \infty}}}$$

is equal to

[2021, 18 March shift-I]

Options:

A. $1.5 + \sqrt{3}$

B. $2 + \sqrt{3}$

C. $3 + 2\sqrt{3}$

D. $4 + \sqrt{3}$

Answer: A

Solution:

$$\text{Let } x = 3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \dots}}}}$$

$$\text{So, } x = 3 + \frac{1}{4 + \frac{1}{x}} = 3 + \frac{1}{\frac{4x+1}{x}} = 3 + \frac{x}{4x+1}$$

$$\Rightarrow (x-3) = \frac{x}{4x+1}$$

$$\Rightarrow (4x+1)(x-3) = x$$

$$11 \Rightarrow 4x^2 - 12x - 3 = 0$$

$$\Rightarrow x = \frac{3 \pm 2\sqrt{3}}{2}$$

$$\Rightarrow x = \frac{3}{2} \pm \sqrt{3} = 1.5 \pm \sqrt{3}$$

But from above, $x > 0$

$\therefore$ Only positive value of x is accepted

$$\therefore x = 1.5 + \sqrt{3}$$

Question 182

Let C be the set of all complex numbers. Let

$$S_1 = \{z \in C \mid |z - 3 - 2i|^2 = 8\},$$

$$S_2 = \{z \in C \mid \operatorname{Re}(z) \geq 5\} \text{ and}$$

$$S_3 = \{z \in C \mid |z - \bar{z}| \geq 8\}.$$

Then, the number of elements in $S_1 \cap S_2 \cap S_3$ is equal to

[2021, 27 July Shift-1]

Options:

A. 1

B. 0

C. 2

D. Infinite

Answer: A

Solution:

$$S_1: |z - 3 - 2i|^2 = 8$$

$$\Rightarrow |(x+iy) - (3+2i)|^2 = 8$$

$$\Rightarrow |(x-3) + i(y-2)|^2 = 8$$

$$\Rightarrow (x-3)^2 + (y-2)^2 = 8$$

$$S_2: \operatorname{Re}(z) \geq 5$$

$$x \geq 5$$

$$S_3: |z - \bar{z}| \geq 8$$

$$|(x+iy) - (x-iy)| \geq 8$$

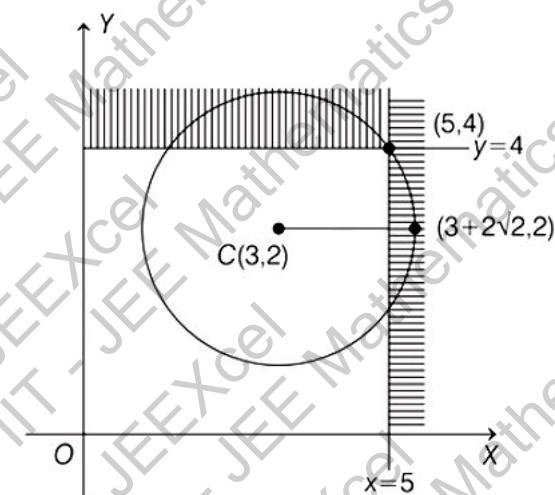
$$\Rightarrow 2y \geq 8$$

$$\Rightarrow y \geq 4$$

$$S_1: (x-3)^2 + (y-2)^2 = 8$$

$$S_2: x \geq 5$$

$$S_3: y \geq 4$$



Circle passes through (5, 4) as shown in the figure.
 $\Rightarrow$ There is exactly one point (5, 4) in $S_1 \cap S_2 \cap S_3$.

Question 183

The point $P(a, b)$ undergoes the following three transformations successively

(A) Reflection about the line $y = x$.

(B) Translation through 2 units along the positive direction of X-axis.

(C) Rotation through angle $\frac{\pi}{4}$

about the origin in the anti-clockwise direction.

If the co-ordinates of the final position of the point P are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$, then the value of $2a + b$ is equal to
 [2021, 27 July Shift-II]

Options:

- A. 13
- B. 9
- C. 5
- D. 7

Answer: B

Solution:

The image of $P(a, b)$ along $y = x$ is $Q(b, a)$. Translating it 2 units along the positive direction of X-axis, it becomes $R(b+2, a)$. Then, rotation through $\frac{\pi}{4}$ about the origin in the anticlockwise direction, the final position of the point P is

$$\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right).$$

Now, applying rotational theorem,

$$-\frac{1}{\sqrt{2}} + \frac{7}{\sqrt{2}}i = [(b+2) + ai] \cdot \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

$$\Rightarrow -\frac{1}{\sqrt{2}} + \frac{7}{\sqrt{2}}i = \left(\frac{b+2}{\sqrt{2}} - \frac{a}{\sqrt{2}}\right) + i\left(\frac{b+2}{\sqrt{2}} + \frac{a}{\sqrt{2}}\right)$$

$$\Rightarrow -\frac{1}{\sqrt{2}} + \frac{7}{\sqrt{2}}i = \left(\frac{b-a+2}{\sqrt{2}}\right) + i\left(\frac{a+b+2}{\sqrt{2}}\right)$$

$$\text{H So, } \frac{b-a+2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow b-a = -3 \quad \dots\dots (i)$$

$$\text{and } \frac{a+b+2}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$\Rightarrow a+b = 5 \quad \dots\dots (ii)$$

Adding Eqs. (i) and (ii),

$$2b = 2 \Rightarrow b = 1$$

Substitute the value of b in Eq. (ii), $a = 4$
 Now, $2a + b = 2 \times 4 + 1 = 9$

Question 184

Let C be the set of all complex numbers.

Let $S_1 = \{z \in C : |z - 2| \leq 1\}$ and $S_2 = \{z \in C : z(1+i) + \bar{z}(1-i) \geq 4\}$.

Then, the maximum value of $z - \frac{5}{2}$ for $z \in S_1 \cap S_2$ is equal to

[2021, 27 July Shift-II]

Options:

A. $\frac{3+2\sqrt{2}}{4}$

B. $\frac{5+2\sqrt{2}}{2}$

C. $\frac{3+2\sqrt{2}}{2}$

D. $\frac{5+2\sqrt{2}}{4}$

Answer: D

Solution:

Let $S_1 = \{z \in C : |z - 2| \leq 1\}$

and $S_2 = \{z \in C : z(1+i) + \bar{z}(1-i) \geq 4\}$

Now $|z - 2| \leq 1$

Let $z = x + iy$

$$\Rightarrow |x + iy - 2| \leq 1$$

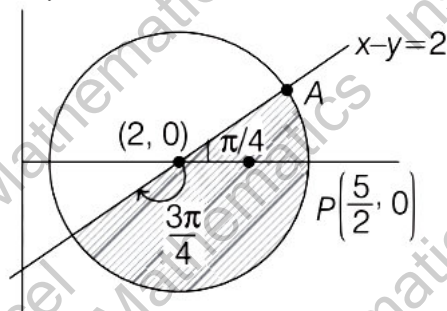
$$\Rightarrow (x-2)^2 + y^2 \leq 1$$

Also, $z(1+i) + \bar{z}(1-i) \geq 4$

$$\Rightarrow (x+iy)(1+i) + (x-iy)(1-i) \geq 4$$

$$\Rightarrow 2x - 2y \geq 4$$

$$\Rightarrow x - y \geq 2$$



Let point on circle be $A(2 + \cos \theta, \sin \theta)$.

$$1 \theta \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$$

$$(AP)^2 = \left(2 + \cos \theta - \frac{5}{2}\right)^2 + \sin^2 \theta$$

$$\Rightarrow (AP)^2 = \cos^2 \theta + \frac{1}{4} - \cos \theta + \sin^2 \theta$$

$$\Rightarrow (AP)^2 = \frac{5}{4} - \cos \theta$$

$$\text{For } (AP)^2 \text{ to be maximum, } \theta = -\frac{3\pi}{4}$$

$$\Rightarrow (AP)^2 = \frac{5}{4} + \frac{1}{\sqrt{2}}$$

$$\Rightarrow (AP)^2 = \frac{5+2\sqrt{2}}{4}$$

Question 185

Let α, β be two roots of the equation $x^2 + (20)^{1/4}x + (5)^{1/2} = 0$. Then, $\alpha^8 + \beta^8$ is equal to
[2021, 27 July Shift-I]

Options:

- A. 10
- B. 50
- C. 100
- D. 160

Answer: B

Solution:

$$x^2 + (20)^{1/4}x + (5)^{1/2} = 0$$

roots α & β .

$$\alpha + \beta = -(20)^{1/4}$$

$$\alpha\beta = (5)^{1/2}$$

$$\alpha^8 + \beta^8 = (\alpha^4)^2 + (\beta^4)^2$$

$$= (\alpha^4 - \beta^4)^2 + 2(\alpha\beta)^4 \dots\dots\dots (i)$$

$$\Rightarrow (\alpha + \beta)^2 = (\alpha^2 + \beta^2) + 2\alpha\beta$$

$$\Rightarrow (20)^{1/2} = (\alpha^2 + \beta^2) + 2 \cdot 5^{1/2}$$

$$\Rightarrow 2 \cdot (5)^{1/2} = (\alpha^2 + \beta^2) + 2 \cdot 5^{1/2}$$

$$\Rightarrow 0 = (\alpha^2 + \beta^2)$$

From eqn (1)

$$\alpha^8 + \beta^8 = ((\alpha^2 + \beta^2) \cdot (\alpha^2 - \beta^2))^2 + 2 \cdot (5)^{1/2}$$

$$= 0 + 2 \times 5^{1/2}$$

$$= 2 \times 25$$

$$= 50 \text{ (Ans)}$$

Question 186

The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$ is equal to
[2021, 27 July Shift-II]

Answer: 2

Solution:

