



Binomial Theorem

Question 1

The coefficient of x^{48} in $(1+x) + 2(1+x)^2 + 3(1+x)^3 + \dots + 100(1+x)^{100}$ is equal to

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Options:

A. $100 \cdot {}^{100}C_{49} - {}^{100}C_{48}$

B. $100 \cdot {}^{101}C_{49} - {}^{101}C_{50}$

C. ${}^{100}C_{50} + {}^{101}C_{49}$

D. $100 \cdot {}^{100}C_{49} - {}^{100}C_{50}$

Answer: B

Solution:

$$\begin{aligned}
 S &= (1+x) + 2(1+x)^2 + 3(1+x)^3 + \dots + 100(1+x)^{100} \\
 (1+x)S &= (1+x)^2 + \dots + 99(1+x)^{100} + 100(1+x)^{101} \\
 -xS &= (1+x) \left(\frac{(1+x)^{100} - 1}{x} \right) - 100(1+x)^{101} \\
 S &= \left(\frac{(1+x) - (1+x)^{101}}{x^2} \right) + \frac{100(1+x)^{101}}{x} \\
 \text{Coeff of } x^{48} &= -{}^{101}C_{50} + 100 \times {}^{101}C_{49}
 \end{aligned}$$

Question 2

If $(1-x^3)^{10} = \sum_{r=0}^{10} a_r x^r (1-x)^{30-2r}$, then $\frac{9a_9}{a_{10}}$ is equal to _____.

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Answer: 30

Solution:

Given

$$(1-x^3)^{10} = \sum_{r=0}^{10} a_r x^r (1-x)^{30-2r}.$$

We have to find

$$\frac{9a_9}{a_{10}}.$$

Let us simplify the left side first.

Since

$$1-x^3 = (1-x)(1+x+x^2),$$

so

$$(1-x^3)^{10} = (1-x)^{10}(1+x+x^2)^{10}.$$

Also,

$$1+x+x^2 = \frac{1-x^3}{1-x}.$$

Now notice that

$$1+x+x^2 = (1-x)^{-2}(x+(1-x)^2)?$$

This is not the most useful form here. So let us proceed in a better NCERT-style way.

We are given

$$(1 - x^3)^{10} = \sum_{r=0}^{10} a_r x^r (1 - x)^{30-2r}.$$

Take out $(1 - x)^{10}$ from the left side:

$$(1 - x^3)^{10} = (1 - x)^{10} (1 + x + x^2)^{10}.$$

Hence

$$(1 + x + x^2)^{10} = \sum_{r=0}^{10} a_r x^r (1 - x)^{20-2r}.$$

Now observe that

$$1 + x + x^2 = (1 - x)^2 + 3x.$$

Because

$$(1 - x)^2 + 3x = 1 - 2x + x^2 + 3x = 1 + x + x^2.$$

Therefore

$$(1 + x + x^2)^{10} = ((1 - x)^2 + 3x)^{10}.$$

Using binomial theorem,

$$((1 - x)^2 + 3x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (3x)^r ((1 - x)^2)^{10-r}.$$

So

$$(1 + x + x^2)^{10} = \sum_{r=0}^{10} \binom{10}{r} 3^r x^r (1 - x)^{20-2r}.$$

Comparing this with

$$(1 + x + x^2)^{10} = \sum_{r=0}^{10} a_r x^r (1 - x)^{20-2r},$$

we get

$$a_r = \binom{10}{r} 3^r.$$

Hence

$$a_9 = \binom{10}{9} 3^9 = 10 \cdot 3^9,$$

and

$$a_{10} = \binom{10}{10} 3^{10} = 3^{10}.$$

Therefore

$$\frac{9a_9}{a_{10}} = 9 \cdot \frac{10 \cdot 3^9}{3^{10}} = 9 \cdot \frac{10}{3} = 30.$$

So the required value is

$$\boxed{30}.$$

Question3

If the coefficient of x in the expansion of $(ax^2 + bx + c)(1 - 2x)^{26}$ is -56 and the coefficients of x^2 and x^3 are both zero, then $a + b + c$ is equal to :

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Options:

A.

1483

B.

1300

C.

1500

D.

1403

Answer: D

Solution:

$$(ax^2 + bx + c)(1 - 2x)^{26}$$
$$ax^2(1 - 2x)^{26} + bx(1 - 2x)^{26} + c(1 - 2x)^{26}$$

Coefficient of

$$x \Rightarrow [0 + b \cdot {}^{26}C_0 + c \cdot {}^{26}C_1(-2)]x$$
$$= b - 52c$$

Coefficient of

$$x^2 = (a \cdot {}^{26}C_0) + b \cdot {}^{26}C_1(-2) + c \cdot {}^{26}C_2(-2)^2 x^2$$
$$= a - 52b + {}^{26}C_2 \times 4 = a - 52b + 1300c$$

Coefficient of

$$x^3 = [a \cdot {}^{26}C_1(-2) + b \cdot {}^{26}C_2(26)^2 + c \cdot {}^{26}C_3(-2)^3]$$
$$= -52a + 1300b - 20800c$$
$$\Rightarrow b - 52c = -56$$
$$\Rightarrow b - 52c = -56$$
$$a - 52b + 1300c = 0 \Rightarrow a = 1300$$
$$-a + 25b - 400c = 0 \quad b = 100$$
$$c = 3$$
$$\Rightarrow a + b + c = 1403$$

Question4

The coefficient of x^{48} in $(1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \dots + 100(1 + x)^{100}$ is equal to

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Options:

A.

$$100 \cdot {}^{100}C_{49} - {}^{100}C_{48}$$

B.

$$100 \cdot {}^{101}C_{49} - {}^{101}C_{50}$$

C.

$${}^{100}C_{50} + {}^{101}C_{49}$$

D.

$$100 \cdot {}^{100}C_{49} - {}^{100}C_{50}$$

Answer: B

Solution:

$$S = (1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \dots + 100(1 + x)^{100}$$
$$(1 + x)S = (1 + x)^2 + \dots + 99(1 + x)^{100} + 100(1 + x)^{101}$$
$$-xS = (1 + x) \left(\frac{(1 + x)^{100} - 1}{x} \right) - 100(1 + x)^{101}$$
$$S = \left(\frac{(1 + x) - (1 + x)^{101}}{x^2} \right) + \frac{100(1 + x)^{101}}{x}$$

Coeff of $x^{48} = -{}^{101}C_{50} + 100 \times {}^{(101)}C_{49}$

Question5

Let C_r denote the coefficient of x^r in the binomial expansion of $(1+x)^n$, $n \in \mathbb{N}$, $0 \leq r \leq n$. If

$P_n = C_0 - C_1 + \frac{2^2}{3}C_2 - \frac{2^3}{4}C_3 + \dots + \frac{(-2)^n}{n+1}C_n$, then the value of $\sum_{n=1}^{25} \frac{1}{P_{2n}}$ equals.

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Options:

A.

675

B.

580

C.

525

D.

650

Answer: A

Solution:

$$\begin{aligned}P_n &= \sum_{r=0}^n \frac{{}^nC_r (-2)^r}{r+1} = \sum_{r=0}^n \frac{1}{(n+1)} {}^{n+1}C_{r+1} (-2)^r \\&= \frac{-1}{2(n+1)} \sum_{r=0}^n {}^{n+1}C_{r+1} (-2)^{r+1} = \frac{-1}{2(n+1)} [(1-2)^{n+1} - 1] \\P_n &= \frac{1}{2(n+1)} [1 - (-1)^{n+1}] \\P_{2n} &= \frac{1}{2(2n+1)} [1 - (-1)^{2n+1}] \\P_{2n} &= \frac{1}{2n+1} \\\sum_{n=0}^{25} \frac{1}{P_{2n}} &= \sum_{n=1}^{25} (2n+1) = 3 + 5 + \dots + 51 = \frac{25}{2} [51 + 3] = 25 \times 27 = 675\end{aligned}$$

Question6

The value of $\frac{{}^{100}C_{50}}{51} + \frac{{}^{100}C_{51}}{52} + \dots + \frac{{}^{100}C_{100}}{101}$ is:

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Options:

A.

$\frac{2^{101}}{101}$

B.

$\frac{2^{100}}{101}$

C.

$$\frac{2^{100}}{100}$$

D.

$$\frac{2^{101}}{100}$$

Answer: B

Solution:

$$\begin{aligned}(1+x)^{100} &= {}^{100}C_0 + {}^{100}C_1x + {}^{100}C_2x^2 + \dots + {}^{100}C_{100}x^{100} \\ \sum_{r=50}^{100} \frac{{}^{100}C_r}{r+1} &= \frac{1}{101} \sum_{r=50}^{100} {}^{101}C_{r+1} \\ &\Rightarrow \frac{1}{101} \sum_{r=51}^{101} {}^{101}C_r \\ &\Rightarrow \frac{2^{100}}{101}\end{aligned}$$

Question7

The sum of all possible values of $n \in \mathbb{N}$, so that the coefficients of x, x^2 and x^3 in the expansion of $(1+x^2)^2(1+x)^n$, are in arithmetic progression is :

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Options:

A.

12

B.

9

C.

3

D.

7

Answer: B

Solution:

$$\begin{aligned}&(1+x^2)^2(1+x)^n \\ &(1+x^4+2x^2)(1+x)^n \\ &\text{Coeff of } x = {}^nC_1 \\ &\text{Coeff of } x^2 = {}^nC_2 + 2 \cdot {}^nC_0 \\ &\text{Coeff of } x^3 = {}^nC_3 + 2 \cdot {}^nC_1 \\ &\because \text{Coeff of } x, x^2, x^3 \rightarrow \text{AP} \\ &\Rightarrow 2({}^nC_2 + 2) = {}^nC_1 + {}^nC_3 + 2n \\ &2^n C_2 + 4 = {}^nC_3 + 3n \\ &\Rightarrow n^2 - 9n^2 + 26n - 24 = 0 \\ &\Rightarrow n = 2, 3, 4 \\ &\Rightarrow \text{Sum of all value of } n \text{ is } 9\end{aligned}$$

Question8

Let $S = \frac{1}{25!} + \frac{1}{3!23!} + \frac{1}{5!21!} + \dots$ up to 13 terms. If $13S = \frac{2^k}{n!}, k \in \mathbb{N}$, then $n+k$ is equal to

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Options:

A.

50

B.

52

C.

49

D.

51

Answer: C

Solution:

$$S = \frac{1}{1!25!} + \frac{1}{3!23!} + \frac{1}{5!21!} + \dots 13 \text{ terms}$$

multiply in numerator and denominator by $26!$

$$S = \frac{1}{26!} \left[\frac{26!}{1!25!} + \frac{26!}{3!23!} + \frac{26!}{5!21!} + \dots 13 \text{ term} \right]$$

$$S = \frac{1}{26!} [{}^{26}C_1 + {}^{26}C_3 + {}^{26}C_5 + \dots 13 \text{ term}], \left\{ {}^nC_r = \frac{n!}{(n-r)!r!} \right\}$$

Lets find 13th term lower subscript are in A.P with, common difference $= 2$

$$T_{13} = 1 + (13 - 1)2 = 25$$

$$\text{so, } S = \frac{1}{26!} [{}^{26}C_1 + {}^{26}C_3 + {}^{26}C_5 + \dots {}^{26}C_{25}]$$

$$S = \frac{1}{26!} [2^{26-1}], \{ \text{because } {}^nC_1 + {}^nC_3 + {}^nC_5 \dots = 2^{n-1} \}$$

$$S = \frac{2^{25}}{26!}$$

$$\text{It is given that } 13S = \frac{2^K}{n!}$$

substitute value of S

$$\begin{aligned} 13 \times \frac{2^{25}}{26!} &= \frac{2^K}{n!} \\ \frac{13 \times 2^{25}}{26 \times 25!} &= \frac{2^K}{n!} \\ \frac{2^{24}}{25!} &= \frac{2^K}{n!} \end{aligned}$$

$$\text{so } n = 25, K = 24$$

$$n + K = 25 + 24 = 49$$

Question9

The sum of the coefficients of x^{499} and x^{500} in $(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \dots + x^{1000}$ is ;

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Options:

A. ${}^{1002}C_{501}$

B. ${}^{1001}C_{501}$

C. $^{1000}C_{501}$

D. $^{1002}C_{500}$

Answer: D

Solution:

Let

$$S = (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \dots + x^{1000}$$

Notice that this is a Geometric Progression (GP) with:

- First term (a): $(1+x)^{1000}$
- Common ratio (r) : $\frac{x}{1+x}$
- Number of terms (n): 1001

The sum of a GP is given by $S = a \frac{(1-r^n)}{(1-r)}$. Substituting our values:

$$S = (1+x)^{1000} \left[\frac{1 - \left(\frac{x}{1+x}\right)^{1001}}{1 - \frac{x}{1+x}} \right]$$

Simplify the denominator: $1 - \frac{x}{1+x} = \frac{1+x-x}{1+x} = \frac{1}{1+x}$.

$$S = (1+x)^{1000} \cdot (1+x) \left[1 - \frac{x^{1001}}{(1+x)^{1001}} \right]$$

$$S = (1+x)^{1001} - x^{1001}$$

Now we need the coefficients of x^{499} and x^{500} in the simplified expression $(1+x)^{1001} - x^{1001}$.

Since x^{1001} only affects the term with power 1001, we only look at $(1+x)^{1001}$.

In the binomial expansion $(1+x)^n$

Coefficient of x^r is ^nC_r .