Given equation,

$$e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$$

$$\text{Let } e^x = t > 0$$

$$t^4 - t^3 - 4t^2 - t + 1 = 0$$

$$\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$$

$$\Rightarrow t^2 + \frac{1}{t^2} + 2 - \left(t + \frac{1}{t}\right) - 6 = 0$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right) - 6 = 0$$

$$\text{Let } \alpha = t + \frac{1}{t} \geq 2$$

$$1 \Rightarrow \alpha^2 - \alpha - 6 = 0$$

$$\Rightarrow \alpha^2 - 3\alpha + 2\alpha - 6 = 0$$

$$\Rightarrow \alpha(\alpha - 3) + 2(\alpha - 3) = 0$$

$$\Rightarrow (\alpha - 3)(\alpha + 2) = 0$$

$$\Rightarrow \alpha = 3 \text{ or } \alpha = -2 \text{ (not possible)}$$

$$\Rightarrow t + \frac{1}{t} = 3$$

$$\Rightarrow t^2 - 3t + 1 = 0$$

$\therefore$ The number of real roots = 2

Question 187

The number of real roots of the equation

$$e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0 \text{ is}$$

[2021, 25 July Shift-1]

Options:

A. 2

B. 4

C. 6

D. 1

Answer: A

Solution:

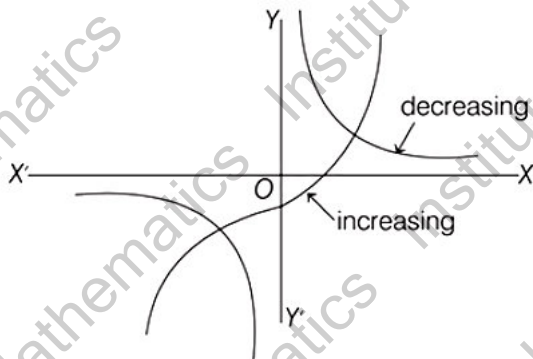
$$e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$$

$$\Rightarrow (e^{3x} - 1)^2 - e^x(e^{3x} - 1) = 12e^{2x}$$

$$\Rightarrow (e^{3x} - 1)(e^{3x} - e^x - 1) = 12e^{2x}$$

$$\Rightarrow (e^{3x} - 1)(e^x - e^{-x} - e^{-2x}) = 12$$

$$\Rightarrow e^x - e^{-x} - e^{-2x} = \frac{12}{e^{3x} - 1}$$



Hence, the number of real roots is 2.

Question 188

If α, β are roots of the equation.

$$x^2 + 5(\sqrt{2})x + 10 = 0, \alpha > \beta \text{ and}$$

$$P_n = \alpha^n - \beta^n \text{ for each positive}$$

integer n , then the value of

$$\left(\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2} \right) \text{ is equal to}$$

[2021, 25 July Shift-1]

Answer: 1

Solution:

$$x^2 + 5\sqrt{2}x + 10 = 0$$

$$P_n = \alpha^n - \beta^n$$

$$\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2} = \frac{P_{17}(5\sqrt{2}P_{19} + P_{20})}{P_{18}(5\sqrt{2}P_{18} + P_{19})}$$

$$\Rightarrow x^{18}(x^2 + 5\sqrt{2}x + 16) = 0$$

$$\Rightarrow x^{20} + 5\sqrt{2}x^{19} + x^{18} = 0$$

$$(\alpha^{20} - \beta^{20}) + 5\sqrt{2}(\alpha^{19} - \beta^{19}) + (\alpha^{18} - \beta^{18}) = 0$$

$$P_{20} + 5\sqrt{2}P_{19} + P_{18} = 0$$

Similarly,

$$P_{19} + 5\sqrt{2}P_{18} + P_{17} = 0$$

$$\text{So, } \frac{P_{17}(5\sqrt{2}P_{19} + P_{20})}{P_{18}(5\sqrt{2}P_{18} + P_{19})} = \frac{P_{17}(-P_{18})}{P_{18}(-P_{17})} = 1$$

Question 189

The number of real solutions of the equation $x^2 - |x| - 12 = 0$ is
[2021, 25 July Shift-II]

Options:

- A. 2
- B. 3
- C. 1
- D. 4

Answer: A

Solution:

Given equation,

$$x^2 - |x| - 12 = 0$$

$$\Rightarrow |x|^2 - |x| - 12 = 0$$

$$\Rightarrow |x|^2 - 4|x| + 3|x| - 12 = 0$$

$$\Rightarrow (|x| - 4)(|x| + 3) = 0$$

$$\text{So } |x| - 4 = 0 \text{ or } |x| + 3 = 0$$

$$|x| = 4 \text{ or } |x| = -3 \text{ (not possible)}$$

$$x = \pm 4$$

Hence, the number of real solutions = 2

Question 190

Let $[x]$ denote the greatest integer less than or equal to x . Then, the values of $x \in \mathbb{R}$ satisfying the equation $[e^x]^2 + [e^x + 1] - 3 = 0$ lie in the interval
[2021, 22 July Shift-II]

Options:

- A. $\left[0, \frac{1}{e}\right)$
- B. $[\log_e 2, \log_e 3)$
- C. $[1, e)$
- D. $[0, \log_e 2)$

Answer: D

Solution:

$$\begin{aligned} [e^x]^2 + [e^x + 1] - 3 &= 0 \\ \Rightarrow [e^x]^2 + [e^x] + 1 - 3 &= 0 \\ \Rightarrow [e^x]^2 + [e^x] - 2 &= 0 \\ \Rightarrow ([e^x] - 1)([e^x] + 2) &= 0 \\ [e^x] &= 1 \text{ or } [e^x] = -2 \\ \text{Not possible as } e^x &> 0. \\ \Rightarrow [e^x] &= 1 \\ \Rightarrow 1 \leq e^x < 2 \\ \Rightarrow 0 \leq x < \log_e 2 \end{aligned}$$

Question 191

If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to
[2021, 20 July Shift-1]

Options:

- A. 56×3^{25}
- B. 56×3^{24}
- C. 52×3^{24}
- D. 28×3^{25}

Answer: C

Solution:

$$\begin{aligned} x^2 + 3^{1/4}x + 3^{1/2} &= 0 \\ \therefore x &= \frac{-3^{1/4} \pm \sqrt{3^{1/2} - 4 \cdot 3^{1/2}}}{2} \\ &= \frac{3^{1/4}(-1 \pm \sqrt{3}i)}{2} \\ &= 3^{1/4} \left(\frac{-1 + \sqrt{3}i}{2} \right) \text{ or } 3^{1/4} \left(\frac{-1 - \sqrt{3}i}{2} \right) \\ &= 3^{1/4}\omega \text{ or } 3^{1/4}\omega^2 \\ &= 3^{1/4}\omega \text{ or } 3^{1/4}\omega^2 \\ \text{Now, } \alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1) &= \alpha^{108} - \alpha^{96} + \beta^{108} - \beta^{96} \\ &= (\alpha^{108} + \beta^{108}) - (\alpha^{96} + \beta^{96}) \\ &= \{(3^{1/4}\omega)^{108} + (3^{1/4}\omega^2)^{108}\} \\ &\quad - \{(3^{1/4}\omega)^{96} + (3^{1/4}\omega^2)^{96}\} \\ &= 3^{27}(\omega^{108} + \omega^{216}) - 3^{24}(\omega^{96} + \omega^{192}) \\ &= 3^{27}(2) - 3^{24}(2) = 3^{24}(54) - 3^{24}(2) \\ &= 3^{24}(52) = 52 \times 3^{24} \end{aligned}$$

Question 192

The number of solutions of the equation $\log_{(x+1)}(2x^2 + 7x + 5) +$

$$\log_{(2x+5)}(x+1)^2 - 4 = 0$$

$x > 0$, is

[2021, 20 July Shift-II]

Answer: 1

Solution:

$$\begin{aligned} & \log_{(x+1)}(2x^2 + 7x + 5) \\ & + \log_{(2x+5)}(x+1)^2 - 4 = 0 \\ & = \log_{(x+1)}\{(2x+5)(x+1)\} \\ & + 2\log_{(2x+5)}(x+1) - 4 = 0 \\ & = \log_{(x+1)}(2x+5) + \log_{(x+1)}(x+1) \\ & + 2\log_{(2x+5)}(x+1) - 4 = 0 \\ & = \log_{(x+1)}(2x+5) + 2\log_{(2x+5)}(x+1) - 3 = 0 \\ & [\because \log_a a = 1] \end{aligned}$$

$$= \log_{(x+1)}(2x+5) + 2 \frac{\log_{(x+1)}(x+1)}{\log_{(x+1)}(2x+5)} = 3$$

$$\text{Let } \log_{(x+1)}(2x+5) = t$$

$$t + \frac{2}{t} = 3 \Rightarrow t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$\Rightarrow t = 1, t = 2$$

$$\Rightarrow \log_{(x+1)}(2x+5) = 1 \text{ and}$$

$$\log_{(x+1)}(2x+5) = 2$$

$$2x+5 = (x+1)$$

$$\text{and } 2x+5 = (x+1)^2$$

$$x = -4$$

$$\text{and } 2x+5 = x^2 + 1 + 2x$$

$$\text{i.e., } x^2 = 4$$

$$\Rightarrow x = +2, -2$$

Given, $x > 0$

$x = -4, x = -2$ are discarded

$\therefore x = 2$ is only solution.

Question 193

If the real part of the complex number $z = \frac{3+2i \cos \theta}{1-3i \cos \theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is

zero, then the value of $\sin^2 3\theta + \cos^2 \theta$ is equal to

[2021, 27 July Shift-II]

Answer: 1

Solution:

We have,

$$1z = \frac{3+2i \cos \theta}{1-3i \cos \theta} = \frac{3+2i \cos \theta}{1-3i \cos \theta} \times \frac{1+3i \cos \theta}{1+3i \cos \theta}$$

$$= \frac{(3-6\cos^2 \theta) + i(9\cos \theta + 2\cos \theta)}{1+9\cos^2 \theta}$$

$$z = \frac{(3-6\cos^2 \theta) + (11\cos \theta)i}{1+9\cos^2 \theta}$$

Given, $\text{Re}(z) = 0$

$$\Rightarrow \frac{3-6\cos^2 \theta}{1+9\cos^2 \theta} = 0$$

$$\Rightarrow 3 - 6\cos^2\theta = 0$$

$$\Rightarrow \cos^2\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4} \left\{ \theta \in \left(0, \frac{\pi}{2} \right) \right\}$$

$$\Rightarrow = \frac{1}{2} + \frac{1}{2} = 1$$

Hence, $\sin^2 3\theta + \cos^2 \theta$

$$= \sin^2 \frac{3\pi}{4} + \cos^2 \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

Question 194

Let n denote the number of solutions of the equation $z^2 + 3\bar{z} = 0$, where z is a complex number. Then, the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to
[2021, 22 July Shift-11]

Options:

A. 1

B. $\frac{4}{3}$

C. $\frac{3}{2}$

D. 2

Answer: B

Solution:

$$z^2 + 3\bar{z} = 0$$

$$z = x + iy$$

$$\Rightarrow (x^2 - y^2) + i(2xy) + 3(x - iy) = 0$$

$$\Rightarrow (x^2 - y^2 + 3x) + i(2xy - 3y) = 0$$

$$\begin{cases} x^2 - y^2 + 3x = 0 \\ y(2x - 3) = 0. \end{cases}$$

$$y = 0 \text{ or } x = \frac{3}{2}$$

$$\text{If } y = 0,$$

$$\Rightarrow x(x + 3) = 0$$

$$\Rightarrow x = 0, -3$$

$$\Rightarrow \text{So, } (0, 0) \text{ and } (-3, 0) \text{ are solutions, when } y = 0.$$

$$\text{When } x = \frac{3}{2}, \quad \frac{9}{4} - y^2 + \frac{9}{2} = 0 \Rightarrow y^2 = \frac{27}{4}$$

$$\Rightarrow y = \pm \frac{3\sqrt{3}}{2}$$

$$\therefore \left(\frac{3}{2}, \frac{3\sqrt{3}}{2} \right) \text{ and } \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2} \right)$$

There are 4 solutions.

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\frac{1}{n^k} \right) &= 1 + \frac{1}{4} + \frac{1}{4^2} + \dots \infty \\ &= \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \end{aligned}$$

Question 195

If the real part of the complex number

$(1 - \cos \theta + 2i \sin \theta)^{-1}$ is $\frac{1}{5}$ or $\theta \in (0, \pi)$, then the value of the integral $\int_0^{\theta} \sin x \, dx$ is equal to

[2021, 22 July Shift-II]

Options:

- A. 1
- B. 2
- C. -1
- D. 0

Answer: A

Solution:

$$\text{Let } z = (1 - \cos \theta + 2i \sin \theta)^{-1}$$

$$\begin{aligned}\Rightarrow z &= \frac{1}{1 - \cos \theta + 2i \sin \theta} \\&= \frac{1}{1 - \cos \theta + 2i \sin \theta} \times \frac{1 - \cos \theta - 2i \sin \theta}{1 - \cos \theta - 2i \sin \theta} \\&= \frac{(1 - \cos \theta) - 2i \sin \theta}{(1 - \cos \theta)^2 - (2i \sin \theta)^2} \\&= \frac{2 \sin^2 \frac{\theta}{2} - 4i \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{4 \sin^4 \frac{\theta}{2} + 16 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \\&= \frac{2 \sin \frac{\theta}{2} \left(\sin \frac{\theta}{2} - 2i \cos \frac{\theta}{2} \right)}{4 \sin^2 \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + 4 \cos^2 \frac{\theta}{2} \right)} \\&= \frac{\sin \frac{\theta}{2} - 2i \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + 4 \cos^2 \frac{\theta}{2} \right)}\end{aligned}$$

$$\begin{aligned}\text{Now, } \operatorname{Re}(z) &= \frac{\sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + 4 \cos^2 \frac{\theta}{2} \right)} \\&= \frac{1}{2 \left(1 + 3 \cos^2 \frac{\theta}{2} \right)}\end{aligned}$$

$$\text{Given, } \operatorname{Re}(z) = \frac{1}{5}$$

$$\begin{aligned}\Rightarrow \frac{1}{2 \left(1 + 3 \cos^2 \frac{\theta}{2} \right)} &= \frac{1}{5} \\ \Rightarrow 1 + 3 \cos^2 \frac{\theta}{2} &= \frac{5}{2} \Rightarrow \cos^2 \frac{\theta}{2} = \frac{1}{2} \\ \Rightarrow \cos \frac{\theta}{2} &= \pm \frac{1}{\sqrt{2}} \\ \therefore \frac{\theta}{2} &= n\pi \pm \frac{\pi}{4} \\ \therefore \theta &= 2n\pi \pm \frac{\pi}{2}\end{aligned}$$

Given, range is $\theta \in (0, \pi)$.

$$\therefore \theta = \frac{\pi}{2}$$

$$\text{Now, } \int_0^{\theta} \sin x \, dx = \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin x \, dx &= -\cos x \Big|_0^{\pi/2} \\&= -\left(\cos \frac{\pi}{2} - \cos 0 \right) \\&= -(0 - 1) = 1\end{aligned}$$

Question 196

If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$, then $\arg\left(\frac{1 - 2z\bar{\omega}}{1 + 3z\bar{\omega}}\right)$ is
(Here, $\arg(z)$ denotes the principal argument of complex number z) [2021, 20 July Shift-1]

Options:

A. $\frac{\pi}{4}$

B. $-\frac{3\pi}{4}$

C. $-\frac{\pi}{4}$

D. $\frac{3\pi}{4}$

Answer: B

Solution:

$$|zW| = 1, \arg(z) - \arg(w) = \frac{3\pi}{2}$$

$$\text{Let } z = re^{i\theta}$$

$$w = \frac{1}{r} e^{i\left(\theta - \frac{3\pi}{2}\right)} \Rightarrow \bar{z} = re^{-i\theta}$$

$$w\bar{z} = \frac{1}{r} e^{i\left(\theta - \frac{3\pi}{2}\right)} \cdot re^{-i\theta}$$

$$\Rightarrow w\bar{z} = e^{i\left(\theta - \frac{3\pi}{2} - \theta\right)} = e^{-i\frac{3\pi}{2}}$$

$$\Rightarrow w\bar{z} = \cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)$$

$$\Rightarrow w\bar{z} = 0 + i$$

$$\Rightarrow w\bar{z} = i$$

$$\left(\frac{1-2w\bar{z}}{1+3w\bar{z}}\right) = \left(\frac{1-2i}{1+3i} \times \frac{1-3i}{1-3i}\right)$$

$$= \frac{1-2i-3i+6i^2}{10} = \frac{-5-5i}{10}$$

$$\therefore \arg = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

Question 197

Let Z_1 and Z_2 be two complex numbers such that $\arg(Z_1 - Z_2) = \frac{\pi}{4}$ and Z_1, Z_2 satisfy the equation

$|Z - 3| = \operatorname{Re}(Z)$. Then, the imaginary part of $Z_1 + Z_2$ is equal to

[2021, 27 Aug. Shift-11]

Answer: 6

Solution:

$$\text{Let } Z_1 = a_1 + ib_1, Z_2 = a_2 + ib_2$$

$$Z_1 - Z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

$$\arg(Z_1 - Z_2) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{b_1 - b_2}{a_1 - a_2}\right) = \frac{\pi}{4}$$

$$\Rightarrow b_1 - b_2 = a_1 - a_2$$

$$\text{Also, } |Z_1 - 3| = \operatorname{Re}(Z_1)$$

$$\Rightarrow (a_1 - 3)^2 + b_1^2 = a_1^2$$

$$\text{and } |Z_2 - 3| = \operatorname{Re}(Z_2)$$

$$\Rightarrow (a_2 - 3)^2 + b_2^2 = a_2^2$$

$$\Rightarrow (a_1 - 3)^2 - (a_2 - 3)^2 + b_1^2 - b_2^2$$

$$= a_1^2 - a_2^2$$

$$\Rightarrow (a_1 - a_2)(a_1 + a_2 - 6) + (b_1 - b_2)(b_1 + b_2)$$

$$= (a_1 - a_2)(a_1 + a_2)$$

$$\Rightarrow a_1 + a_2 - 6 + b_1 + b_2 = a_1 + a_2$$

$$\Rightarrow b_1 + b_2 = 6$$

$$\Rightarrow 1m(Z_1 + Z_2) = 6$$

[using Eq. (i).]

Question 198

The sum of the roots of the equation $x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$ is

[2021, 31 Aug. Shift-II]

Options:

A. $\log_2 14$

B. $\log_2 11$

C. $\log_2 12$

D. $\log_2 13$

Answer: B

Solution:

$$x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$$

$$\Rightarrow$$

$$x + 1 - 2\log_2(3 + 2^x) + \log_2\left(\frac{10 \cdot 2^x - 1}{2^x}\right) = 0$$

$$\Rightarrow x + 1 - 2\log_2(3 + 2^x) + \log_2(10 \cdot 2^x - 1) = 0$$

$$\Rightarrow 1 + \log_2\left(\frac{10 \cdot 2^x - 1}{(3 + 2^x)^2}\right) = 0$$

$$\Rightarrow \frac{10 \cdot 2^x - 1}{9 + (2^x)^2 + 6 \cdot 2^x} = \frac{1}{2}$$

$$\Rightarrow (2^x)^2 - 14 \cdot 2^x + 11 = 0$$

$$\text{Let } 2^x = y$$

$$\Rightarrow y^2 - 14y + 11 = 0$$

$$\text{Let } 2^x = y$$

$$\Rightarrow y^2 - 14y + 11 = 0$$

$$y = \frac{14 \pm \sqrt{152}}{2} = 7 \pm \frac{\sqrt{152}}{2}$$

$$y_1 = 7 + \frac{\sqrt{152}}{2},$$

$$y_2 = 7 - \frac{\sqrt{152}}{2}$$

$$\Rightarrow 2^{x_1} = 7 + \frac{\sqrt{152}}{2},$$

$$2^{x_2} = 7 - \frac{\sqrt{152}}{2}$$

$$\Rightarrow x_1 = \log_2\left(7 + \frac{\sqrt{152}}{2}\right)$$

$$x_2 = \log_2\left(7 - \frac{\sqrt{152}}{2}\right)$$

$$\therefore \text{Sum of roots} = x_1 + x_2$$

$$= \log_2\left(49 - \frac{152}{4}\right) = \log_2 11$$

Question 199

The number of distinct real roots of the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$ is