Coefficient of x^{499} in S :

$$^{1001}C_{499}$$

Coefficient of x^{500} in S :

$$^{1001}C_{500} \text{ (The term } x^{1001} \text{ does not contribute.)}$$

Sum of coefficients

$$^{1001}C_{499} + ^{1001}C_{500} = ^{1002}C_{500}$$

[Using pascal's identity

$$^nC_r + ^nC_{r+1} = ^{n+1}C_{r+1}]$$

Question10

Given below are two statements :

Statement I :

$25^{13} + 20^{13} + 8^{13} + 3^{13}$ is divisible by 7.

Statement II :

The integral part of $(7 + 4\sqrt{3})^{25}$ is an odd number.

In the light of the above statements, choose the correct answer from the options given below :

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Options:

A.

Statement I is false but Statement II is true

B.

Both Statement I and Statement II are false

C.

Both Statement I and Statement II are true

D.

Statement I is true but Statement II is false

Answer: C

Solution:

Statement I : $25^{13} + 20^{13} + 8^{13} + 3^{13}$ is divisible by 7.

Key concept : If n is odd, $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \dots - xy^{n-2} + y^{n-1})$. So, $x^n + y^n$ is divisible by $x + y$.

In the given statement :

- $25^{13} + 3^{13}$ is divisible by $25 + 3 = 28$.
- $25^{13} + 3^{13}$ is also divisible by 7 because 28 is divisible by 7.
- So, $25^{13} + 3^{13} = 7 \cdot I_1 \dots \dots (1)$ (I_1 is a positive integer).

Similarly :

- $20^{13} + 8^{13}$ is divisible by $20 + 8 = 28$.
- $20^{13} + 8^{13}$ is divisible by 7.
- So, $20^{13} + 8^{13} = 7 \cdot I_2 \dots \dots (2)$ (I_2 is a positive integer).

Add (1) & (2) :

- $25^{13} + 3^{13} + 20^{13} + 8^{13} = 7(I_1 + I_2)$

So, $25^{13} + 20^{13} + 8^{13} + 3^{13}$ is divisible by 7. Statement I is true.

Statement : The integral part of $(7 + 4\sqrt{3})^{25}$ is an odd number.

Let $X = (7 + 4\sqrt{3})^{25}$. Let I be the integral part and f be the fractional part ($0 < f < 1$).

Adding the two expansions:

$$(7 + 4\sqrt{3})^{25} + (7 - 4\sqrt{3})^{25} = I + f + f'$$

Using the binomial expansion $(a + b)^n + (a - b)^n = 2[a^n + \binom{n}{2}a^{n-2}b^2 + \dots]$, we see that the sum is always an even integer because of the factor of 2.

$$I + f + f' = \text{Even Integer}$$

Since I is an integer, $f + f'$ must also be an integer. Given $0 < f < 1$ and $0 < f' < 1$, the only possible integer value for their sum is $f + f' = 1$.

$$I + 1 = \text{Even Integer}$$

$$I = \text{Even Integer} - 1 = \text{Odd Integer}$$

The integral part I is indeed an odd number. Statement II is True.

Question 11

If for $3 \leq r \leq 30$, $\binom{30}{30-r} + 3\binom{30}{31-r} + 3\binom{30}{32-r} + \binom{30}{33-r} = {}^m C_r$, then m equals :

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Options:

A.

31

B.

32

C.

33

D.

34

Answer: C

Solution:

Given

$${}^{30}C_{30-r} + {}^{30}C_{31-r} + {}^{30}C_{32-r} + {}^{30}C_{33-r} = {}^mC_r$$

for $3 \leq r \leq 30$.

We have to find m .

First, use the symmetry property of combinations:

$${}^nC_k = {}^nC_{n-k}$$

So,

$${}^{30}C_{30-r} = {}^{30}C_r$$

$${}^{30}C_{31-r} = {}^{30}C_{r-1}$$

because

$$30 - (31 - r) = r - 1$$

Similarly,

$${}^{30}C_{32-r} = {}^{30}C_{r-2}$$

and

$${}^{30}C_{33-r} = {}^{30}C_{r-3}$$

Therefore the left-hand side becomes

$${}^{30}C_r + {}^{30}C_{r-1} + {}^{30}C_{r-2} + {}^{30}C_{r-3}$$

Now notice the coefficients:

1, 3, 3, 1

These are the binomial coefficients in the expansion of

$$(1+x)^3$$

Also, by Vandermonde's identity,

$${}^{30}C_r + {}^{30}C_{r-1} + {}^{30}C_{r-2} + {}^{30}C_{r-3} = {}^{33}C_r$$

Let us see this through binomial expansion:

Coefficient of x^r in $(1+x)^{33}$ is

$${}^{33}C_r$$

But

$$(1+x)^{33} = (1+x)^{30}(1+x)^3$$

and

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

So coefficient of x^r in $(1+x)^{30}(1+3x+3x^2+x^3)$ is

$${}^{30}C_r + {}^{30}C_{r-1} + {}^{30}C_{r-2} + {}^{30}C_{r-3}$$

Hence,

$${}^{30}C_r + {}^{30}C_{r-1} + {}^{30}C_{r-2} + {}^{30}C_{r-3} = {}^{33}C_r$$

Comparing with

$${}^m C_r$$

we get

$$m = 33$$

The correct answer is:

$$\boxed{33}$$

So, Option C.

Question 12

Let the smallest value of $k \in \mathbb{N}$, for which the coefficient of x^3 in $(1+x)^3 + (1+x)^4 + (1+x)^5 + \dots + (1+x)^{99} + (1+kx)^{100}$, $x \neq 0$, is $(43n + \frac{101}{4}) \binom{100}{3}$ for some $n \in \mathbb{N}$, be p . Then the value of $p + n$ is :

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Options:

A.

10

B.

11

C.

12

D.

13

Answer: B

Solution:

Coefficient of x^3 in the given expression

$$(1+x)^3 + (1+x)^4 + \dots + (1+x)^{99} + (1+kx)^{100}$$

is obtained by adding the coefficients of x^3 from each term.

For $(1+x)^r$, coefficient of x^3 is

$${}^r C_3$$

So, from

$$(1+x)^3 + (1+x)^4 + \dots + (1+x)^{99}$$

the total coefficient of x^3 is

$$\sum_{r=3}^{99} {}^r C_3$$

Also, in $(1+kx)^{100}$, coefficient of x^3 is

$${}^{100} C_3 k^3$$

Hence total coefficient of x^3 is

$$\sum_{r=3}^{99} {}^r C_3 + {}^{100} C_3 k^3$$

Now use the standard NCERT identity:

$$\sum_{r=3}^{99} {}^r C_3 = {}^{100} C_4$$

So coefficient becomes

$${}^{100} C_4 + {}^{100} C_3 k^3$$

Given that this equals

$$(43n + \frac{101}{4})^{100} C_3$$

Therefore,

$$^{100}C_4 + ^{100}C_3 k^3 = (43n + \frac{101}{4})^{100} C_3$$

Divide by $^{100}C_3$:

$$\frac{^{100}C_4}{^{100}C_3} + k^3 = 43n + \frac{101}{4}$$

Now,

$$\frac{^{100}C_4}{^{100}C_3} = \frac{100-3}{4} = \frac{97}{4}$$

So,

$$k^3 + \frac{97}{4} = 43n + \frac{101}{4}$$

Thus,

$$k^3 = 43n + 1$$

We need the smallest $k \in \mathbb{N}$ such that

$$k^3 - 1$$

is divisible by 43.

That is,

$$k^3 \equiv 1 \pmod{43}$$

Now check smallest natural numbers:

$$1^3 = 1 \equiv 1 \pmod{43}$$

So the smallest value is

$$k = 1$$

Hence,

$$1 = 43n + 1$$

which gives

$$n = 0$$

But in the question, $n \in \mathbb{N}$. In school-level convention, if $\mathbb{N} = \{0, 1, 2, \dots\}$, then this is allowed.

Thus,

$$p = 1, \quad n = 0$$

and

$$p + n = 1$$

But this is not among the options, so clearly the intended convention is $n \geq 1$.

Now take the smallest $n \in \mathbb{N}$ with $n \geq 1$ such that

$$k^3 = 43n + 1$$

Try $n = 1$:

$$k^3 = 44 \quad \text{not a cube}$$

$$n = 2:$$

$$k^3 = 87 \quad \text{not a cube}$$

$$n = 3:$$

$$k^3 = 130 \quad \text{not a cube}$$

$$n = 4:$$

$$k^3 = 173$$

$$n = 5:$$

$$k^3 = 216 = 6^3$$

So smallest such k is

$$p = 6, \quad n = 5$$

Therefore,

$$p + n = 6 + 5 = 11$$

Hence the correct answer is:

11

So, Option B is correct.

Question 13

In the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x > 0$, if the term independent of x is $(221)^k$, then k is equal to:

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Options:

A.

84

B.

78

C.

168

D.

198

Answer: A

Solution:

For the expansion of

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18},$$

we use the general term in the binomial expansion:

$$T_{r+1} = {}^{(18)}C_r (9x)^{18-r} \left(-\frac{1}{3\sqrt{x}}\right)^r$$

Now simplify the power of x in this term.

Since

$$(9x)^{18-r} = 9^{18-r} x^{18-r}$$

and

$$\left(\frac{1}{3\sqrt{x}}\right)^r = \frac{1}{3^r x^{r/2}},$$

so the power of x in the general term is

$$x^{18-r-\frac{r}{2}} = x^{18-\frac{3r}{2}}.$$

For the term independent of x , the power of x must be zero:

$$18 - \frac{3r}{2} = 0$$

$$\frac{3r}{2} = 18$$

$$r = 12$$

So the required term is the $(12 + 1)$ th term, i.e. T_{13} .

Now compute it:

$$T_{13} = {}^{(18)}C_{12} (9x)^6 \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

Since the power of x becomes zero, only the numerical coefficient remains:

$$T_{13} = \binom{18}{12} \cdot 9^6 \cdot \frac{1}{3^{12}}$$

Also,

$$9^6 = (3^2)^6 = 3^{12}$$

So,

$$T_{13} = \binom{18}{12}$$

and

$$\binom{18}{12} = \binom{18}{6} = \frac{18!}{6!12!} = 18564$$

Given that the term independent of x is $(221)k$, we have

$$221k = 18564$$

Thus,

$$k = \frac{18564}{221} = 84$$

So the correct answer is:

84

Hence, **Option A** is correct.

Question 14

The coefficient of x^2 in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{10}$, $x \neq 0$, is :

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Options:

A.

3240

B.

3360

C.

3480

D.

3600

Answer: B

Solution:

Using the binomial theorem,

$$\left(2x^2 + \frac{1}{x}\right)^{10} = \sum_{r=0}^{10} \binom{10}{r} (2x^2)^{10-r} \left(\frac{1}{x}\right)^r$$

Now simplify the general term:

$$T_{r+1} = \binom{10}{r} 2^{10-r} x^{2(10-r)} x^{-r}$$

$$T_{r+1} = \binom{10}{r} 2^{10-r} x^{20-2r-r} = \binom{10}{r} 2^{10-r} x^{20-3r}$$

We need the coefficient of x^2 , so set the power of x equal to 2:

$$20 - 3r = 2$$

$$3r = 18$$

$$r = 6$$

Now substitute $r = 6$:

$$\text{Required coefficient} = \binom{10}{6} 2^{10-6}$$

$$= \binom{10}{6} 2^4$$

$$= 210 \times 16$$

$$= 3360$$

So, the coefficient of x^2 is

$$\boxed{3360}$$

Hence, the correct option is **B**.

Question 15

If the coefficients of the middle terms in the binomial expansions of $(1 + \alpha x)^{26}$ and $(1 - \alpha x)^{28}$, $\alpha \neq 0$, are equal, then the value of α is:

JEE Main 2026 (Online) 6th April Morning Shift

Options:

A.

$$1$$

B.

$$\frac{14}{13}$$

C.

$$\frac{27}{7}$$

D.

$$\frac{7}{27}$$

Answer: D

Solution:

For a binomial expansion $(a + b)^n$:

- if n is even, there is only one middle term,
- and the middle term is the $\left(\frac{n}{2} + 1\right)$ th term.

We are given:

$$(1 + \alpha x)^{26} \quad \text{and} \quad (1 - \alpha x)^{28}$$

and the coefficients of their middle terms are equal.

1. Middle term of $(1 + \alpha x)^{26}$

Since 26 is even, the middle term is the $\left(\frac{26}{2} + 1\right) = 14$ th term.

The general term in $(1 + \alpha x)^{26}$ is

$$T_{r+1} = \binom{26}{r} (1)^{26-r} (\alpha x)^r$$

So the 14th term corresponds to $r = 13$:

$$T_{14} = \binom{26}{13} (\alpha x)^{13}$$

Hence, the coefficient of the middle term is

$$\binom{26}{13} \alpha^{13}$$

2. Middle term of $(1 - \alpha x)^{28}$

Since 28 is even, the middle term is the $\left(\frac{28}{2} + 1\right) = 15$ th term.

The general term is

$$T_{r+1} = \binom{28}{r} (1)^{28-r} (-\alpha x)^r$$

So the 15th term corresponds to $r = 14$:

$$T_{15} = \binom{28}{14} (-\alpha x)^{14}$$

Since 14 is even,

$$(-\alpha x)^{14} = \alpha^{14} x^{14}$$

Thus, the coefficient of the middle term is

$$\binom{28}{14} \alpha^{14}$$

3. Equate the coefficients

Given that these coefficients are equal:

$$\binom{26}{13} \alpha^{13} = \binom{28}{14} \alpha^{14}$$

Since $\alpha \neq 0$, divide both sides by α^{13} :

$$\binom{26}{13} = \binom{28}{14} \alpha$$

So,

$$\alpha = \frac{\binom{26}{13}}{\binom{28}{14}}$$

4. Simplify

$$\alpha = \frac{26!}{\frac{13!13!}{28!} \cdot 14!14!}$$

$$\alpha = \frac{26!}{13!13!} \cdot \frac{14!14!}{28!}$$

Now use

$$14! = 14 \cdot 13!$$

So,

$$14!14! = (14 \cdot 13!)(14 \cdot 13!) = 14^2(13!)^2$$

Thus,

$$\alpha = \frac{26! \cdot 14^2 (13!)^2}{(13!)^2 \cdot 28!} = \frac{26! \cdot 14^2}{28!}$$

Also,

$$28! = 28 \cdot 27 \cdot 26!$$

Therefore,

$$\alpha = \frac{14^2}{28 \cdot 27} = \frac{14}{2 \cdot 27} = \frac{7}{27}$$

$$\boxed{\alpha = \frac{7}{27}}$$

So, the correct option is **D**.