[2021, 27 Aug. Shift-I]

Answer: 4

Solution:

$$\text{Let } f(x) = 3x^4 + 4x^3 - 12x^2 + 4 = 0$$

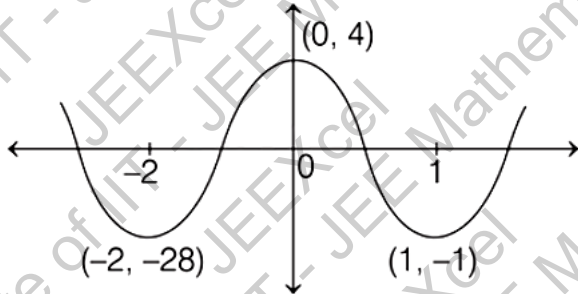
Differentiating w.r.t. x ,

$$f'(x) = 12x^3 + 12x^2 - 24x = 0$$

$$\Rightarrow 12x(x^2 + x - 2) = 0$$

$$\Rightarrow x(x+2)(x-1) = 0$$

Critical point $x = 0, 1, -2$



Graph of $y = f(x)$

Number of real roots = 4

Question 200

The set of all values of $k > -1$, for which the equation $(3x^2 + 4x + 3)^2 - (k+1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$ has real roots, is
[2021, 27 Aug. Shift-II]

Options:

A. $\left[1, \frac{5}{2}\right]$

B. $[2, 3]$

C. $\left[-\frac{1}{2}, 1\right)$

D. $\left[\frac{1}{2}, \frac{3}{2}\right] - \{1\}$

Answer: A

Solution:

Given,

$$(3x^2 + 4x + 3)^2 - (k+1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$$

$$(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$$

$$\text{Let } y = 3x^2 + 4x + 2$$

Then, given equation becomes

$$(y+1)^2 - (k+1)y(y+1) + ky^2 = 0$$

$$\Rightarrow y^2 + 2y + 1 - ky^2 - ky - y^2 - y + ky^2 = 0$$

$$\Rightarrow y + 1 - ky = 0$$

$$\Rightarrow y(1-k) = -1$$

$$\Rightarrow y = \frac{1}{k-1}$$

$$\Rightarrow 3x^2 + 4x + 2 - \frac{1}{k-1} = 0$$

For real roots, $D \geq 0$

$$\Rightarrow 16 - 4 \cdot 3 \cdot \left(2 - \frac{1}{k-1}\right) \geq 0$$

$$\Rightarrow -8 + \frac{12}{k-1} \geq 0 \Rightarrow \frac{3}{k-1} \geq 2$$

$$\Rightarrow \frac{3-2k+2}{k-1} \leq 0 \Rightarrow \frac{2k-5}{k-1} \leq 0$$

$$\Rightarrow k \in \left(1, \frac{5}{2}\right] \quad [\because k \neq 1]$$

Question 201

The sum of all integral values of k ($k \neq 0$) for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is
[2021, 26 Aug. Shift-I]

Answer: 66

Solution:

$$\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k} \Rightarrow x \in \mathbb{R} - \{1, 2\}$$

$$k(2x - 4 - x + 1) = 2(x^2 - 3x + 2)$$

$$k(x - 3) = 2(x^2 - 3x + 2)$$

$$2x^2 - (6+k)x + 3k + 4 = 0$$

$$\text{For no real roots } b^2 - 4ac < 0$$

$$\therefore (k+6)^2 - 8 \cdot (3k+4) < 0$$

$$\Rightarrow k^2 - 12k - 4 < 0$$

$$\Rightarrow (k-6)^2 - 32 < 0$$

$$\Rightarrow (k-6)^2 < 32$$

$$\Rightarrow -4\sqrt{2} < k-6 < 4\sqrt{2}$$

$$\Rightarrow 6-4\sqrt{2} < k < 6+4\sqrt{2}$$

$$\text{Integral } k \in \{1, 2, 3, 4, \dots, 11\}$$

$$\text{Sum} = 66$$

Question 202

If $(\sqrt{3} + i)^{100} = 2^{99}(p + iq)$, then p and q are roots of the equation
[2021, 26 Aug. Shift-III]

Options:

A. $x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0$

B. $x^2 + (\sqrt{3} + 1)x + \sqrt{3} = 0$

C. $x^2 + (\sqrt{3} - 1)x - \sqrt{3} = 0$

D. $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$

Answer: A

Solution:

$$(\sqrt{3} + i)^{100} = 2^{99}(p + iq)$$

$$2^{100} e^{i100 \frac{\pi}{6}} = 2^{99}(p + iq)$$

$$\Rightarrow 2e^{i \frac{2\pi}{3}} = p + iq$$

$$\Rightarrow 2 \left[\cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right) \right] = p + iq$$

$$\Rightarrow (-1 + i\sqrt{3}) = p + iq$$

$$\Rightarrow p = -1 \text{ and } q = \sqrt{3}$$

$$\text{Equation whose roots are } -1 \text{ and } \sqrt{3}i \text{ is}$$

$$\Rightarrow (x+1)(x-\sqrt{3})=0$$

$$x^2-(\sqrt{3}-1)x-\sqrt{3}=0$$

Question 203

Let $\lambda \neq 0$ be in $\mathbb{R}$. If α and β are the roots of the equation $x^2 - x + 2\lambda = 0$ and α and γ are the roots of equation $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to
[2021, 26 Aug. Shift-II]

Answer: 18

Solution:

We have, α is common root of the equations $x^2 - x + 2\lambda = 0$ and $3x^2 - 10x + 27\lambda = 0$.

Now, common root of these equations is $(3\alpha^2 - 10\alpha + 27\lambda) - (3\alpha^2 - 3\alpha + 6\lambda) = 0 \Rightarrow -7\alpha + 21\lambda = 0$

$$\Rightarrow \alpha = 3\lambda$$

Again, α is root of $x^2 - x + 2\lambda = 0$

$$\therefore \alpha^2 - \alpha + 2\lambda = 0$$

$$\Rightarrow (3\lambda)^2 - 3\lambda + 2\lambda = 0$$

$$\Rightarrow 9\lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda(9\lambda - 1) = 0$$

$$\Rightarrow \lambda = 0, \frac{1}{9}$$

$$\Rightarrow \lambda = \frac{1}{9} \quad [\because \lambda \neq 0]$$

$$\therefore \alpha = 3\lambda = 3 \times \frac{1}{9} = \frac{1}{3}$$

Again, α and β are roots of the equation

$$x^2 - x + 2\lambda = 0$$

$$11 \therefore \alpha + \beta = \frac{-(-1)}{1} = 1$$

$$\Rightarrow \beta = 1 - \alpha = 1 - \frac{1}{3} = \frac{2}{3}$$

And α and γ are the roots of the equation $3x^2 - 10x + 27\lambda = 0$

$$\therefore \alpha + \gamma = \frac{-(-10)}{3} = \frac{10}{3}$$

$$\Rightarrow \gamma = \frac{10}{3} - \alpha = \frac{10}{3} - \frac{1}{3} = \frac{9}{3} = 3$$

$$\therefore \frac{\beta\gamma}{\lambda} = \frac{\left(\frac{2}{3}\right) \times (3)}{\left(\frac{1}{9}\right)} = 18$$

Question 204

If $S = \left\{ z \in \mathbb{C} : \frac{z-i}{z+2i} \in \mathbb{R} \right\}$, then

[2021, 27 Aug. Shift-1]

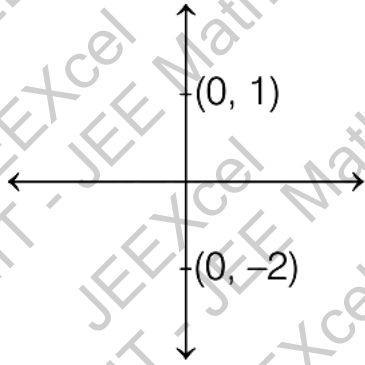
Options:

- A. S contains exactly two elements.
- B. S contains only one element.
- C. S is a circle in the complex plane.
- D. S is a straight line in the complex plane.

Answer: D

Solution:

Given, $\frac{z-i}{z+2i} \in \mathbb{R}$



$\Rightarrow \arg\left(\frac{z-i}{z+2i}\right) = 0 \text{ or } \pi$

$\Rightarrow i, -2i, z$ are collinear.

$\Rightarrow S$ is a straight line in the complex plane.

Question 205

Let $z = \frac{1-i\sqrt{3}}{2}$ and $i = \sqrt{-1}$. Then the value of $21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 + \dots + \left(z^{21} + \frac{1}{z^{21}}\right)^3$ is
[2021, 26 Aug. Shift-1]

Answer: 13

Solution:

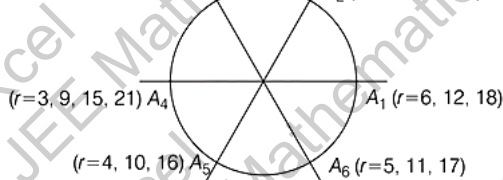
$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{-i\pi/3}$

Again, $z^r + \frac{1}{z^r} = z^r + \bar{z}^r = 2\operatorname{Re}(z^r)$

$[\because |z^r| = 1] = 2 \cos\left(\frac{r\pi}{3}\right)$

$21 + \sum_{r=1}^{21} \left(z^r + \frac{1}{z^r}\right)^3 = 21 + \sum_{r=1}^{21} 8 \cos^3\left(\frac{r\pi}{3}\right)$

$(r=2, 8, 14, 20) A_3 \quad A_2 (r=1, 7, 13, 19)$



Now, all the diametric ends will cancel out each other. Only a single value at A_1 will remain which is -1 .

So, $21 + 8(-1) = 13$

Question 206

The equation $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents a circle with
[2021, 26 Aug. Shift-I]

Options:

- A. centre at $(0, -1)$ and radius $\sqrt{2}$
- B. centre at $(0, 1)$ and radius $\sqrt{2}$
- C. centre at $(0, 0)$ and radius $\sqrt{2}$
- D. centre at $(0, 1)$ and radius 2

Answer: B

Solution:

$$\text{We have, } \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \arg(z-1) - \arg(z+1) = \frac{\pi}{4}$$

$$\text{Let } z = x + iy$$

$$\arg[(x-1) + iy] - \arg[(x+1) + iy] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1} \cdot \frac{y}{x+1}} \right) = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{y(x+1) - y(x-1)}{(x^2-1) + y^2} = 1$$

$$\Rightarrow 2y = x^2 + y^2 - 1$$

$$\Rightarrow x^2 + y^2 - 2y - 1 = 0$$

$$\Rightarrow x^2 + (y-1)^2 = 2$$

$$\Rightarrow x^2 + (y-1)^2 = (\sqrt{2})^2$$

Which is a circle with Centre $(0, 1)$ and Radius $= \sqrt{2}$ units

Question 207

A point z moves in the complex plane such that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$, then the minimum value of

$|z - 9\sqrt{2} - 2i|^2$ equal to
[2021, 31 Aug. Shift-I]

Answer: 98

Solution:

$$\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

If

$$z = x + iy$$

$$\arg[(x-2) + iy] - \arg[(x+2) + iy] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-2}\right) - \tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{y}{x-2} - \frac{y}{x+2}}{1 + \frac{y}{x-2} \cdot \frac{y}{x+2}} = \tan\left(\frac{\pi}{4}\right)$$

$$1 \Rightarrow \frac{xy + 2y - xy + 2y}{x^2 + y^2 - 4} = 1$$

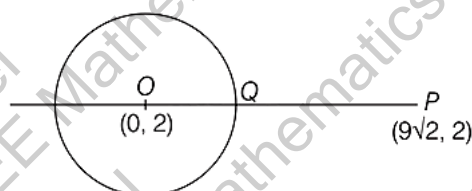
$$\Rightarrow 4y = x^2 + y^2 - 4$$

$$\Rightarrow x^2 + y^2 - 4y - 4 = 0$$

z is a circle.

Centre $= (0, 2)$, Radius $= (2\sqrt{2})$

$|z - 9\sqrt{2} - 2i|^2$ is the distance of $(9\sqrt{2}, 2)$ from any point on circle. Distance will be minimum when $(9\sqrt{2}, 2)$ will lie on the line joining the centre.



$$PQ = OP - OQ$$

$$= 9\sqrt{2} - 2\sqrt{2} = 7\sqrt{2}$$

$$PO^2 = (7\sqrt{2})^2 = 98$$

Question 208

If z is a complex number such that $\frac{z-i}{z-1}$ is purely imaginary, then the minimum value of $|z - (3 + 3i)|$ is [2021, 31 Aug. Shift-II]

Options:

- A. $2\sqrt{2} - 1$
- B. $6\sqrt{2}$
- C. $3\sqrt{2}$
- D. $2\sqrt{2}$

Answer: D

Solution:

Let $z = x + iy$

$$\frac{z-i}{z-1} = \frac{x+i(y-1)}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

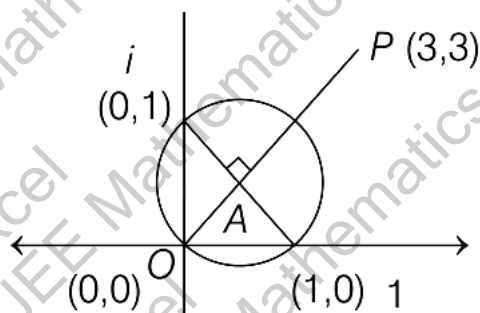
$$= \frac{x(x-1)+y(y-1)+i \left[\frac{(x-1)(y-1)-xy}{(x-1)^2+y^2} \right]}$$

As $\frac{z-i}{z-1}$ is purely imaginary,

$$x^2 + y^2 - x - y = 0$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 0$$

This is a circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$, radius = $\frac{1}{2}$ which passes through origin as shown in the figure.



$$\text{Minimum } |z - (3 + 3i)| = OP - OA$$

$$= \sqrt{(3-0)^2 + (3-0)^2} - \sqrt{\left(\frac{1}{2}-0\right)^2 + \left(\frac{1}{2}-0\right)^2}$$

$$= 3\sqrt{2} - \sqrt{2}$$

$$= 2\sqrt{2}$$

Question 209

The least positive integer n such that $\frac{(2i)^n}{(1-i)^{n-2}}$, $i = \sqrt{-1}$, is a positive integer, is [2021, 26 Aug. Shift-11]

Answer: 6

Solution:

We have,

$$\begin{aligned}\frac{(2i)^n}{(1-i)^{n-2}} &= \frac{(2i)^n}{(1-i)^n(1-i)^{-2}} \\&= \left(\frac{2i}{1-i}\right)^n (1-i)^2 \\&= \left[\frac{2i(1+i)}{(1-i)(1+i)}\right]^n (1+i^2-2i) \\&= \left(\frac{2i-2}{2}\right)^n (1-1-2i) \\&= (i-1)^n(-2i)\end{aligned}$$

$$\text{If } n=1, (i-1)(-2i) = -2i^2 + 2i = 2 + 2i$$

$$\text{If } n=2, -2i(i-1)^2 = -2i(-2i) = -4$$

$$\text{If } n=4, -2i(i-1)^4 = -2i(-2i)(-2i) = 8i$$

$$\text{If } n=6, -2i(i-1)^6 = -2i(-2i)(-2i)(-2i) = 16$$

So, least value of n for which given complex is positive is 6.

Question 210

The numbers of pairs (a, b) of real numbers, such that whenever α is a root of the equation $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of this equation, is [2021, 01 Sep. Shift-II]

Options:

- A. 6
- B. 2
- C. 4
- D. 8

Answer: A

Solution:

Given equation $x^2 + ax + b = 0$

It has two roots (not necessarily real α and β)

$\Rightarrow$ Either $\alpha = \beta$ or $\alpha \neq \beta$

$$1. \text{ If } \alpha = \beta \Rightarrow \alpha = \alpha^2 - 2 \Rightarrow \alpha = -1, 2$$

When $\alpha = -1$, then $(a, b) = (2, 1)$

When $\alpha = 2$, then $(a, b) = (-4, 4)$

II. If $\alpha \neq \beta$, then

$$(a) \alpha = \alpha^2 - 2 \text{ and } \beta = \beta^2 - 2$$

Here, $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$

Hence $(a, b) = (-\alpha - \beta, \alpha\beta) = (-1, -2)$

$$(b) \alpha = \beta^2 - 2 \text{ and } \beta = \alpha^2 - 2$$

$$\text{Then } \alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$$

$$\therefore \alpha \neq \beta$$

$$\Rightarrow \alpha + \beta = \beta^2 + \alpha^2 - 4$$

$$\text{or } \alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta - 4$$

$$\Rightarrow -1 = 1 - 2\alpha\beta - 4 \Rightarrow \alpha\beta = -1$$

$$\Rightarrow (a, b) = (-\alpha - \beta, \alpha\beta) = (1, -1)$$

$$(c) \alpha = \alpha^2 - 2 = \beta^2 - 2 \text{ and } \alpha \neq \beta \Rightarrow \alpha = -\beta$$

Thus, $\alpha = 2, \beta = -2$

or $\alpha = -1, \beta = 1$

$$\therefore (a, b) = (0, -4) \text{ and } (0, -1)$$

$$(d) \beta = \alpha^2 - 2 = \beta^2 - 2 \text{ and } \alpha \neq \beta \text{ (as in (c))}$$

$\Rightarrow$ We get 6 pairs of (a, b)

They are (2, 1), (-4, 4), (-1, -2), (1, -1), (0, -4), and (0, -1).

Question 211

If $\frac{3 + i \sin \theta}{4 - i \cos \theta}$, $\theta \in [0, 2\pi]$, is a real number, then an argument of $\sin \theta + i \cos \theta$ is:

[Jan. 7, 2020 (II)]

Options:

A. $\pi - \tan^{-1}\left(\frac{4}{3}\right)$

B. $\pi - \tan^{-1}\left(\frac{3}{4}\right)$

C. $-\tan^{-1}\left(\frac{3}{4}\right)$

D. $\tan^{-1}\left(\frac{4}{3}\right)$

Answer: A

Solution:

Let $z = \frac{3 + i \sin \theta}{4 - i \cos \theta}$, after rationalising

$$z = \frac{(3 + i \sin \theta) \times (4 + i \cos \theta)}{(4 - i \cos \theta)(4 + i \cos \theta)}$$

As z is purely real

$$\Rightarrow 3 \cos \theta + 4 \sin \theta = 0 \Rightarrow \tan \theta = -\frac{3}{4}$$

$$\arg(\sin \theta + i \cos \theta) = \pi + \tan^{-1}\left(\frac{\cos \theta}{\sin \theta}\right)$$

$$= \pi + \tan^{-1}\left(-\frac{4}{3}\right) = \pi - \tan^{-1}\left(\frac{4}{3}\right)$$

Question 212

Let z be a complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|Z| = \frac{5}{2}$. Then the value of $|Z + 3i|$ is:

[Jan. 9, 2020 (I)]

Options:

A. $\sqrt{10}$

B. $\frac{7}{2}$

C. $\frac{15}{4}$

D. $2\sqrt{3}$

Answer: B

Solution:

Let $z = x + iy$

$$\text{Then, } \left|\frac{z-i}{z+2i}\right| = 1 \Rightarrow x^2 + (y-1)^2$$

$$= x^2 + (y+2)^2 \Rightarrow -2y + 1 = 4y + 4$$

$$\Rightarrow 6y = -3 \Rightarrow y = -\frac{1}{2}$$

$$\therefore |z| = \frac{5}{2} \Rightarrow x^2 + y^2 = \frac{25}{4}$$

$$\Rightarrow x^2 = \frac{24}{4} = 6$$

$$\therefore z = x + iy \Rightarrow z = \pm\sqrt{6} - \frac{i}{2}$$

$$|z + 3i| = \sqrt{6 + \frac{25}{4}} = \sqrt{\frac{49}{4}}$$

$$\Rightarrow |z + 3i| = \frac{7}{2}$$

Question 213

If z be a complex number satisfying $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| = 4$, then $|z|$ cannot be:

[Jan. 9, 2020 (II)]

Options:

A. $\sqrt{\frac{17}{2}}$

B. $\sqrt{10}$

C. $\sqrt{7}$

D. $\sqrt{8}$

Answer: C

Solution:

$$z = x + iy \quad |x| + |y| = 4$$

$$|z| = \sqrt{x^2 + y^2}$$

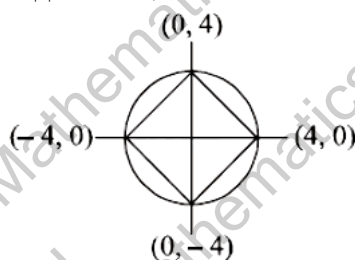
Minimum value of

$$|z| = 2\sqrt{2}$$

Maximum value of

$$|z| = 4 \quad |z| \in [\sqrt{8}, \sqrt{16}]$$

So, $|z|$ can't be $\sqrt{7}$.