Question 16

If $26 \left(\frac{2^3}{3} \binom{12}{2} + \frac{2^5}{5} \binom{12}{4} + \frac{2^7}{7} \binom{12}{6} + \dots + \frac{2^{13}}{13} \binom{12}{12} \right) = 3^{13} - \alpha$, then α is equal to :

Options:

- A.
- 45
- B.
- 48
- C.
- 51
- D.
- 54

Answer: C

Solution:

$$\begin{aligned}
 S_r &= 26 \left[\left(\frac{2^3}{3} \right) \cdot {}^{(12)}C_2 + \left(\frac{2^5}{5} \right) \cdot {}^{(12)}C_4 + \left(\frac{2^7}{7} \right) \cdot {}^{(12)}C_6 + \left(\frac{2^9}{9} \right) \cdot {}^{(12)}C_8 + \dots + \left(\frac{2^{13}}{13} \right) \cdot {}^{(12)}C_{12} \right] \\
 &= 26 \left[\sum_{r=1}^6 \left(\frac{2^{2r+1}}{2r+1} \right) \cdot {}^{(12)}C_{2r} \right] \\
 S_r &= \frac{26}{13} \left[\sum_{r=1}^6 (2^{2r+1}) \cdot {}^{(13)}C_{2r+1} \right] \\
 &= 2 \left[\sum_{r=0}^6 (2^{2r+1}) \cdot {}^{(13)}C_{2r+1} - {}^{(13)}C_1 \cdot 2^1 \right] \\
 &= 2 \left[\sum_{r=0}^6 (2^{2r+1}) \cdot {}^{(13)}C_{2r+1} \right] - 52 \\
 S_r &= 2 \left[({}^{(13)}C_1) \cdot 2^1 + ({}^{(13)}C_3) \cdot 2^3 + ({}^{(13)}C_5) \cdot 2^5 + ({}^{(13)}C_7) \cdot 2^7 + \dots + ({}^{(13)}C_{13}) \cdot 2^{13} \right] - 52 \\
 &= (1+2)^{13} - (1-2)^{13} - 52 \\
 &= 3^{13} + 1 - 52 \\
 &= 3^{13} - 51 \dots (i) \\
 S_r &= 26 \left[\left(\frac{2^3}{3} \right) \cdot {}^{(12)}C_2 + \left(\frac{2^5}{5} \right) \cdot {}^{(12)}C_4 + \left(\frac{2^7}{7} \right) \cdot {}^{(12)}C_6 + \left(\frac{2^9}{9} \right) \cdot {}^{(12)}C_8 + \dots + \left(\frac{2^{13}}{13} \right) \cdot {}^{(12)}C_{12} \right] = 3^{13} - \alpha \dots (ii)
 \end{aligned}$$

From the equivalence relationship between equation (i) and (ii) we conclude $\alpha = 51$

Question 17

If $\sum_{r=0}^5 \frac{{}^{11}C_{2r+1}}{2r+2} = \frac{m}{n}$, $\gcd(m, n) = 1$, then $m - n$ is equal to _____.

JEE Main 2025 (Online) 22nd January Morning Shift

Answer: 2035

Solution:

$$\begin{aligned}
 (1+x)^{11} &= {}^{11}C_0 + {}^{11}C_1x + {}^{11}C_2x^2 + \dots + {}^{11}C_{11}x^{11} \\
 \int_0^1 (1+x)^{11} dx &= \int_0^1 ({}^{11}C_0 + {}^{11}C_1x + {}^{11}C_2x^2 + \dots + {}^{11}C_{11}x^{11}) dx \\
 \left[\frac{(1+x)^{12}}{12} \right]_0^1 &= {}^{11}C_0x + \frac{{}^{11}C_1x^2}{2} + \frac{{}^{11}C_2x^3}{3} + \dots + \frac{{}^{11}C_{11}x^{12}}{12} \Bigg|_0^1 \\
 \frac{2^{12}-1}{12} &= C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_{11}}{12} \dots \dots (1)
 \end{aligned}$$

Now,

$$\int_{-1}^0 (1+x)^{11} dx = \int_{-1}^0 ({}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + \dots + {}^{11}C_{11} x^{11}) dx$$

$$\left[\frac{(1+x)^{12}}{12} \right]_{-1}^0 = {}^{11}C_0 x + \frac{{}^{11}C_1 x^2}{2} + \frac{{}^{11}C_2 x^3}{3} + \dots + \frac{{}^{11}C_{11} x^{12}}{12} \Bigg]_{-1}^0$$

$$\frac{1}{12} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots \dots \dots (2)$$

$$(1) - (2)$$

$$= \frac{2^{12} - 2}{12} = 2 \left[\frac{C_1}{2} + \frac{C_3}{4} + \dots \right]$$

$$\Rightarrow \sum_{r=0}^5 \frac{C_{2r+1}}{2r+2} = \frac{2^{11} - 1}{12} = \frac{2047}{12} = \frac{m}{n}$$

$$= 2047 - 12 = 2035$$

Question18

If $\sum_{r=1}^{30} \frac{r^2 ({}^{30}C_r)^2}{{}^{30}C_{r-1}} = \alpha \times 2^{29}$, then α is equal to _____.

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Answer: 465

Solution:

$$\sum_{r=1}^{30} \frac{r^2 ({}^{30}C_r)^2}{{}^{30}C_{r-1}}$$

$$= \sum_{r=1}^{30} r^2 \left(\frac{31-r}{r} \right) \cdot \frac{30!}{r!(30-r)!}$$

$$\left(\because \frac{{}^{30}C_r}{{}^{30}C_{r-1}} = \frac{30-r+1}{r} = \frac{31-r}{r} \right)$$

$$= \sum_{r=1}^{30} \frac{(31-r)30!}{(r-1)!(30-r)!}$$

$$= 30 \sum_{r=1}^{30} \frac{(31-r)29!}{(r-1)!(30-r)!}$$

$$= 30 \sum_{r=1}^{30} (30-r+1)^{29} C_{30-r}$$

$$= 30 \left(\sum_{r=1}^{30} (31-r)^{29} C_{30-r} + \sum_{r=1}^{30} {}^{29}C_{30-r} \right)$$

$$= 30 (29 \times 2^{28} + 2^{29}) = 30(29+2)2^{28}$$

$$= 15 \times 31 \times 2^{29}$$

$$= 465 (2^{29})$$

$$\alpha = 465$$

Question19

The sum of all rational terms in the expansion of $(1 + 2^{1/3} + 3^{1/2})^6$ is equal to _____.

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Answer: 612

Solution:

$$\left(1 + 2^{\frac{1}{3}} + 3^{\frac{1}{7}}\right)^6$$

$$= \frac{6}{r_1} \frac{6}{r_2} \frac{6}{r_3} (1)^{r_1} (2)^{\frac{r_2}{3}} (3)^{\frac{r_3}{7}}$$

r_1	r_2	r_3
6	0	0
4	0	2
2	0	4
0	0	6
3	3	0
1	3	2
0	6	0

$$= \frac{6}{6 \ 0 \ 0} + \frac{6}{4 \ 0 \ 2} (3) + \frac{6}{2 \ 0 \ 4} (3)^2 + \frac{6}{0 \ 0 \ 6} (3)^3 + \frac{6}{3 \ 3 \ 0} (2) + \frac{6}{1 \ 3 \ 2} (2)^1 (3)^1 + \frac{6}{0 \ 6 \ 0} (2)^2$$

$$= 1 + 45 + 135 + 27 + 40 + 360 + 4 = 612$$

Question 20

If $\alpha = 1 + \sum_{r=1}^6 (-3)^{r-1} {}^{12}C_{2r-1}$, then the distance of the point $(12, \sqrt{3})$ from the line $\alpha x - \sqrt{3}y + 1 = 0$ is _____.

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Answer: 5

Solution:

$$\alpha = 1 + \sum_{r=1}^6 (-1)^{r-1} {}^{12}C_{2r-1} 3^{r-1}$$

$$\alpha = 1 + \sum_{r=1}^6 {}^{12}C_{2r-1} \frac{(\sqrt{3}i)^{2r-1}}{\sqrt{3}i} \quad i = \text{iota, let } \sqrt{3}i = x$$

$$\alpha = 1 + \frac{1}{\sqrt{3}i} ({}^{12}C_1 x + {}^{12}C_3 x^3 + \dots + {}^{12}C_{11} x^{11})$$

$$= 1 + \frac{1}{\sqrt{3}i} \left(\frac{(1 + \sqrt{3}i)^{12} - (1 - \sqrt{3}i)^{12}}{2} \right)$$

$$= 1 + \frac{1}{\sqrt{3}i} \left(\frac{(-2\omega^2)^{12} - (2\omega)^{12}}{2} \right) = 1$$

so distance of $(12, \sqrt{3})$ from $x - \sqrt{3}y + 1 = 0$ is

$$\frac{12 - 3 + 1}{2} = 5$$

Question 21

Let $(1 + x + x^2)^{10} = a_0 + a_1 x + a_2 x^2 + \dots + a_{20} x^{20}$. If $(a_1 + a_3 + a_5 + \dots + a_{19}) - 11a_2 = 121k$, then k is equal to _____.

JEE Main 2025 (Online) 3rd April Evening Shift

Answer: 239

Solution:

$$\text{Let } f(x) = (1 + x + x^2)^{10} = \sum_{r=0}^{20} a_r x^r$$

$$\text{The sum of odd coefficients: } S_{\text{odd}} = a_1 + a_3 + a_5 + \dots + a_{19}$$

Subtracting $11a_2$ from above will give the answer

$$S_{\text{odd}} = \frac{f(1) - f(-1)}{2}$$

$$f(1) = (1 + 1 + 1)^{10} = 3^{10}$$

$$f(-1) = (1 - 1 + 1)^{10} = (1)^{10} = 1$$

$$S_{\text{odd}} = \sum_{\text{odd } r} a_r = \frac{3^{10} - 1}{2}$$

Now for a_2

$$1 + x + x^2 = \frac{1-x^3}{1-x} \Rightarrow f(x) = \left(\frac{1-x^3}{1-x} \right)^{10} = \frac{(1-x^3)^{10}}{(1-x)^{10}}$$

Now use:

$$(1 - x^3)^{10} = \sum_{x=0}^{10} (-1)^k \binom{10}{k} x^{3k}$$

$$(1 - x)^{-10} = \sum_{r=0}^{\infty} \binom{r+9}{9} x^r$$

So

$$f(x) = \left(\sum_{k=0}^{10} (-1)^k \binom{10}{k} x^{3k} \right) \cdot \left(\sum_{r=0}^{\infty} \binom{r+9}{9} x^r \right)$$

Only the term with x^0 from the first sum (i.e., $k = 0$) can contribute to x^2 , since all other $k \geq 1$ gives $x^{3k} \geq x^3$

From $(1 - x^3)^{10}$: the x^0 term is $\binom{10}{0} = 1$

From $(1 - x)^{-10}$: the coefficient of x^2 is

$$\binom{2+9}{9} = \binom{11}{9} = 55$$

$$\text{Hence, } a_2 = 1.55 = 55$$

$$\text{Now, } S_{\text{odd}} - 11a_2 = \frac{3^{10} - 1}{2} - 11 \cdot 55 = 121k$$

$$3^{10} = 59049$$

So:

$$S = \frac{59049 - 1}{2} - 605 = \frac{59048}{2} - 605 = 29524 - 605 = 28919$$

So:

$$121k = 28919 \Rightarrow k = \frac{28919}{121} = 239$$

Question 22

The sum of the series

$2 \times 1 \times {}^{20}C_4 - 3 \times 2 \times {}^{20}C_5 + 4 \times 3 \times {}^{20}C_6 - 5 \times 4 \times {}^{20}C_7 + \dots + 18 \times 17 \times {}^{20}C_{20}$, is equal to _____.