Question 214

Let $\alpha = \frac{-1 + i\sqrt{3}}{2}$. If $a = (1 + \alpha) \sum_{k=0}^{100} \alpha^{2k}$ and $b = \sum_{k=0}^{100} \alpha^{3k}$, then a and b are the roots of the quadratic equation:

[Jan. 8, 2020 (II)]

Options:

A. $x^2 + 101x + 100 = 0$

B. $x^2 - 102x + 101 = 0$

C. $x^2 - 101x + 100 = 0$

D. $x^2 + 102x + 101 = 0$

Answer: B

Solution:

$$\begin{aligned}\text{Let } \alpha &= \omega, b = 1 + \omega^3 + \omega^6 + \dots = 101 \\ a &= (1 + \omega)(1 + \omega^2 + \omega^4 + \dots + \omega^{198} + \omega^{200}) \\ &= (1 + \omega) \frac{(1 - (\omega^2)^{101})}{1 - \omega^2} = \frac{(\omega + 1)(\omega^{202} - 1)}{(\omega^2 - 1)} \\ \Rightarrow a &= \frac{(1 + \omega)(1 - \omega)}{1 - \omega^2} = 1\end{aligned}$$

$$\begin{aligned}\text{Required equation} &= x^2 - (101 + 1)x + (101) \times 1 = 0 \\ \Rightarrow x^2 - 102x + 101 &= 0\end{aligned}$$

Question 215

If $\operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x, y) lies on a :

[Jan. 7, 2020 (I)]

Options:

- A. circle whose centre is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$
- B. straight line whose slope is $-\frac{2}{3}$.
- C. straight line whose slope is $\frac{3}{2}$.
- D. circle whose diameter is $\frac{\sqrt{5}}{2}$.

Answer: D

Solution:

$$\begin{aligned}\because z &= x + iy \\ \left(\frac{z-1}{2z+i}\right) &= \frac{(x-1)+iy}{2(x+iy)+i} \\ &= \frac{(x-1)+iy}{2x+(2y+1)i} \times \frac{2x-(2y+1)i}{2x-(2y+1)i} \\ \operatorname{Re}\left(\frac{z-1}{2z+i}\right) &= \frac{2x(x-1)+y(2y+1)}{(2x)^2+(2y+1)^2} = 1 \\ \Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 &= \left(\frac{\sqrt{5}}{4}\right)^2\end{aligned}$$

Question 216

The number of real roots of the equation, $e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$ is:

[Jan. 9, 2020 (I)]

Options:

- A. 1
- B. 3
- C. 2
- D. 4

Answer: A

Solution:

Let $e^x = t \in (0, \infty)$

Given equation

$$t^4 + t^3 - 4t^2 + t + 1 = 0$$

$$\Rightarrow t^2 + t - 4 + \frac{1}{t} + \frac{1}{t^2} = 0$$

$$\Rightarrow \left(t^2 + \frac{1}{t^2}\right) + \left(t + \frac{1}{t}\right) - 4 = 0$$

$$\text{Let } t + \frac{1}{t} = y$$

$$(y^2 - 2) + y - 4 = 0 \Rightarrow y^2 + y - 6 = 0$$

$$y^2 + y - 6 = 0 \Rightarrow y = -3, 2$$

$$\Rightarrow y = 2 \Rightarrow t + \frac{1}{t} = 2$$

$$\Rightarrow e^x + e^{-x} = 2$$

$x = 0$, is the only solution of the equation

Hence, there only one solution of the given equation.

Question 217

The least positive value of 'a' for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is

[Jan. 8, 2020 (I)]

Answer: 8

Solution:

Since, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots,

$$\therefore D \geq 0$$

$$\Rightarrow (a - 10)^2 - 4(2)\left(\frac{33}{2} - 2a\right) \geq 0$$

$$\Rightarrow (a - 10)^2 - 4(33 - 4a) \geq 0$$

$$\Rightarrow a^2 - 4a - 32 \geq 0$$

$$\Rightarrow (a - 8)(a + 4) \geq 0$$

$$\Rightarrow a \leq -4 \cup a \geq 8$$

$$\Rightarrow a \in (-\infty, -4] \cup [8, \infty)$$

Question 218

If the equation, $x^2 + bx + 45 = 0$ ($b \in \mathbb{R}$) has conjugate complex roots and they satisfy $|z + 1| = 2\sqrt{10}$, then:

[Jan. 8, 2020 (I)]

Options:

A. $b^2 - b = 30$

B. $b^2 + b = 72$

C. $b^2 - b = 42$

D. $b^2 + b = 12$

Answer: A

Solution:

Let $z = \alpha \pm i\beta$ be the complex roots of the equation

So, sum of roots $= 2\alpha = -b$ and

Product of roots $= \alpha^2 + \beta^2 = 45$

$(\alpha + 1)^2 + \beta^2 = 40$

Given, $|z + 1| = 2\sqrt{10}$

$\Rightarrow (\alpha + 1)^2 - \alpha^2 = -5$ [$\because \beta^2 = 45 - \alpha^2$]

$\Rightarrow 2\alpha + 1 = -5 \Rightarrow 2\alpha = -6$

Hence, $b = 6$ and $b^2 - b = 30$

Question 219

Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If $p_k = (\alpha)^k + (\beta)^k$, $k \geq 1$, then which one of the following statements is not true?

[Jan. 7, 2020 (II)]

Options:

A. $p_3 = p_5 - p_4$

B. $p_5 = 11$

C. $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

D. $p_5 = p_2 \cdot p_3$

Answer: D

Solution:

$$\alpha^5 = 5\alpha + 3$$

$$\beta^5 = 5\beta + 3$$

$$p_5 = 5(\alpha + \beta) + 6 = 5(1) + 6$$

$$\left[\because \text{from } x^2 - x - 1 = 0, \alpha + \beta = \frac{-b}{a} = 1 \right]$$

$$p_5 = 11 \text{ and } p_5 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1$$

$$p_2 = 3 \text{ and } p_3 = \alpha^3 + \beta^3 = 2\alpha + 1 + 2\beta + 1$$

$$= 2(1) + 2 = 4$$

$$p_2 \times p_3 = 12 \text{ and } p_5 = 11 \Rightarrow p_5 \neq p_2 \times p_3$$

Question 220

Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2} \cdot \lambda \tan x = (1 - k)$, where $k (\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is:

[Jan. 7, 2020 (I)]

Options:

A. $10\sqrt{2}$

B. 10

C. 5

D. $5\sqrt{2}$

Answer: B

Solution:

$$(k+1)\tan^2 x - \sqrt{2}\lambda \tan x + (k-1) = 0$$

$$\tan \alpha + \tan \beta = \frac{\sqrt{2}\lambda}{k+1} \quad [\text{Sum of roots}]$$

$$\tan \alpha \cdot \tan \beta = \frac{k-1}{k+1} \quad [\text{Product of roots}]$$

$$\therefore \tan(\alpha + \beta) = \frac{\frac{\sqrt{2}\lambda}{k+1}}{1 - \frac{k-1}{k+1}} = \frac{\sqrt{2}\lambda}{2} = \frac{\lambda}{\sqrt{2}}$$

$$\tan^2(\alpha + \beta) = \frac{\lambda^2}{2} = 50$$

$$\lambda = 10$$

Question 221

Let $a, b \in \mathbb{R}$, $a \neq 0$ be such that the equation, $ax^2 - 2bx + 5 = 0$ has a repeated root α , which is also a root of the equation, $x^2 - 2bx - 10 = 0$. If β is the other root of this equation, then $\alpha^2 + \beta^2$ is equal to :
[Jan. 9, 2020 (II)]

Options:

- A. 25
- B. 26
- C. 28
- D. 24

Answer: A

Solution:

$$ax^2 - 2bx + 5 = 0$$

If α and α are roots of equations, then sum of roots

$$2\alpha = \frac{2b}{a} \Rightarrow \alpha = \frac{b}{a}$$

$$\text{and product of roots} = \alpha^2 = \frac{5}{a} \Rightarrow \frac{b^2}{a^2} = \frac{5}{a}$$

$$\Rightarrow b^2 = 5a \quad (a \neq 0) \quad \dots (i)$$

$$\text{For } x^2 - 2bx - 10 = 0$$

$$\alpha + \beta = 2b \quad \dots (ii)$$

$$\text{and } \alpha\beta = -10 \quad \dots (iii)$$

$$\alpha = \frac{b}{a} \text{ is also root of } x^2 - 2bx - 10 = 0$$

$$\Rightarrow b^2 - 2ab^2 - 10a^2 = 0$$

$$\text{By eqn. (i)} \Rightarrow 5a - 10a^2 - 10a^2 = 0$$

$$\Rightarrow 20a^2 = 5a$$

$$\Rightarrow a = \frac{1}{4} \text{ and } b^2 = \frac{5}{4}$$

$$\alpha^2 = 20 \text{ and } \beta^2 = 5$$

$$\text{Now, } \alpha^2 + \beta^2 = 5 + 20 = 25$$

Question 222

If the four complex numbers $z, \bar{z}, \bar{z} - 2\text{Re}(\bar{z})$ and $z - 2\text{Re}(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to :
[Sep. 05, 2020 (I)]

Options:

- A. $4\sqrt{2}$
- B. 4
- C. $2\sqrt{2}$

D. 2

Answer: C

Solution:

$$D(z - 2\operatorname{Re}(z)) \quad C(\bar{z} - 2\operatorname{Re}(\bar{z}))$$



Let $z = x + iy$.