JEE Main 2025 (Online) 7th April Evening Shift

Answer: 34

Solution:

$$(1-x)^{20} = {}^{20}C_0 - {}^{20}C_1x + {}^{20}C_2x^2 - \dots + {}^{20}C_{20}x^{20}$$

$$\frac{(1-x)^{20}}{x^2} = \frac{{}^{20}C_0}{x^2} - \frac{{}^{20}C_1}{x} + {}^{20}C_2 - {}^{20}C_3x + {}^{20}C_4x^2 - \dots$$

Diff twice and put $x = 1$

$$= 6 - {}^{20}C_1(2) + A$$

$$A = 40 - 6 = 34$$

Question23

The product of the last two digits of $(1919)^{1919}$ is

JEE Main 2025 (Online) 8th April Evening Shift

Answer: 63

Solution:

$$(1919)^{1919} = (1920 - 1)^{1919}$$

$$= {}^{1919}C_0(1920)^{1919} - {}^{1919}C_1(1920)^{1918} + \dots$$

$$+ {}^{1919}C_{1918}(1920)^1 - {}^{1919}C_{1919}$$

$$= 100\lambda + 1919 \times 1920 - 1$$

$$= 100\lambda + 3684480 - 1$$

$$= 100\lambda + \dots \dots \dots 79 \text{ (last two digit)}$$

$$\Rightarrow \text{Number having last two digit 79}$$

$$\therefore \text{Product of last two digit 63}$$

Question24

Let α, β, γ and δ be the coefficients of x^7, x^5, x^3 and x respectively in the expansion of

$$\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5, x \geq 1. \text{ If } u \text{ and } v \text{ satisfy the equations}$$

$$\alpha u + \beta v = 18,$$

$$\gamma u + \delta v = 20,$$

then $u + v$ equals :

JEE Main 2025 (Online) 22nd January Evening Shift

Options:

- A. 4
- B. 3
- C. 5
- D. 8

Answer: C

Solution:

To find the sum of u and v , we first need to expand the expression:

$$\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$$

Using the Binomial Theorem, the expansion yields:

$$= 2({}^5C_0 \cdot x^5 + {}^5C_2 \cdot x^3(x^3 - 1) + {}^5C_4 \cdot x(x^3 - 1)^2)$$

Simplifying this, we obtain:

$$= 2(5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x)$$

From this expansion, we can identify the coefficients:

The coefficient of x^7 is $\alpha = 10$

The coefficient of x^5 is $\beta = 2$

The coefficient of x^3 is $\gamma = -20$

The coefficient of x is $\delta = 10$

Given the equations:

$$\alpha u + \beta v = 18$$

$$\gamma u + \delta v = 20$$

Substituting in the coefficients:

$$10u + 2v = 18$$

$$-20u + 10v = 20$$

By solving these equations, we find:

$$\text{From } 10u + 2v = 18, \text{ simplify to } 5u + v = 9.$$

$$\text{From } -20u + 10v = 20, \text{ simplify to } -2u + v = 2.$$

Solving these linear equations simultaneously, we find:

Subtracting equation 2 from equation 1:

$$5u + v = 9$$

$$-(-2u + v = 2)$$

This yields:

$$7u = 7 \Rightarrow u = 1$$

Substitute $u = 1$ back into $5u + v = 9$:

$$5(1) + v = 9 \Rightarrow v = 4$$

Thus, the sum $u + v$ is:

$$u + v = 1 + 4 = 5$$

Question 25

If in the expansion of $(1 + x)^p(1 - x)^q$, the coefficients of x and x^2 are 1 and -2, respectively, then $p^2 + q^2$ is equal to :

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Options:

A. 8

B. 20

C. 13

D. 18

Answer: C

Solution:

$$(1+x)^p(1-x)^q = ({}^pC_0 + {}^pC_1x + {}^pC_2x^2 + \dots) ({}^qC_0 - {}^qC_1x + {}^qC_2x^2 + \dots)$$

$$\text{coff of } x \equiv {}^pC_0 {}^qC_1 - {}^pC_1 {}^qC_0 = 1$$

$$p - q = 1$$

$$\text{coff of } x^2 \equiv {}^pC_0 {}^qC_2 - {}^pC_1 {}^qC_1 + {}^pC_2 {}^qC_0 = -2$$

$$\frac{q(q-1)}{2} - pq + \frac{p(p-1)}{2} = 2$$

$$q^2 - q - 2pq + p^2 - p = -4$$

$$(p-q)^2 - (p+q) = -4$$

$$p+q=5$$

$$p=3$$

$$q=2$$

$$\text{so } p^2 + q^2 = 13$$

Question 26

For some $n \neq 10$, let the coefficients of the 5th, 6th and 7th terms in the binomial expansion of $(1+x)^{n+4}$ be in A.P. Then the largest coefficient in the expansion of $(1+x)^{n+4}$ is:

JEE Main 2025 (Online) 24th January Morning Shift

Options:

A. 10

B. 35

C. 70

D. 20

Answer: B

Solution:

$$(1+x)^{n+4}$$

$${}^{n+4}C_4, {}^{n+4}C_5, {}^{n+4}C_6 \rightarrow \text{A.P.}$$

$$\Rightarrow 2 \times {}^{n+4}C_5 = {}^{n+4}C_4 + {}^{n+4}C_6$$

$$\Rightarrow 4 \times {}^{n+4}C_5 = ({}^{n+4}C_4 + {}^{n+4}C_5) + ({}^{n+4}C_5 + {}^{n+4}C_6)$$

$$\Rightarrow 4 \times {}^{n+4}C_5 = {}^{n+5}C_5 + {}^{n+5}C_6$$

$$\Rightarrow 4 \times \frac{(n+4)!}{5!(n-1)!} = \frac{(n+6)!}{6!n!}$$

$$\Rightarrow 4 = \frac{(n+6)(n+5)}{6n}$$

$$\Rightarrow n^2 + 11n + 30 = 24n$$

$$\Rightarrow n^2 - 13n + 30 = 0$$

$$\Rightarrow n = 3, 10 \text{ (rejected)}$$

$$\therefore n \neq 10$$

$\therefore$ Largest binomial coefficient in expansion of

$$(1+x)^7$$

$$(\because n+4=7)$$

is coeff. of middle term

$$\Rightarrow {}^7C_4 = {}^7C_3 = 35$$

N.T.A. Ans Option (2)

Question 27

Suppose A and B are the coefficients of 30th and 12th terms respectively in the binomial expansion of $(1+x)^{2n-1}$. If $2A = 5B$, then n is equal to:

JEE Main 2025 (Online) 24th January Evening Shift

Options:

- A. 20
- B. 19
- C. 22
- D. 21

Answer: D

Solution:

$$\begin{aligned} A &= {}^{2n-1}C_{29} \quad B = {}^{2n-1}C_{11} \\ {}^{2n-1}C_{29} &= 5 {}^{2n-1}C_{11} \\ 2 \frac{(2n-1)!}{29!(2n-30)!} &= 5 \frac{(2n-1)!}{(2n-12)!11!} \\ \frac{1}{29 \cdot \dots \cdot 12 \cdot 5} &= \frac{1}{(2n-12)(2n-13) \dots (2n-29)2} \\ \frac{1}{30 \cdot 29 \cdot \dots \cdot 12} &= \frac{1}{(2n-12)(2n-13) \dots (2n-29)12} \\ 2n-12 &= 30 \\ n &= 21 \end{aligned}$$

Question 28

Let the coefficients of three consecutive terms T_r , T_{r+1} and T_{r+2} in the binomial expansion of $(a+b)^{12}$ be in a G.P. and let p be the number of all possible values of r . Let q be the sum of all rational terms in the binomial expansion of $(\sqrt[4]{3} + \sqrt[3]{4})^{12}$. Then $p+q$ is equal to:

JEE Main 2025 (Online) 28th January Evening Shift

Options:

- A. 295
- B. 283
- C. 299
- D. 287

Answer: B

Solution:

$$(a + b)^{\frac{1}{2}}$$

$$T_r, T_{r+1}, T_{r+2} \rightarrow GP$$

$$\text{So, } \frac{T_{r+1}}{T_r} = \frac{T_{r+2}}{T_{r+1}}$$

$$\frac{{}^{12}C_r}{{}^{12}C_{r-1}} = \frac{{}^{12}C_{r+1}}{{}^{12}C_r}$$

$$\frac{12-r+1}{r} = \frac{12-(r+1)+1}{r+1}$$

$$(13-r)(r+1) = (12-r)(r)$$

$$-r + 12r + 13 = 12r - r^2$$

$$13 = 0$$

No value of r possible

$$\text{So } P = 0$$

$$\left(3^{\frac{1}{4}} + 4^{\frac{1}{3}}\right)^{12} = \sum {}^{12}C_r \left(3^{\frac{1}{4}}\right)^{12-r} \left(4^{\frac{1}{3}}\right)^r$$

Exponent of $\left(3^{\frac{1}{4}}\right)$ exponent of $\left(4^{\frac{1}{3}}\right)$ term

$$\begin{array}{ccc} 12 & 0 & 27 \\ 0 & 12 & 256 \end{array}$$

$$q = 27 + 256 = 283$$

$$p + q = 0 + 283 = 283$$

Question29

The least value of n for which the number of integral terms in the Binomial expansion of $(\sqrt[3]{7} + \sqrt[12]{11})^n$ is 183, is :

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Options:

A.

$$2184$$

B.

$$2172$$

C.

$$2196$$

D.

$$2148$$

Answer: A

Solution:

$$\text{General term} = {}^nC_r \left(7^{1/3}\right)^{n-r} \left(11^{1/12}\right)^r$$

$$= {}^nC_r (7)^{\frac{n-r}{3}} (11)^{r/12}$$

For integral terms, r must be multiple of 12

$$\therefore r = 12k, k \in W$$

$$\text{Total values of } r = 183$$

$$\text{Hence max } r = 12(182)$$

$$= 2184$$

$$\text{Min value of } n = 2184$$

Question30

The remainder, when 7^{103} is divided by 23, is equal to:

JEE Main 2025 (Online) 29th January Evening Shift

Options:

A.

9

B.

6

C.

14

D.

17

Answer: C

Solution:

$$7^{103} = 7(7^{102}) = 7(343)^{34} = 7(345 - 2)^{34}$$

$$7^{103} = 23 K_1 + 7 \cdot 2^{34}$$

$$\text{Now } 7 \cdot 2^{34} = 7 \cdot 2^2 \cdot 2^{32}$$

$$= 28 \cdot (256)^4$$

$$= 28(253 + 3)^4$$

$$\therefore 28 \times 81 \Rightarrow (23 + 5)(69 + 12)$$

$$23 K_2 + 60$$

$$\therefore \text{Remainder} = 14$$

Question31

The term independent of x in the expansion of $\left(\frac{(x+1)}{(x^{2/3}+1-x^{1/3})} - \frac{(x-1)}{(x-x^{1/2})} \right)^{10}$, $x > 1$, is :

JEE Main 2025 (Online) 2nd April Morning Shift

Options:

A. 240

B. 120

C. 150

D. 210

Answer: D

Solution:

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}} \right)^{10}$$

$$\begin{aligned}
 &= \left(\frac{(x^{1/3})^3 + (1^{1/3})^3}{x^{2/3} - x^{1/3} + 1} - \frac{(\sqrt{x})^2 - (1)^2}{x - x^{1/2}} \right)^{10} \\
 &= \left(\frac{(x^{1/3} + 1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{\sqrt{x}(\sqrt{x} - 1)} \right)^{10} \\
 &= \left((x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}} \right)^{10} \\
 &= \left((x^{1/3} + 1) - \left(1 + \frac{1}{\sqrt{x}} \right) \right)^{10} \\
 &= \left(x^{1/3} - \frac{1}{x^{1/2}} \right)^{10}
 \end{aligned}$$

[Note:

For $(x^\alpha \pm \frac{1}{x^\beta})^n$ the $(r+1)^{\text{th}}$ term with power m of x is

$$r = \frac{n\alpha - m}{\alpha + \beta}]$$

Here $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ and $m = 0$

$$\text{then } r = \frac{10 \times \frac{1}{3} - 0}{\frac{1}{3} + \frac{1}{2}} = \frac{10}{3} \times \frac{6}{5} = 4$$

$\therefore T_5$ is the term independent of x.

$$\therefore T_5 = {}^{10}C_4 = 210$$

Question 32

The largest $n \in \mathbb{N}$ such that 3^n divides $50!$ is :

JEE Main 2025 (Online) 2nd April Morning Shift

Options:

- A. 22
- B. 20
- C. 21
- D. 23

Answer: A

Solution:

$$V_3(50!) = \left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{9} \right\rfloor + \left\lfloor \frac{50}{27} \right\rfloor + \left\lfloor \frac{50}{81} \right\rfloor + \dots + \left\lfloor \frac{50}{3^n} \right\rfloor \\ = 16 + 5 + 1 + 0 + 0 + 0 \dots = 22$$

Question33

If $\sum_{r=0}^{10} \left(\frac{10^{r+1}-1}{10^r} \right) \cdot {}^{11}C_{r+1} = \frac{\alpha 11 - 11^{11}}{10^{10}}$, then α is equal to :

JEE Main 2025 (Online) 2nd April Evening Shift

Options:

- A. 11
- B. 20
- C. 24
- D. 15

Answer: B

Solution:

$$\sum_{r=0}^{10} \left(\frac{10^{r+1}-1}{10^r} \right) {}^{11}C_{r+1} \\ = \sum_{r=0}^{10} \left(10 - \frac{1}{10^r} \right) {}^{11}C_{r+1} \\ = 10 \sum_{r=0}^{10} {}^{11}C_{r+1} - 10 \sum_{r=0}^{10} \left({}^{11}C_{r+1} \left(\frac{1}{10} \right)^{r+1} \right) \\ = 10 \left[{}^{11}C_1 + {}^{11}C_2 + \dots + {}^{11}C_{11} \right] \\ - 10 \left[{}^{11}C_1 \left(\frac{1}{10} \right)^1 + {}^{11}C_2 \left(\frac{1}{10} \right)^2 + \dots + {}^{11}C_{11} \left(\frac{1}{10} \right)^{11} \right] \\ = 10 \left[2^{11} - 1 \right] - 10 \left[\left(1 + \frac{1}{10} \right)^{11} - 1 \right] \\ = 10(2)^{11} - 10 - \frac{11^{11}}{10^{10}} + 10 \\ = \frac{(20)^{11} - 11^{11}}{10^{10}} \\ \therefore \alpha = 20$$

Question34

If $\sum_{r=1}^9 \left(\frac{r+3}{2^r}\right) \cdot {}^9C_r = \alpha \left(\frac{3}{2}\right)^9 - \beta, \alpha, \beta \in \mathbb{N}$, then $(\alpha + \beta)^2$ is equal to

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

- A. 27
- B. 81
- C. 18
- D. 9

Answer: B

Solution:

$$\begin{aligned} \sum_{r=1}^9 \left(\frac{r+3}{2^r}\right) \cdot {}^9C_r &= \sum_{r=1}^9 \frac{r}{2^r} \cdot \frac{9}{r} \cdot {}^8C_{r-1} + \sum_{r=1}^9 3 \cdot {}^9C_r \left(\frac{1}{2}\right)^r \\ &= \frac{9}{2} \sum_{r=1}^9 {}^8C_{r-1} \left(\frac{1}{2}\right)^{r-1} + 3 \sum_{r=1}^9 {}^9C_r \left(\frac{1}{2}\right)^r \\ &= \frac{9}{2} \sum_{r=0}^8 {}^8C_r \left(\frac{1}{2}\right)^r + 3 \sum_{r=1}^9 {}^9C_r \left(\frac{1}{2}\right)^r \\ &= \frac{9}{2} \left(1 + \frac{1}{2}\right)^8 + 3 \left[\left(1 + \frac{1}{2}\right)^9 - {}^9C_0 \left(\frac{1}{2}\right)^0\right] \\ &= \frac{9}{2} \cdot \frac{3^8}{2^8} + 3 \left[\frac{3^9}{2^9} - 1\right] \\ &= \frac{3^{10}}{2^9} + \frac{3^{10}}{2^9} - 3 = 4 \cdot \frac{3^{10}}{2^{10}} - 3 \\ &= 4 \left(\frac{3}{2}\right)^{10} - 3 \\ &= 6 \left(\frac{3}{2}\right)^9 - 3 \\ \alpha &= 6, \beta = 3 \Rightarrow (\alpha + \beta)^2 = 81 \end{aligned}$$

Question 35

The sum of all rational terms in the expansion of $(2 + \sqrt{3})^8$ is :

JEE Main 2025 (Online) 3rd April Morning Shift

Options:

- A. 16923
- B. 18817
- C. 3763
- D. 33845

Answer: B

Solution:

To find the sum of all rational terms in the expansion of $(2 + \sqrt{3})^8$, we consider the binomial expansion:

$$S = {}^8C_0(2)^8 + {}^8C_1(2)^7(\sqrt{3}) + \dots + {}^8C_8(\sqrt{3})^8$$

We need to identify and sum only the rational terms. In the binomial expansion, a term is rational if the exponent of $\sqrt{3}$ is even.

Thus, the rational terms are:

$${}^8C_0(2)^8 + {}^8C_2(2)^6(\sqrt{3})^2 + {}^8C_4(2)^4(\sqrt{3})^4 + {}^8C_6(2)^2(\sqrt{3})^6 + {}^8C_8(\sqrt{3})^8$$

Evaluating these terms:

$${}^8C_0(2)^8 = 256$$

$${}^8C_2(2)^6(3) = {}^8C_2 \times 64 \times 3 = 1344$$

$${}^8C_4(2)^4(3)^2 = {}^8C_4 \times 16 \times 9 = 3024$$

$${}^8C_6(2)^2(3)^3 = {}^8C_6 \times 4 \times 27 = 4032$$

$${}^8C_8(3)^4 = 1 \times 81 = 81$$

Summing these rational terms gives:

$$256 + 1344 + 3024 + 4032 + 81 = 18817$$

Therefore, the sum of all rational terms in the expansion is 18,817.

Question 36

In the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, $n \in \mathbb{N}$, if the ratio of 15th term from the beginning to the 15th term from the end is $\frac{1}{6}$, then the value of ${}^nC_3$ is

JEE Main 2025 (Online) 4th April Morning Shift

Options:

A. 4960

B. 2300

C. 1040

D. 4060

Answer: B

Solution:

In the expansion of $(a + b)^n$

$$15^{\text{th}} \text{ term from beginning: } T_{15} = {}^nC_{14}a^{n-14}b^{14}$$

$$15^{\text{th}} \text{ term from end: } T'_{15} = {}^nC_{14}b^{n-14}a^{14}$$

$$\therefore \frac{T_{15}}{T'_{15}} = \frac{1}{6}$$

$$\left(\frac{a}{b}\right)^{n-28} = \frac{1}{6}$$

$$\left(6^{\frac{1}{3}}\right)^{n-28} = 6^{-1}$$

$$\Rightarrow \frac{n-28}{3} = -1$$

$$n = 25$$

$$\therefore {}^{25}C_3 = 2300$$

Question 37

For an integer $n \geq 2$, if the arithmetic mean of all coefficients in the binomial expansion of $(x + y)^{2n-3}$ is 16, then the distance of the point P $(2n - 1, n^2 - 4n)$ from the line $x + y = 8$ is

JEE Main 2025 (Online) 4th April Morning Shift

Options:

- A. $\sqrt{2}$
- B. $2\sqrt{2}$
- C. $5\sqrt{2}$
- D. $3\sqrt{2}$

Answer: D

Solution:

$$\begin{aligned}\text{Mean} &= \frac{{}^{2n-3}C_0 + {}^{2n-3}C_1 + {}^{2n-3}C_2 + \dots + {}^{2n-3}C_{2n-3}}{2n-2} = 16 \\ &= 2^{2n-3} = 16(2n-2) \\ &= 2^{2n-3} = 2^5(n-1) \\ &\Rightarrow n = 5 \\ \therefore P(9, 5) \\ d &= \left| \frac{9+5-8}{\sqrt{2}} \right| = \frac{6}{\sqrt{2}} = 3\sqrt{2}\end{aligned}$$

Question38

If $1^2 \cdot ({}^{15}C_1) + 2^2 \cdot ({}^{15}C_2) + 3^2 \cdot ({}^{15}C_3) + \dots + 15^2 \cdot ({}^{15}C_{15}) = 2^m \cdot 3^n \cdot 5^k$, where $m, n, k \in \mathbf{N}$, then $m + n + k$ is equal to :

JEE Main 2025 (Online) 4th April Evening Shift

Options:

- A. 20
- B. 19
- C. 18
- D. 21

Answer: B

Solution:

$$\begin{aligned}\sum_{r=1}^{15} r^2 \cdot {}^{15}C_r &= (r^n C_r = n^{n-1} C_{r-1}) \\ &= 15 \sum_{r=1}^{15} r \cdot {}^{14}C_{r-1} \\ &= 15 \sum_{r=1}^{15} (r-1+1) {}^{14}C_{r-1} \\ &= 15 \cdot \sum_{r=1}^{15} (r-1) {}^{14}C_{r-1} + 15 \cdot \sum_{r=1}^{15} {}^{14}C_{r-1} \\ &= 15 \cdot 14 \cdot 2^{13} + 15 \cdot 2^{14} \\ &= 15 \cdot 2^{14} (7+1) \\ &= 5 \cdot 3 \cdot 2^{17} \\ n+m+k &= 17+1+1 = 19\end{aligned}$$

Question39

The remainder when $((64)^{(64)})^{(64)}$ is divided by 7 is equal to

JEE Main 2025 (Online) 7th April Morning Shift

Options:

- A. 4
- B. 6
- C. 3
- D. 1

Answer: D

Solution:

$$64^{64} \Rightarrow (63 + 1)^{64} = 63\lambda + 1$$

$$64^{64^{64}} \Rightarrow (63 + 1)^{64^{64}}$$

$$63\lambda_1 + 1$$

Required remainder when divided 7 is 1.

Question40

The number of integral terms in the expansion of $\left(5^{\frac{1}{2}} + 7^{\frac{1}{8}}\right)^{1016}$ is:

JEE Main 2025 (Online) 8th April Evening Shift

Options:

- A.
127
- B.
128
- C.
130
- D.
129

Answer: B

Solution:

$$T_r = {}^{1016}C_r (5)^{\frac{1016-r}{2}} (7)^{\frac{r}{8}}$$

$$\Rightarrow r = 0, 8, 16, 24, \dots, 1016$$

$$1016 = 0 + (n-1)8$$

$$\Rightarrow n-1 = \frac{1016}{8} = 127$$

$$\text{So, } n = 128.$$

Question41

$${}^{n-1}C_r = (k^2 - 8){}^nC_{r+1} \text{ if and only if :}$$

[27-Jan-2024 Shift 1]

Options:

A.

$$2\sqrt{2} < k \leq 3$$

B.

$$2\sqrt{3} < k \leq 3\sqrt{2}$$

C.

$$2\sqrt{3} < k < 3\sqrt{3}$$

D.

$$2\sqrt{2} < k < 2\sqrt{3}$$

Answer: A

Solution:

$${}^{n-1}C_r = (k^2 - 8){}^nC_{r+1}$$

$$\underbrace{r+1 \geq 0, r \geq 0}_{r \geq 0}$$

$$\frac{{}^{n-1}C_r}{{}^nC_{r+1}} = k^2 - 8$$

$$\frac{r+1}{n} = k^2 - 8$$

$$\Rightarrow k^2 - 8 > 0$$

$$(k - 2\sqrt{2})(k + 2\sqrt{2}) > 0$$

$$k \in (-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty) \dots\dots\dots (i)$$

$$\therefore n \geq r+1, \frac{r+1}{n} \leq 1$$

$$\Rightarrow k^2 - 8 \leq 1$$

$$k^2 - 9 \leq 0$$

$$-3 \leq k \leq 3 \dots\dots\dots (ii)$$

From equation (i) and (ii) we get

$$k \in [-3, -2\sqrt{2}) \cup (2\sqrt{2}, 3]$$

Question 42

If A denotes the sum of all the coefficients in the expansion of $(1 - 3x + 10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1 + x^2)^n$, then :

[27-Jan-2024 Shift 1]

Options:

A.

$$A = B^3$$

B.

$$3A = B$$

C.

$$B = A^3$$

D.

$$A = 3B$$

Answer: A

Solution:

Sum of coefficients in the expansion of

$$(1 - 3x + 10x^2)^n = A$$

$$\text{then } A = (1 - 3 + 10)^n = 8^n \text{ (put } x = 1)$$

and sum of coefficients in the expansion of

$$(1 + x^2)^n = B$$

$$\text{then } B = (1 + 1)^n = 2^n$$

Question 43

The coefficient of x^{2012} in the expansion of $(1 - x)^{2008} (1 + x + x^2)^{2007}$ is equal to

[27-Jan-2024 Shift 2]

Answer: 0

Solution:

$$(1 - x)(1 - x)^{2007} (1 + x + x^2)^{2007}$$

$$(1 - x)(1 - x^3)^{2007}$$

$$(1 - x)(^{2007}C_0 - ^{2007}C_1(x^3) + \dots)$$

General term

$$(1 - x)((-1)^{2007}C_r x^{3r})$$

$$(-1)^{2007}C_r x^{3r} - (-1)^{2007}C_r x^{3r+1}$$

$$3r = 2012$$

$$r \neq \frac{2012}{3}$$

$$3r + 1 = 2012$$

$$3r = 2011$$

$$r \neq \frac{2011}{3}$$

Hence there is no term containing x^{2012} .

So coefficient of $x^{2012} = 0$

Question44

If $\frac{{}^{11}C_1}{2} + \frac{{}^{11}C_2}{3} + \dots + \frac{{}^{11}C_9}{10} = \frac{n}{m}$ with $\gcd(n, m) = 1$, then $n + m$ is equal to

[29-Jan-2024 Shift 1]

Answer: 2041

Solution:

$$\begin{aligned} & \sum_{r=1}^9 \frac{{}^{11}C_r}{r+1} \\ &= \frac{1}{12} \sum_{r=1}^9 {}^{12}C_{r+1} \\ &= \frac{1}{12} [2^{12} - 26] = \frac{2035}{6} \end{aligned}$$

$$\therefore m + n = 2041$$

Question45

Remainder when $64^{32^{32}}$ is divided by 9 is equal to

[29-Jan-2024 Shift 2]

Answer: 1

Solution:

$$\begin{aligned} \text{Let } 32^{32} &= t \\ 64^{32^{32}} &= 64^t = 8^{2t} = (9-1)^{2t} \\ &= 9k + 1 \end{aligned}$$

Hence remainder = 1

Question46

Number of integral terms in the expansion of $\left(7^{\left(\frac{1}{2}\right)} + 11^{\left(\frac{1}{6}\right)}\right)^{824}$ is equal to

[30-Jan-2024 Shift 1]

Answer: 138

Solution:

General term in expansion of $((7)^{1/2} + (11)^{1/6})^{824}$ is $t_{r+1} = {}^{824}C_r (7)^{\frac{824-r}{2}} (11)^{r/6}$

For integral term, r must be multiple of 6.

Hence $r = 0, 6, 12, \dots, 822$

Question 47

Suppose $2 - p$, p , $2 - a$, a are the coefficient of four consecutive terms in the expansion of $(1 + x)^n$. Then the value of $p^2 - a^2 + 6a + 2p$ equals

[30-Jan-2024 Shift 2]

Options:

A.

4

B.

10

C.

8

D.