$\therefore$ Length of side of square = 4 units

Then, $|z - \bar{z}| = 4 \Rightarrow |2iy| = 4 \Rightarrow |y| = 2$

Also, $|z - (z - 2\operatorname{Re}(z))| = 4$

$\Rightarrow |2\operatorname{Re}(z)| = 4 \Rightarrow |2x| = 4 \Rightarrow |x| = 2$

$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = 2\sqrt{2}$

Question 223

The value of $\left(\frac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is:

[Sep. 05, 2020 (II)]

Options:

A. -2^{15}

B. $2^{15}i$

C. $-2^{15}i$

D. 6^5

Answer: C

Solution:

$$\because -1 + \sqrt{3}i = 2 \cdot e^{\frac{2\pi i}{3}} \text{ and } 1 - i = \sqrt{2} \cdot e^{-\frac{i\pi}{4}}$$

$$\therefore \left(\frac{-1 + \sqrt{3}i}{1 - i}\right)^{30} = \left(\frac{2}{\sqrt{2}} e^{\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)i}\right)^{30}$$

$$= 2^{15} \cdot e^{-\frac{\pi i}{2}} = -2^{15}i.$$

Question 224

If $\left(\frac{1+i}{1-i}\right)^m = \left(\frac{1+i}{1-i}\right)^n = 1$, ($n, m \in \mathbb{N}$), then the greatest common divisor of the least values of m and n is _____.

[Sep. 03, 2020 (I)]

Answer: 4

Solution:

$$\begin{aligned}\text{Given that } \left(\frac{1+i}{1-i}\right)^{m/2} &= \left(\frac{1+i}{1-i}\right)^{n/3} = 1 \\ \Rightarrow \left(\frac{(1+i)^2}{2}\right)^{m/2} &= \left(\frac{(1+i)^2}{-2}\right)^{n/3} = 1 \\ \Rightarrow i^{m/2} &= (-i)^{n/3} = 1 \\ m(\text{least}) &= 8, n(\text{least}) = 12 \\ \text{GCD}(8, 12) &= 4\end{aligned}$$

Question 225

If Z_1, Z_2 are complex numbers such that $\text{Re}(z_1) = |Z_1 - 1|$, $\text{Re}(Z_2) = |Z_2 - 1|$ and $\arg(Z_1 - Z_2) = \frac{\pi}{6}$, then $\text{Im}(Z_1 + Z_2)$ is equal to:
[Sep. 03, 2020 (II)]

Options:

- A. $\frac{2}{\sqrt{3}}$
- B. $2\sqrt{3}$
- C. $\frac{\sqrt{3}}{2}$
- D. $\frac{1}{\sqrt{3}}$

Answer: B

Solution:

$$\text{Let } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

$$\because |z_1 - 1| = \text{Re}(z_1)$$

$$\Rightarrow (x_1 - 1)^2 + y_1^2 = x_1^2$$

$$\Rightarrow y_1^2 - 2x_1 + 1 = 0 \dots\dots(i)$$

$$|z_2 - 1| = \text{Re}(z_2) \Rightarrow (x_2 - 1)^2 + y_2^2 = x_2^2$$

$$\Rightarrow y_2^2 - 2x_2 + 1 = 0 \dots\dots(ii)$$

$$\text{From eqn. (i) - (ii)}$$

$$y_1^2 - y_2^2 - 2(x_1 - x_2) = 0$$

$$\Rightarrow y_1 + y_2 = 2 \left(\frac{x_1 - x_2}{y_1 - y_2} \right) \dots\dots(iii)$$

$$\because \arg(Z_1 - Z_2) = \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1} \left(\frac{y_1 - y_2}{x_1 - x_2} \right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{2}{y_1 + y_2} = \frac{1}{\sqrt{3}} \left[\text{From, } \frac{y_1 - y_2}{x_1 - x_2} = \frac{2}{y_1 + y_2} \right]$$

$$\therefore y_1 + y_2 = 2\sqrt{3} \Rightarrow \text{Im}(z_1 + z_2) = 2\sqrt{3}$$

Question 226

Let $z = x + iy$ be a non-zero complex number such that $z^2 = i |z|^2$, where $i = \sqrt{-1}$, then z lies on the:
[Sep. 06, 2020 (II)]

Options:

- A. line, $y = -x$
- B. imaginary axis
- C. line, $y = x$
- D. real axis

Answer: C

Solution:

$$\begin{aligned}\text{Let } z &= x + iy \\ \because z^2 &= i \mid z|^2 \\ \therefore x^2 - y^2 + 2ixy &= i(x^2 + y^2) \\ \Rightarrow x^2 - y^2 &= 0 \text{ and } 2xy = x^2 + y^2 \\ \Rightarrow (x - y)(x + y) &= 0 \text{ and } (x - y)^2 = 0 \\ \Rightarrow x &= y\end{aligned}$$

Question 227

If a and b are real numbers such that $(2 + \alpha)^4 = a + b\alpha$, where $\alpha = \frac{-1 + i\sqrt{3}}{2}$, then $a + b$ is equal to :
[Sep. 04, 2020 (II)]

Options:

- A. 9
- B. 24
- C. 33
- D. 57

Answer: A

Solution:

$$\begin{aligned}\text{Given that, } \alpha &= \frac{-1 + i\sqrt{3}}{2} = \omega \\ \therefore (2 + \omega)^4 &= a + b\omega \Rightarrow (4 + \omega^2 + 4\omega)^2 = a + b\omega \\ \Rightarrow (\omega^2 + 4(1 + \omega))^2 &= a + b\omega \\ \Rightarrow (\omega^2 - 4\omega^2)^2 &= a + b\omega \\ [\because 1 + \omega &= -\omega^2] \\ \Rightarrow (-3\omega^2)^2 &= a + b\omega \Rightarrow 9\omega^4 = a + b\omega \\ \Rightarrow 9\omega &= a + b\omega \quad (\because \omega^3 = 1) \\ \text{On comparing, } a &= 0, b = 9 \\ \Rightarrow a + b &= 0 + 9 = 9\end{aligned}$$

Question 228

The value of $\left(\frac{1 + \sin \frac{2\pi}{9} + i \cos \frac{2\pi}{9}}{1 + \sin \frac{2\pi}{9} - i \cos \frac{2\pi}{9}} \right)^3$ is :

[Sep. 02, 2020 (I)]

Options:

- A. $\frac{1}{2}(1 - i\sqrt{3})$

- B. $\frac{1}{2}(\sqrt{3} - i)$
 C. $-\frac{1}{2}(\sqrt{3} - i)$
 D. $-\frac{1}{2}(1 - i\sqrt{3})$

Answer: C

Solution:

$$\begin{aligned} & \left(\frac{1 + \cos \frac{5\pi}{18} + i \sin \frac{5\pi}{18}}{1 + \cos \frac{5\pi}{18} - i \sin \frac{5\pi}{18}} \right)^3 \\ &= \left(\frac{2 \cos^2 \frac{5\pi}{36} + i 2 \sin \frac{5\pi}{36} \cdot \cos \frac{5\pi}{36}}{2 \cos^2 \frac{5\pi}{36} - i 2 \sin \frac{5\pi}{36} \cdot \cos \frac{5\pi}{36}} \right)^3 \\ &= \left(\frac{\cos \frac{5\pi}{36} + i \sin \frac{5\pi}{36}}{\cos \frac{5\pi}{36} - i \sin \frac{5\pi}{36}} \right)^3 = \left(\cos \frac{5\pi}{36} + i \sin \frac{5\pi}{36} \right)^6 \\ &= \cos \left(6 \times \frac{5\pi}{36} \right) + i \sin \left(6 \times \frac{5\pi}{36} \right) = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \\ &= -\frac{\sqrt{3}}{2} + i \frac{1}{2} = -\frac{1}{2}(\sqrt{3} - i) \end{aligned}$$

Question 229

The imaginary part of $(3 + 2\sqrt{-54})^{1/2} - (3 - 2\sqrt{-54})^{1/2}$ can be:
 [Sep. 02, 2020 (II)]

Options:

- A. $-\sqrt{6}$
 B. $-2\sqrt{6}$
 C. 6
 D. $\sqrt{6}$

Answer: B

Solution:

$$\begin{aligned} 3 + 2\sqrt{-54} &= 3 + 6\sqrt{6}i \\ \text{Let } \sqrt{3 + 6\sqrt{6}i} &= a + ib \\ \Rightarrow a^2 - b^2 &= 3 \text{ and } ab = 3\sqrt{6} \\ \Rightarrow a^2 + b^2 &= \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = 15 \\ \text{So, } a &= \pm 3 \text{ and } b = \pm \sqrt{6} \\ \sqrt{3 + 6\sqrt{6}i} &= \pm(3 + \sqrt{6}i) \\ \text{Similarly, } \sqrt{3 - 6\sqrt{6}i} &= \pm(3 - \sqrt{6}i) \\ \text{Im}(\sqrt{3 + 6\sqrt{6}i} - \sqrt{3 - 6\sqrt{6}i}) &= \pm 2\sqrt{6} \end{aligned}$$

Question 230

If α and β be two roots of the equation $x^2 - 64x + 256 = 0$.