2

Answer: D

Solution:

$2 - p$, p , $2 - a$, a

Binomial coefficients are

$\Rightarrow {}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}, {}^nC_{r+3}$ respectively

$\Rightarrow {}^nC_r + {}^nC_{r+1} = 2$

${}^{n-1}C_{r+1} = 2 \dots \dots (1)$

Also ${}^nC_{r+2} + {}^nC_{r+3} = 2$

$\Rightarrow {}^{n+1}C_{r+3} = 2 \dots \dots (2)$

From (1) and (2)

${}^{n-1}C_{r+1} = {}^{n+1}C_{r+3}$

$2r + 4 = n + 1$

$n = 2r + 3$

${}^{2r+4}C_{r+1} = 2$

Data Inconsistent

Question48

Let $\alpha = \sum_{k=0}^n \left(\frac{{}^nC_k}{k+1} \right)$ and $\beta = \sum_{k=0}^{n-1} \left(\frac{{}^nC_k \cdot {}^nC_{k+1}}{k+2} \right)$ If $5\alpha = 6\beta$, then n equals

[30-Jan-2024 Shift 2]

Answer: 10

Solution:

$$\alpha = \sum_{k=0}^n \frac{{}^nC_k \cdot {}^nC_k}{k+1} \cdot \frac{n+1}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=0}^n {}^{n+1}C_{k+1} \cdot {}^nC_{n-k}$$

$$\alpha = \frac{1}{n+1} \cdot {}^{2n+1}C_{n+1}$$

$$\beta = \sum_{k=0}^{n-1} {}^nC_k \cdot \frac{{}^nC_{k+1}}{k+2} \cdot \frac{n+1}{n+1}$$

$$= \frac{1}{n+1} \sum_{k=0}^{n-1} {}^nC_{n-k} \cdot {}^{n+1}C_{k+2}$$

$$= \frac{1}{n+1} \cdot {}^{2n+1}C_{n+2}$$

$$\frac{\beta}{\alpha} = \frac{{}^{2n+1}C_{n+2}}{{}^{2n+1}C_{n+1}} = \frac{2n+1-(n+2)+1}{n+2}$$

$$\frac{\beta}{\alpha} = \frac{n}{n+2} = \frac{5}{6}$$

$$n = 10$$

Question49

In the expansion of $(1+x)(1-x^2) \left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right)^5$, $x \neq 0$, the sum of the coefficient of x^3 and x^{-13} is equal to

[31-Jan-2024 Shift 1]

Answer: 118

Solution:

$$(1+x)(1-x^2)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5$$

$$= (1+x)(1-x^2)\left(\left(1+\frac{1}{x}\right)^3\right)^5$$

$$= \frac{(1+x)^2(1-x)(1+x)^{15}}{x^{15}}$$

$$= \frac{(1+x)^{17}-x(1+x)^{17}}{x^{15}}$$

$$= \text{coeff}(x^3) \text{ in the expansion } \approx \text{coeff}(x^{18}) \text{ in } (1+x)^{17}-x(1+x)^{17}$$

$$= 0 - 1$$

$$= -1$$

$$\text{coeff}(x^{-13}) \text{ in the expansion } \approx \text{coeff}(x^2) \text{ in } (1+x)^{17}-x(1+x)^{17}$$

$$= \binom{17}{2} - \binom{17}{1}$$

$$= 17 \times 8 - 17$$

$$= 17 \times 7$$

$$= 119$$

$$\text{Hence Answer} = 119 - 1 = 118$$

Question 50

If for some m, n ; ${}^6C_m + 2({}^6C_{m+1}) + {}^6C_{m+2} > {}^8C_3$ and ${}^{n-1}P_3 : {}^nP_4 = 1 : 8$, then ${}^nP_{m+1} + {}^{n+1}C_m$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

380

B.

376

C.

384

D.

372

Answer: D

Solution:

$${}^6C_m + 2({}^6C_{m+1}) + {}^6C_{m+2} > {}^8C_3$$

$${}^7C_{m+1} + {}^7C_{m+2} > {}^8C_3$$

$${}^8C_{m+2} > {}^8C_3$$

$$\therefore m = 2$$

$$\text{And } {}^{n-1}P_3 : {}^nP_4 = 1 : 8$$

$$\frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)} = \frac{1}{8}$$

$$\therefore n = 8$$

$$\therefore {}^nP_{m+1} + {}^{n+1}C_m = {}^8P_3 + {}^9C_2$$

$$= 8 \times 7 \times 6 + \frac{9 \times 8}{2}$$

$$= 372$$

Question51

Let the coefficient of x^r in the expansion of

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) +$$

$$(x+3)^{n-3}(x+2)^2 + \dots + (x+2)^{n-1}$$

be α_r . If $\sum_{r=0}^n \alpha_r = \beta^n - \gamma^n$, $\beta, \gamma \in N$, then the value of $\beta^2 + \gamma^2$ equals

[31-Jan-2024 Shift 2]

Answer: 25

Solution:

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \dots + (x+2)^{n-1}$$

$$\sum \alpha_r = 4^{n-1} + 4^{n-2} \times 3 + 4^{n-3} \times 3^2 + \dots + 3^{n-1}$$

$$= 4^{n-1} \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots + \left(\frac{3}{4}\right)^{n-1} \right]$$

$$= 4^{n-1} \times \frac{1 - \left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}}$$

$$= 4^n - 3^n = \beta^n - \gamma^n$$

$$\beta = 4, \gamma = 3$$

$$\beta^2 + \gamma^2 = 16 + 9 = 25$$

Question52

If the Coefficient of x^{30} in the expansion of $(1 + 1/x)^6(1 + x^2)^7(1 - x^3)^8$; $x \neq 0$ is α , then $|\alpha|$ equals _____

[1-Feb-2024 Shift 1]

Answer: 678

Solution:

$$\text{coeff of } x^{30} \text{ in } \frac{(x+1)^6(1+x^2)^7(1-x^3)^8}{x^6}$$

$$\text{coeff. of } x^{36} \text{ in } (1+x)^6(1+x^2)^7(1-x^3)^8$$

General term

$${}^6C_{r_1} {}^7C_{r_2} {}^8C_{r_3} (-1)^{r_3} x^{r_1+2r_2+3r_3}$$

$$r_1 + 2r_2 + 3r_3 = 36$$

Case - I :

r_1	r_2	r_3
0	6	8
2	5	8
4	4	8
6	3	8

$$r_1 + 2r_2 = 12 \quad (\text{Taking } r_3 = 8)$$

Case - II :

r_1	r_2	r_3
1	7	7
3	6	7
5	5	7

$$r_1 + 2r_2 = 15 \quad (\text{Taking } r_3 = 7)$$

Case - III :

r_1	r_2	r_3
4	7	6
6	6	6

$$r_1 + 2r_2 = 18 \quad (\text{Taking } r_3 = 6)$$

$$\text{Coeff.} = 7 + (15 \times 21) + (15 \times 35) + (35) - (6 \times 8) - (20 \times 7 \times 8) - (6 \times 21 \times 8) + (15 \times 28) + (7 \times 28) = -678 = a$$

$$|a| = 678$$

Question 53

Let m and n be the coefficients of seventh and thirteenth terms respectively in the expansion of

$$\left(\frac{1}{3}x^{\frac{1}{3}} + \frac{1}{2x^{\frac{2}{3}}} \right)^{18}.$$

Then

$(n/m)^{1/3}$ is:

[1-Feb-2024 Shift 2]

Options:

A.

$\frac{4}{9}$

B.

$\frac{1}{9}$

C.

$\frac{1}{4}$

D.

$\frac{9}{4}$

Answer: D

Solution:

$$\left(\frac{x^{\frac{1}{3}}}{3} + \frac{x^{-\frac{2}{3}}}{x} \right)^{18}$$

$$t_7 = {}^{18}C_6 \left(\frac{x^{\frac{1}{3}}}{3} \right)^{12} \left(\frac{x^{-\frac{2}{3}}}{x} \right)^6 = {}^{18}C_6 \frac{1}{(3)^{12}} \cdot \frac{1}{2^6}$$

$$t_{13} = {}^{18}C_{12} \left(\frac{x^{\frac{1}{3}}}{3} \right)^6 \left(\frac{x^{-\frac{2}{3}}}{x} \right)^{12} = {}^{18}C_{12} \frac{1}{(3)^6} \cdot \frac{1}{2^{12}} \cdot x^{-6}$$

$$m = {}^{18}C_6 \cdot 3^{-12} \cdot 2^{-6} : n = {}^{18}C_{12} \cdot 2^{-12} \cdot 3^{-6}$$

$$\left(\frac{n}{m} \right)^{\frac{1}{3}} = \left(\frac{2^{-12} \cdot 3^{-6}}{3^{-12} \cdot 2^{-6}} \right)^{\frac{1}{3}} = \left(\frac{3}{2} \right)^2 = \frac{9}{4}$$

Question54

The value $\sum_{r=0}^{22} {}^{22}C_r \cdot {}^{23}C_r$ is
[24-Jan-2023 Shift 1]

Options:

A. ${}^{45}C_{23}$

B. ${}^{44}C_{23}$

C. ${}^{45}C_{24}$

D. ${}^{44}C_{22}$

Answer: A

Solution:

Solution:

$$\sum_{r=0}^{22} {}^{22}C_r \cdot {}^{23}C_r = \sum_{r=0}^{22} {}^{22}C_r \cdot {}^{23}C_{23-r} \\ = {}^{45}C_{23}$$

Question55

Suppose $\sum_{r=0}^{2023} r^2 \cdot {}^{2023}C_r = 2023 \times \alpha \times 2^{2022}$. Then the value of α is
[24-Jan-2023 Shift 1]

Answer: 1012

Solution:

using result

$$\sum_{r=0}^n r^2 C_r = n(n+1) \cdot 2^{n-2}$$

$$\text{Then } \sum_{r=0}^{2023} r^2 \cdot {}^{2023}C_r = 2023 \times 2024 \times 2^{2021}$$

$$= 2023 \times \alpha \times 2^{2022} \text{ So,}$$

$$\alpha = 1012$$

Question56

If $({}^{30}C_1)^2 + 2({}^{30}C_2)^2 + 3({}^{30}C_3)^2 + \dots + 30({}^{30}C_{30})^2 = \frac{\alpha 60!}{(30!)^2}$, then α is equal to

[24-Jan-2023 Shift 2]

Options:

A. 30

B. 60

C. 15

D. 10

Answer: C

Solution:

$$S = 0 \cdot ({}^{30}C_0)^2 + 1 \cdot ({}^{30}C_1)^2 + 2 \cdot ({}^{30}C_2)^2 + \dots + 30 \cdot ({}^{30}C_{30})^2$$

$$S = 30 \cdot ({}^{30}C_0)^2 + 29 \cdot ({}^{30}C_1)^2 + 28 \cdot ({}^{30}C_2)^2 + \dots + 0 \cdot ({}^{30}C_{30})^2$$

$$2S = 30 \cdot ({}^{30}C_0^2 + {}^{30}C_1^2 + \dots + {}^{30}C_{30}^2)$$

$$S = 15 \cdot {}^{60}C_{30} = 15 \cdot \frac{60!}{(30!)^2}$$

$$\frac{15 \cdot 10!}{(30!)^2} = \frac{\alpha \cdot 60!}{(30!)^2}$$

$$\Rightarrow \alpha = 15$$

Question57

Let the sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^n$, $x \neq 0$, $n \in \mathbb{N}$, be 376

. Then the coefficient of x^4 is _____

[24-Jan-2023 Shift 2]

Answer: 405

Solution:

Given Binomial $\left(x - \frac{3}{x^2}\right)^n$, $x \neq 0$, $n \in \mathbb{N}$,

Sum of coefficients of first three terms

$${}^nC_0 - {}^nC_1 \cdot 3 + {}^nC_2 3^2 = 376$$

$$\Rightarrow 3n^2 - 5n - 250 = 0$$

$$\Rightarrow (n-10)(3n+25) = 0$$

$$\Rightarrow n = 10$$

Now general term ${}^{10}C_r x^{10-r} \left(\frac{-3}{x^2}\right)^r$

$$= {}^{10}C_r x^{10-r} (-3)^r \cdot x^{-2r}$$

$$= {}^{10}C_r (-3)^r \cdot x^{10-3r}$$

Coefficient of $x^4 \Rightarrow 10 - 3r = 4$

$$\Rightarrow r = 2$$

$${}^{10}C_2 (-3)^2 = 405$$

Question 58

If a_r is the coefficient of x^{10-r} in the Binomial expansion of $(1+x)^{10}$, then $\sum_{r=1}^{10} r^3 \left(\frac{a_r}{a_{r-1}}\right)^2$ is equal to

[25-Jan-2023 Shift 1]

Options:

A. 4895

B. 1210

C. 5445

D. 3025

Answer: B

Solution:

$$a_r = {}^{10}C_{10-r} = {}^{10}C_r$$

$$\Rightarrow \sum_{r=1}^{10} r^3 \left(\frac{{}^{10}C_r}{{}^{10}C_{r-1}}\right)^2$$

$$= \sum_{r=1}^{10} r^3 \left(\frac{11-r}{r}\right)^2$$

$$= \sum_{r=1}^{10} r(11-r)^2$$

$$= \sum_{r=1}^{10} (121r + r^3 - 22r^2) = 1210$$

Question 59

The constant term in the expansion of

$$\left(2x + \frac{1}{x} + 3x^2\right)^5 \text{ is } \underline{\hspace{2cm}}.$$

[25-Jan-2023 Shift 1]

Answer: 1080

Solution:

General term is $\sum \frac{5!(2x)^{n_1}(x^{-7})^{n_2}(3x^2)^{n_3}}{n_1!n_2!n_3!}$

For constant term,

$$n_1 + 2n_3 = 7n_2$$

$$n_1 + n_2 + n_3 = 5$$

Only possibility $n_1 = 1, n_2 = 1, n_3 = 3$

$\Rightarrow$ constant term = 1080

Question60

$\sum_{k=0}^6 {}^{51-k}C_3$ is equal to

[25-Jan-2023 Shift 2]

Options:

A. ${}^{51}C_4 - {}^{45}C_4$

B. ${}^{51}C_3 - {}^{45}C_3$

C. ${}^{52}C_4 - {}^{45}C_4$

D. ${}^{52}C_3 - {}^{45}C_3$

Answer: C

Solution:

$$\begin{aligned} & \sum_{k=0}^6 {}^{51-k}C_3 \\ &= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + \dots + {}^{45}C_3 \\ &= {}^{45}C_3 + {}^{46}C_3 + \dots + {}^{51}C_3 \\ &= {}^{45}C_4 + {}^{45}C_3 + {}^{46}C_3 + \dots + {}^{51}C_3 - {}^{45}C_4 \\ & \quad ({}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r) \\ &= {}^{52}C_4 - {}^{45}C_4 \end{aligned}$$

Question61

The remainder when $(2023)^{2023}$ is divided by 35 is _____.