Then the value of $\left(\frac{\alpha^3}{\beta^5}\right)^{\frac{1}{8}} + \left(\frac{\beta^3}{\alpha^5}\right)^{\frac{1}{8}}$ is:

[Sep. 06, 2020 (I)]

Options:

- A. 2
- B. 3
- C. 1
- D. 4

Answer: A

Solution:

$$\because \alpha + \beta = 64, \alpha\beta = 256$$
$$\frac{\alpha^{3/8}}{\beta^{5/8}} + \frac{\beta^{3/8}}{\alpha^{5/8}} = \frac{\alpha + \beta}{(\alpha\beta)^{5/8}} = \frac{64}{(2^8)^{5/8}} = \frac{64}{32} = 2$$

Question 231

If α and β are the roots of the equation $2x(2x + 1) = 1$, then β is equal to:

[Sep. 06, 2020 (II)]

Options:

- A. $2\alpha(\alpha + 1)$
- B. $-2\alpha(\alpha + 1)$
- C. $2\alpha(\alpha - 1)$
- D. $2\alpha^2$

Answer: B

Solution:

Let α and β be the roots of the given quadratic equation,

$$2x^2 + 2x - 1 = 0$$

$$\text{Then, } \alpha + \beta = -\frac{1}{2} \Rightarrow -1 = 2\alpha + 2\beta$$

$$\text{and } 4\alpha^2 + 2\alpha - 1 = 0 \quad [\because \alpha \text{ is root of eq. (i)}]$$

$$\Rightarrow 4\alpha^2 + 2\alpha + 2\alpha + 2\beta = 0 \Rightarrow \beta = -2\alpha(\alpha + 1)$$

Question 232

The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$, is:

[Sep. 05, 2020 (I)]

Options:

- A. $\frac{5}{9}$
- B. $\frac{25}{81}$

C. $\frac{5}{27}$

D. $\frac{25}{9}$

Answer: B

Solution:

Let $|x| = y$ then

$$9y^2 - 18y + 5 = 0$$

$$\Rightarrow 9y^2 - 15y - 3y + 5 = 0$$

$$\Rightarrow (3y - 1)(3y - 5) = 0$$

$$\Rightarrow y = \frac{1}{3} \text{ or } \frac{5}{3} \Rightarrow |x| = \frac{1}{3} \text{ or } \frac{5}{3}$$

$$\text{Roots are } \pm \frac{1}{3} \text{ and } \pm \frac{5}{3}$$

$$\text{Product} = \frac{25}{81}$$

Question 233

If α and β are the roots of the equation, $7x^2 - 3x - 2 = 0$ the the value of $\frac{\alpha}{1-\alpha^2} + \frac{\beta}{1-\beta^2}$ is equal to :

[Sep. 05, 2020 (II)]

Options:

A. $\frac{27}{32}$

B. $\frac{1}{24}$

C. $\frac{3}{8}$

D. $\frac{27}{16}$

Answer: D

Solution:

Let α and β be the roots of the quadratic equation $7x^2 - 3x - 2 = 0$

$$\therefore \alpha + \beta = \frac{3}{7}, \alpha\beta = \frac{-2}{7}$$

$$\begin{aligned} \text{Now, } \frac{\alpha}{1-\alpha^2} + \frac{\beta}{1-\beta^2} &= \frac{\alpha - \alpha\beta(\alpha + \beta) + \beta}{1 - (\alpha^2 + \beta^2) + (\alpha\beta)^2} \\ &= \frac{(\alpha + \beta) - \alpha\beta(\alpha + \beta)}{1 - (\alpha + \beta)^2 + 2\alpha\beta + (\alpha\beta)^2} \\ &= \frac{\frac{3}{7} + \frac{2}{7} \times \frac{3}{7}}{1 - \frac{9}{49} + 2 \times \frac{-2}{7} + \frac{4}{49}} = \frac{27}{16} \end{aligned}$$

Question 234

Let $u = \frac{2z+i}{z-ki}$, $z = x + iy$ and $k > 0$. If the curve represented by $\text{Re}(u) + \text{Im}(u) = 1$ intersects the y -axis at the points P and Q where $PQ = 5$, then the value of k is :
[Sep. 04, 2020 (I)]

Options:

A. $3/2$

B. $1/2$

C. 4

D. 2

Answer: D

Solution:

$$u = \frac{2(x+iy)+i}{(x+iy)-ki} = \frac{2x+i(2y+1)}{x+i(y-k)}$$

$$\text{Real part of } u = \text{Re}(u) = \frac{2x^2+(y-K)(2y+1)}{x^2+(y-K)^2}$$

Imaginary part of u

$$= \text{Im}(u) = \frac{-2x(y-K)+x(2y+1)}{x^2+(y-K)^2}$$

$$\because \text{Re}(u) + \text{Im}(u) = 1$$

$$\Rightarrow 2x^2 + 2y^2 - 2Ky + y - K - 2xy + 2Kx + 2xy + x$$

$$= x^2 + y^2 + K^2 - 2Ky$$

Since, the curve intersect at y -axis

$$\therefore x = 0$$

$$\Rightarrow y^2 + y - K(K+1) = 0$$

Let y_1 and y_2 are roots of equations if $x = 0$

$$\therefore y_1 + y_2 = -1$$

$$y_1 y_2 = -(K^2 + K)$$

$$\therefore (y_1 - y_2)^2 = (1 + 4K^2 + 4K)$$

$$\text{Given } PQ = 5 \Rightarrow |y_1 - y_2| = 5$$

$$\Rightarrow 4K^2 + 4K - 24 = 0 \Rightarrow K = 2 \text{ or } -3$$

$$\text{as } K > 0, \therefore K = 2$$

Question 235

Let $\lambda \neq 0$ be in $\mathbb{R}$. If α and β are roots of the equation, $x^2 - x + 2\lambda = 0$ and α and γ are the roots of the equation, $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to

[Sep. 04, 2020 (II)]

Options:

A. 27

B. 18

C. 9

D. 36

Answer: B

Solution:

Since α is common root of $x^2 - x + 2\lambda = 0$ and $3x^2 - 10x + 27\lambda = 0$

$$\therefore 3\alpha^2 - 10\alpha + 27\lambda = 0 \dots\dots (i)$$

$$3\alpha^2 - 3\alpha + 6\lambda = 0 \dots\dots (ii)$$

$$\therefore \text{On subtract, we get } \alpha = 3\lambda$$

$$\text{Now, } \alpha\beta = 2\lambda \Rightarrow 3\lambda \cdot \beta = 2\lambda \Rightarrow \beta = \frac{2}{3}$$

$$\Rightarrow \alpha + \beta = 1 \Rightarrow 3\lambda + \frac{2}{3} = 1 \Rightarrow \lambda = \frac{1}{9} \text{ and}$$

$$\alpha\gamma = 9\lambda \Rightarrow 3\lambda \cdot \gamma = 9\lambda \Rightarrow \gamma = 3$$

$$\therefore \frac{\beta\gamma}{\lambda} = 18$$

Question 236

If α and β are the roots of the equation $x^2 + px + 2 = 0$ and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation

$2x^2 + 2qx + 1 = 0$ then $\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$ is equal to

[Sep. 03, 2020 (I)]

Options:-

A. $\frac{9}{4}(9 + q^2)$

B. $\frac{9}{4}(9 - q^2)$

C. $\frac{9}{4}(9 + p^2)$

D. $\frac{9}{4}(9 - p^2)$

Answer: D

Solution:

$$\alpha \cdot \beta = 2 \text{ and } \alpha + \beta = -p \text{ also } \frac{1}{\alpha} + \frac{1}{\beta} = -q$$

$$\Rightarrow p = 2q$$

$$\text{Now } \left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$$

$$= \left[\alpha\beta + \frac{1}{\alpha\beta} - \frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right]\left[\alpha\beta + \frac{1}{\alpha\beta} + 2\right]$$

$$= \frac{9}{2}\left[\frac{5}{2} - \frac{\alpha^2 + \beta^2}{2}\right] = \frac{9}{4}[5 - (p^2 - 4)]$$

$$= \frac{9}{4}(9 - p^2) \quad [\because \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta]$$

Question 237

The set of all real values of λ for which the quadratic equations, $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval $(0, 1)$ is :

[Sep. 03, 2020 (II)]

Options:

A. $(0, 2)$

B. $(2, 4]$

C. $(1, 3]$

D. $(-3, -1)$

Answer: C

Solution:

The given quadratic equation is

$$(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$$

$\therefore$ One root is in the interval $(0, 1)$

$$\therefore f(0)f(1) \leq 0$$

$$\Rightarrow 2(\lambda^2 + 1 - 4\lambda + 2) \leq 0$$

$$\Rightarrow 2(\lambda^2 - 4\lambda + 3) \leq 0$$

$$(\lambda - 1)(\lambda - 3) \leq 0 \Rightarrow \lambda \in [1, 3]$$

But at $\lambda = 1$, both roots are 1 so $\lambda \neq 1$

$$\therefore \lambda \in (1, 3]$$

Question 238

Let α and β be the roots of the equation, $5x^2 + 6x - 2 = 0$.

If $S_n = \alpha^n + \beta^n$, $n = 1, 2, 3, \dots$, then :

[Sep. 02, 2020 (I)]

Options:

A. $6S_6 + 5S_5 = 2S_4$

B. $6S_6 + 5S_5 + 2S_4 = 0$

C. $5S_6 + 6S_5 = 2S_4$

D. $5S_6 + 6S_5 + 2S_4 = 0$

Answer: C

Solution:

Since, α and β are the roots of the equation

$$5x^2 + 6x - 2 = 0$$

$$\text{Then, } 5\alpha^2 + 6\alpha - 2 = 0, 5\beta^2 + 6\beta - 2 = 0$$

$$5\alpha^2 + 6\alpha = 2$$

$$5S_6 + 6S_5 = 5(\alpha^6 + \beta^6) + 6(\alpha^5 + \beta^5)$$

$$= (5\alpha^4 + 6\alpha^5) + (5\beta^4 + 6\beta^5)$$

$$= \alpha^4(5\alpha^2 + 6\alpha) + \beta^4(5\beta^2 + 6\beta)$$

$$= 2(\alpha^4 + \beta^4) = 2S_4$$