[25-Jan-2023 Shift 2]

Answer: 7

Solution:

$$\begin{aligned} & (2023)^{2023} \\ &= (2030 - 7)^{2023} \\ &= (35K - 7)^{2023} \\ &= {}^{2023}C_0 (35K)^{2023} (-7)^0 + {}^{2023}C_1 (35K)^{2022} (-7)^1 + \\ & \quad \dots + \dots + {}^{2023}C_{2023} (-7)^{2023} \\ &= 35N - 7^{2023} \\ & \text{Now, } -7^{2023} = -7 \times 7^{2022} = -7(7^2)^{1011} \\ &= -7(50 - 1)^{1011} \\ &= -7({}^{1011}C_0 50^{1011} - {}^{1011}C_1 (50)^{1010} + \dots + {}^{1011}C_{1011}) \\ &= -7(50 - 1) \end{aligned}$$

$$= -35\lambda + 7$$

$\therefore$ when $(2023)^{2023}$ is divided by 35 remainder is 7

Question62

If the co-efficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to _____.

[29-Jan-2023 Shift 1]

Answer: 1

Solution:

Coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11} = {}^{11}C_6 \cdot \frac{\alpha^5}{\beta^6}$

$\therefore$ Both are equal

$$\therefore \frac{{}^{11}C_6 \cdot \frac{\alpha^5}{\beta^6}}{{}^{11}C_5 \cdot \frac{\alpha^6}{\beta^5}} = 1$$

$$\Rightarrow \frac{1}{\beta} = -\alpha$$

$$\Rightarrow \alpha\beta = -1$$

$$\Rightarrow (\alpha\beta)^2 = 1$$

Question63

Let the coefficients of three consecutive terms in the binomial expansion of $(1+2x)^n$ be in the ratio $2 : 5 : 8$. Then the coefficient of the term, which is in the middle of these three terms, is _____.

[29-Jan-2023 Shift 1]

Answer: 1120

Solution:

$${}^{nC}_{r+1} = {}^{nC}_r (2x)^r$$

$$\Rightarrow \frac{{}^{nC}_{r+1} (2)^{r+1}}{{}^{nC}_r (2)^r} = \frac{2}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{r!(n-r)!}} = \frac{2}{5}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{4}{5} \Rightarrow 5r = 4n - 4r + 4$$

$$\Rightarrow 9r = 4(n+1) \dots (1)$$

$$\Rightarrow \frac{{}^{nC}_r (2)^r}{{}^{nC}_{r+1} (2)^{r+1}} = \frac{5}{8}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r+1)!(n-r-1)!}} = \frac{5}{4} \Rightarrow \frac{r+1}{n-r} = \frac{5}{4}$$

$$\Rightarrow 4r+4 = 5n-5r \Rightarrow 5n-4 = 9r \dots (2)$$

From (1) and (2)

$$\Rightarrow 4n+4 = 5n-4 \Rightarrow n = 8$$

$$(1) \Rightarrow r = 4$$

so, coefficient of middle term is

$${}^8C_4 2^4 = 16 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 16 \times 70 = 1120$$

Question 64

Let K be the sum of the coefficients of the odd powers of x in the expansion of $(1+x)^{99}$. Let a be the middle term in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$. If $\frac{{}^{200}C_{99}K}{a} = \frac{2^l m}{n}$, where m and n are odd numbers, then the ordered pair (l, n) is equal to :
[29-Jan-2023 Shift 2]

Options:

- A. (50, 51)
- B. (51, 99)
- C. (50, 101)
- D. (51, 101)

Answer: C

Solution:

In the expansion of

$$(1+x)^{99} = C_0 + C_1x + C_2x^2 + \dots + C_{99}x^{99}$$

$$K = C_1 + C_3 + \dots + C_{99} = 2^{98}$$

$a \Rightarrow$ Middle in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$

$$T_{\frac{200}{2}+1} = {}^{200}C_{100} (2)^{100} \left(\frac{1}{\sqrt{2}}\right)^{100}$$

$$= {}^{200}C_{100} \cdot 2^{50}$$

$$\text{So, } \frac{{}^{200}C_{99} \times 2^{98}}{{}^{200}C_{100} \times 2^{50}} = \frac{100}{101} \times 2^{48}$$

$$\text{So, } \frac{25}{101} \times 2^{50} = \frac{m \cdot 2^l}{n}$$

$\therefore$ m, n are odd so

(l, n) become (50, 101) Ans.

Question 65

If the coefficient of x^{15} in the expansion of $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15}$ is equal to the coefficient of x^{-15} in the

expansion of $\left(ax^{\frac{1}{3}} - \frac{1}{bx^3}\right)^{15}$, where a and b are positive real numbers, then for each such ordered pair

(a, b) :

[30-Jan-2023 Shift 1]

Options:

- A. $a = b$
- B. $ab = 1$
- C. $a = 3b$
- D. $ab = 3$

Answer: B

Solution:

Option (2)

Coefficient Of x^{15} in $\left(ax^3 + \frac{1}{bx^{1/3}}\right)^{15}$

$$T_{r+1} = {}^{15}C_r (ax^3)^{15-r} \left(\frac{1}{bx^{1/3}}\right)^r$$

$$45 - 3r - \frac{r}{3} = 15$$

$$30 = \frac{10r}{3}$$

$$r = 9$$

$$\text{Coefficient of } x^{15} = {}^{15}C_9 a^6 b^{-9}$$

Coefficient of x^{15} in $\left(ax^{1/3} - \frac{1}{bx^3}\right)^{15}$

$$T_{r+1} = {}^{15}C_r (ax^{1/3})^{15-r} \left(-\frac{1}{bx^3}\right)^r$$

$$5 - \frac{r}{3} - 3r = -15$$

$$\frac{10r}{3} = 20$$

$$r = 6$$

$$\text{Coefficient} = {}^{15}C_6 a^9 \times b^{-6}$$

$$\Rightarrow \frac{a^9}{b^6} = \frac{a^6}{b^9}$$

$$\Rightarrow a^3 b^3 = 1 \Rightarrow ab = 1$$

Question66

The coefficient of x^{301} in $(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \dots + x^{500}$ is:
[30-Jan-2023 Shift 1]

Options:

A. ${}^{501}C_{302}$

B. ${}^{500}C_{301}$

C. ${}^{500}C_{300}$

D. ${}^{501}C_{200}$

Answer: D

Solution:

$$(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \dots + x^{500}$$

$$= (1+x)^{500} \cdot \left\{ \frac{1 - \left(\frac{x}{1+x}\right)^{501}}{1 - \frac{x}{1+x}} \right\}$$

$$= (1+x)^{500} \cdot \frac{(1+x)^{501} - x^{501}}{(1+x)^{501}} \cdot (1+x)$$

$$= (1+x)^{501} - x^{501}$$

Coefficient of x^{301} in $(1+x)^{501} - x^{501}$ is given by

$${}^{501}C_{301} = {}^{501}C_{200}$$

Question67

Let $x = (8\sqrt{3} + 13)^{13}$ and $y = (7\sqrt{2} + 9)^9$. If $[t]$ denotes the greatest integer $\leq t$, then
[30-Jan-2023 Shift 2]

Options:

- A. $[x] + [y]$ is even
- B. $[x]$ is odd but $[y]$ is even
- C. $[x]$ is even but $[y]$ is odd
- D. $[x]$ and $[y]$ are both odd

Answer: A

Solution:

$$\begin{aligned} x &= (8\sqrt{3} + 13) = {}^{13}C_0 \cdot (8\sqrt{3})^{13} + {}^{13}C_1 (8\sqrt{3})^{12} (13)^1 + \dots \\ x' &= (8\sqrt{3} - 13)^{13} = {}^{13}C_0 (8\sqrt{3})^{13} - {}^{13}C_1 (8\sqrt{3})^{12} (13)^1 + \dots \\ x - x' &= 2[{}^{13}C_1 \cdot (8\sqrt{3})^{12} (13)^1 + {}^{13}C_3 (8\sqrt{3})^{10} \cdot (13)^3 \dots] \\ \text{therefore, } x - x' &\text{ is even integer, hence } [x] \text{ is even} \\ \text{Now, } y &= (7\sqrt{2} + 9)^9 = {}^9C_0 (7\sqrt{2})^9 + {}^9C_1 (7\sqrt{2})^8 (9)^1 \\ &\quad + {}^9C_2 (7\sqrt{2})^7 (9)^2 \dots \dots \dots \\ y' &= (7\sqrt{2} - 9)^9 = {}^9C_0 (7\sqrt{2})^9 - {}^9C_1 (7\sqrt{2})^8 (9)^1 \\ &\quad + {}^9C_2 (7\sqrt{2})^7 (9)^2 \dots \dots \dots \\ y - y' &= 2[{}^9C_1 (7\sqrt{2})^8 (9)^1 + {}^9C_3 (7\sqrt{2})^6 (9)^3 + \dots] \\ y - y' &= \text{Even integer, hence } [y] \text{ is even} \end{aligned}$$

Question68

50^{th} root of a number x is 12 and 50^{th} root of another number y is 18. Then the remainder obtained on dividing $(x + y)$ by 25 is _____.
[30-Jan-2023 Shift 2]

Answer: 23

Solution:

$$\begin{aligned} x + y &= 12^{50} + 18^{50} = (150 - 6)^{25} + (325 - 1)^{25} \\ &= 25K - (6^{25} + 1) = 25K - ((5 + 1)^{25} + 1) \\ &= 25K_1 - 2 \quad \text{Remainder} = 23 \end{aligned}$$

Question69

Let $\alpha > 0$, be the smallest number such that the expansion of $\left(x^{\frac{2}{3}} + \frac{2}{x^3}\right)^{30}$ has a term $\beta x^{-\alpha}$, $\beta \in \mathbb{N}$.

Then α is equal to _____.
[31-Jan-2023 Shift 1]

Answer: 2

Solution:

$$T_{r+1} = {}^{30}C_r (x^{2/3})^{30-r} \left(\frac{2}{x}\right)^r$$

$$= {}^{30}C_r \cdot 2 \cdot x^{\frac{60-11r}{3}}$$

$$\frac{60-11r}{3} < 0 \Rightarrow 11r > 60 \Rightarrow r > \frac{60}{11} \Rightarrow r = 6$$

$$T_7 = {}^{30}C_6 \cdot 2^6 x^{-2}$$

We have also observed $\beta = {}^{30}C_6 (2)^6$ is a natural number.

$$\therefore \alpha = 2$$

Question70

The remainder on dividing 5^{99} by 11 is _____.
[31-Jan-2023 Shift 1]

Answer: 9

Solution:

$$\begin{aligned} 5^{99} &= 5^4 \cdot 5^{95} \\ &= 625[5^5]^{19} \\ &= 625[3125]^{19} \\ &= 625[3124 + 1]^{19} \\ &= 625[11k + 19 + 1] \\ &= 625 \times 11k + 19 + 625 \\ &= 11k_1 + 616 + 9 \\ &= 11(k_2) + 9 \\ \text{Remainder} &= 9 \end{aligned}$$

Question71

The Coefficient of x^{-6} , in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x^2}\right)^9$, is _____
[31-Jan-2023 Shift 2]

Answer: 5040

Solution:

$$\left(\frac{4x}{5} + \frac{5}{2x^2}\right)^9$$

$$\text{Now, } T_{r+1} = {}^9C_r \left(\frac{4x}{5}\right)^{9-r} \left(\frac{5}{2x^2}\right)^r$$

$$= {}^9C_r \cdot \left(\frac{4}{5}\right)^{9-r} \left(\frac{5}{2}\right)^r \cdot x^{9-3r}$$

$$\text{Coefficient of } x^{-6} \text{ i.e. } 9 - 3r = -6 \Rightarrow r = 5$$

Question72

If the constant term in the binomial expansion of $\left(\frac{x^2}{2} - \frac{4}{x^l}\right)^9$ is -84 and the Coefficient of x^{-3l} is $2^\alpha \beta$,

where $\beta < 0$ is an odd number, Then $|\alpha l - \beta|$ is equal to _____

[31-Jan-2023 Shift 2]

Answer: 98

Solution:

$$\begin{aligned} \text{In, } \left(\frac{x^2}{2} - \frac{4}{x^l}\right)^9 \\ T_{r+1} = {}^9C_r \frac{(x^{2/2})^{9-r}}{2^{9-r}} \left(\frac{-4}{x^l}\right)^r \\ = (-1)^r \frac{{}^9C_r 4^r x^{\frac{45}{2} - \frac{5r}{2}}}{2^{9-r}} \\ = 45 - 5r - 21r = 0 \\ r = \frac{45}{5+21} \dots (1) \end{aligned}$$

$$\text{Now, according to the question, } (-1)^r \frac{{}^9C_r 4^r}{2^{9-r}} = -84$$

$$= (-1)^9 {}^9C_3 2^{3r-9} = 21 \times 4$$

Only natural value of r possible if $3r - 9 = 0$ $r = 3$ and ${}^9C_3 = 84$

$\therefore 1 = 5$ from equation (1)

Now, coefficient of $x^{-3l} = x^{\frac{45}{2} - \frac{5r}{2} - 1r}$ at $l = 5$, gives
 $r = 5$

$$\therefore {}^9C_5 (-1)^5 \frac{4^5}{2^4} = 2^\alpha \times \beta$$

$$= -63 \times 2^7$$

$$\Rightarrow \alpha = 7, \beta = -63$$

$$\therefore \text{value of } |\alpha l - \beta| = 98$$

Question 73

The value of $\frac{1}{1!50!} + \frac{1}{3!48!} + \frac{1}{5!46!} + \dots + \frac{1}{49!2!} + \frac{1}{51!1!}$ is

[1-Feb-2023 Shift 1]

Options:

A. $\frac{2^{50}}{50!}$

B. $\frac{2^{50}}{51!}$

C. $\frac{2^{51}}{51!}$

D. $\frac{2^{51}}{50!}$

Answer: B

Solution:

$$\begin{aligned} \sum_{r=1}^{26} \frac{1}{(2r-1)!(51-(2r-1))!} &= \sum_{r=1}^{26} {}^{51}C_{(2r-1)} \frac{1}{51!} \\ &= \frac{1}{51!} \{ {}^{51}C_1 + {}^{51}C_3 + \dots + {}^{51}C_{51} \} = \frac{1}{51!} (2^{50}) \end{aligned}$$

Question 74

The remainder when $19^{200} + 23^{200}$ is divided by 49, is _____.
[1-Feb-2023 Shift 1]

Answer: 29

Solution:

$$\begin{aligned} & (21+2)^{200} + (21-2)^{200} \\ & \Rightarrow 2[{}^{100}C_0 21^{200} + {}^{200}C_2 21^{198} \cdot 2^2 + \dots + {}^{200}C_{198} 21^2 \cdot 2^{198} + 2^{200}] \\ & \Rightarrow 2[49I_1 + 2^{200}] = 49I_1 + 2^{201} \\ & \text{Now, } 2^{201} = (8)^{67} = (1+7)^{67} = 49I_2 + {}^{67}C_0 {}^{67}C_1 \cdot 7 = \\ & 49I_2 + 470 = 49I_2 + 49 \times 9 + 29 \end{aligned}$$

Question 75

Let the sixth term in the binomial expansion of $(\sqrt{2^{\log_2(10-3^x)}} + {}^5\sqrt{2^{(x-2)\log_2 3}})^m$, in the increasing powers of $2^{(x-2)\log_2 3}$, be 21. If the binomial coefficients of the second, third and fourth terms in the expansion are respectively the first, third and fifth terms of an A.P., then the sum of the squares of all possible values of x is _____.
[1-Feb-2023 Shift 2]

Answer: 4

Solution:

$$T_6 = {}^mC_5 (10-3^x)^{\frac{m-5}{2}} \cdot (3^{x-2}) = 21$$

${}^mC_1, {}^mC_2, {}^mC_3$ are in A.P.

$$2 \cdot {}^mC_2 = {}^mC_1 + {}^mC_3$$

Solving for m, we get

m = 2 (rejected), 7

Put in equation (1)

$$21 \cdot (10-3^x)^{\frac{3}{2}} = 21$$

$$3^x = 3^0, 3^2$$

$$x = 0, 2$$

Sum of the squares of all possible values of x = 4

Question 76

If the term without x in the expansion of $(x^{\frac{2}{3}} + \frac{a}{x^3})^{22}$ is 7315, then |a| is equal to _____.
[1-Feb-2023 Shift 2]

Answer: 1

Solution:

$$\begin{aligned}T_{r+1} &= {}^{22}C_r \cdot \left(x \frac{2}{3}\right)^{22-r} \cdot (\alpha)^r \cdot x^{-3r} \\&= {}^{22}C_r \cdot x \frac{44}{3} \frac{2r}{3} \frac{1}{3} (\alpha)^r \\&\frac{44}{3} = \frac{11r}{3} \\r &= 4 \\{}^{22}C_4 \cdot \alpha^4 &= 7315 \\&\frac{22 \times 21 \times 20 \times 19}{24} \cdot \alpha^4 = 7315 \\&\alpha = 1\end{aligned}$$

Question 77

If the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^a$ is $\sqrt{6} : 1$, then the third term from the beginning is :

[6-Apr-2023 shift 1]

Options:

- A. $30\sqrt{2}$
- B. $60\sqrt{2}$
- C. $30\sqrt{3}$
- D. $60\sqrt{3}$

Answer: D

Solution:

$$\begin{aligned}\frac{T_5}{T_5} &= \frac{{}^nC_4 \cdot \left(\frac{1}{2}\right)^{n-4} \left(\frac{1}{3}\right)^4}{{}^nC_4 \left(\frac{1}{3}\right)^{n-4} \left(\frac{1}{2}\right)^4} = \frac{\sqrt{6}}{1} \\2 \frac{n-8}{4} \cdot \left(\frac{1}{3}\right)^{-4-4+n} &= \sqrt{6} \\2 \frac{n-8}{4} \cdot 3 \frac{n-8}{4} &= \sqrt{6} \\&\frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8=2 \Rightarrow n=10 \\T_3 &= {}^{10}C_2 \left(\frac{1}{2}\right)^8 \left(\frac{1}{3}\right)^2 \\&= {}^{10}C_2 \cdot 2^2 \cdot 3^{-\frac{1}{2}} = \frac{10 \cdot 9}{2} \cdot 4 \cdot \frac{1}{\sqrt{3}} = 60\sqrt{3}\end{aligned}$$

Question 78

If ${}^{2n}C_3 : {}^nC_3 : 10 : 1$, then the ratio $(n^2 + 3n) : (n^2 - 3n + 4)$ is :
[6-Apr-2023 shift 1]

Options:

- A. 27 : 11
- B. 35 : 16
- C. 2 : 1
- D. 65 : 37

Answer: C

Solution:

$$\begin{aligned}\frac{{}^{2n}C_3}{{}^nC_3} &= 10 \Rightarrow \frac{2n!(n-3)!}{(2n-3)!n!} = 10 \\ \frac{2n(2n-1)(2n-2)}{n(n-1)(n-2)} &= 10 \\ \frac{4(2n-1)}{n-2} &= 10 \Rightarrow 8n-4 = 10n-20 \\ 2n &= 16 \\ \text{Now } \frac{n^2+3n}{n^2-3n+4} &= 2 \\ = \frac{64+24}{64-24+4} &= \frac{88}{44} = 2\end{aligned}$$

Question79

The coefficient of x^{18} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is _____.

[6-Apr-2023 shift 1]

Answer: 5005

Solution:

$$\begin{aligned}\left(x^4 - \frac{1}{x^3}\right)^{15} \\ T_{r+1} &= {}^{15}C_r (x^4)^{15-r} \left(\frac{-1}{x^3}\right)^r \\ 60-7r &= 18 \\ r &= 6 \\ \text{Hence coeff. of } x^{18} &= {}^{15}C_6 = 5005\end{aligned}$$

Question80

If the coefficients of x^7 in $\left(ax^2 + \frac{1}{2bx}\right)^{11}$ and x^{-7} in $\left(ax - \frac{1}{3bx^2}\right)^{11}$ are equal, then :

[6-Apr-2023 shift 2]

Options:

- A. $64ab = 243$
- B. $32ab = 729$
- C. $729ab = 32$
- D. $243ab = 64$

Answer: C

Solution:

$$\left(ax^2 + \frac{1}{2bx}\right)^{11}$$
$$r = \frac{11 \times 2 - 7}{3} = 5$$

$$\text{Coefficient of } x^7 \text{ is } = {}^{11}C_5(a)^6\left(\frac{1}{2b}\right)^5$$

$$\left(ax - \frac{1}{3bx^2}\right)^{11}$$
$$r = \frac{11 \times 1 - (-7)}{3} = 6$$

$$\text{Coefficient of } x^{-7} \text{ is } = {}^{11}C_6 \cdot \frac{a^5}{3^6 b^6}$$

$$\therefore {}^{11}C_5(a^6)\left(\frac{1}{2^5 b^5}\right) = {}^{11}C_6 \cdot \frac{a^5}{3^6 b^6}$$

$$\Rightarrow ab = \frac{2^5}{3^6}$$

$$\Rightarrow 729 ab = 32$$

Ans. Option 3

Question81

Among the statements :

(S1) : $2023^{2022} - 1999^{2022}$ is divisible by 8

(S2) : $13(13)^n - 11n - 13$ is divisible by 144 for infinitely many $n \in \mathbb{N}$

[6-Apr-2023 shift 2]

Options:

A. only (S2) is correct

B. only (S1) is correct

C. both (S1) and (S2) are incorrect

D. both (S1) and (S2) are correct

Answer: D

Solution:

$$\because x^n - y^n = (x - y)[x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1}]$$

$$x^n - y^n \text{ is divisible by } x - y$$

$$\text{Stat 1} \rightarrow (2023)^{2022} - (1999)^{2022}$$

$$(2023) - (1999) = 24$$

$$\text{Stat 2} \rightarrow (2023)^{2022} - (1999)^{2022}$$

$$\text{is divisible by } 8$$

$$13(1 + 12)^n - 11n - 13$$

$$13[1 + {}^nC_1(12) + {}^nC_2(12)^2 + \dots] - 11n - 13$$

$$\Rightarrow (156n - 11n) + 13 \cdot {}^nC_2(12)^2 + 13 \cdot {}^nC_3(12)^3 + \dots$$

$$\Rightarrow 145n + 13 \cdot {}^nC_2(12)^2 + 13 \cdot {}^nC_3(12)^3 + \dots$$

$$\text{If } (n = 144m, m \in \mathbb{N}) \text{ then it is divisible by 144 for infinite values of } n.$$

Ans. Option 4

Question82

Let (t) denote the greatest integer $\leq t$, If the constant term in the expansion of $\left(3x^2 - \frac{1}{2x^5}\right)^7$ is α , then $[\alpha]$

is equal to _____.

[8-Apr-2023 shift 1]

Answer: 1275

Solution:

$$\begin{aligned} & \left(3x^2 - \frac{1}{2x^5}\right)^7 \\ T_{r+1} &= {}^7C_r (3x^2)^{7-r} \left(-\frac{1}{2x^5}\right)^r \\ 14-2r-5r &= 14-7r=0 \\ \therefore r &= 2 \\ \therefore T_3 &= {}^7C_2 \cdot 3^5 \left(-\frac{1}{2}\right)^2 = \frac{21 \times 243}{4} = 1275.75 \\ \therefore [\alpha] &= 1275 \end{aligned}$$

Question83

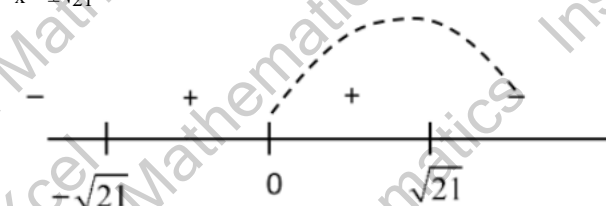
If a_n is the greatest term in the sequence $a_n = \frac{n^3}{n^4 + 147}$, $n = 1, 2, 3, \dots$, then a is equal to _____.

[8-Apr-2023 shift 1]

Answer: 5

Solution:

$$\begin{aligned} f(x) &= \frac{x^3}{x^4 + 147} \\ f'(x) &= \frac{(x^4 + 147)3x^2 - x^3(4x^3)}{(x^4 + 147)^2} \\ &= \frac{3x^6 + 147 \times 3x^2 - 4x^6}{+ve} = x^2(44 - x^4) \\ f'(x) &= 0 \text{ at } x^6 = 147 \times 3x^2 \\ x^2 &= 0, x^4 = 147 \times 3 \\ x &= 0, x^2 = \pm\sqrt{147 \times 3} \\ x^2 &= \pm 21 \\ x &= \pm\sqrt{21} \end{aligned}$$



fmax at $f(4)$ or $f(5)$

$$\begin{aligned} f(4) &= \frac{64}{403} \approx 0.158 \quad f(5) = \frac{125}{772} \approx 0.161 \\ \therefore a &= 5 \end{aligned}$$

Question84

The largest natural number n such that 3^n divides $66!$ is _____.

[8-Apr-2023 shift 1]

Answer: 31

Solution:

$$\left[\frac{66}{3} \right] + \left[\frac{66}{9} \right] + \left[\frac{66}{27} \right]$$
$$22 + 7 + 2 = 31$$

Question85

The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{11}$ is equal to
[8-Apr-2023 shift 2]

Options:

- A. $10^3 - 10$
- B. $11^3 - 11$
- C. $12^3 - 12$
- D. $13^3 - 13$

Answer: C

Solution:

$$T_{r+1} = {}^{11}C_r (2x^2)^{11-r} \left(\frac{1}{2x}\right)^r$$
$$= {}^{11}C_r 2^{11-2r} x^{22-3r}$$
$$22-3r=10 \quad \text{and} \quad 22-3r=7$$
$$r=4 \quad \text{and} \quad r=5$$
$$\text{Coefficient of } x^{10} = {}^{11}C_4 \cdot 2^3$$
$$\text{Coefficient of } x^7 = {}^{11}C_5 \cdot 2^1$$
$$\text{difference} = {}^{11}C_4 \cdot 2^3 - {}^{11}C_5 \cdot 2$$
$$= \frac{11 \times 10 \times 9 \times 8 \times 7}{24} - \frac{11 \times 10 \times 9 \times 8 \times 7}{120} \times 2$$
$$= 11 \times 10 \times 3 \times 8 - 11 \times 3 \times 4 \times 7$$
$$= 11 \times 3 \times 4 \times (20 - 7)$$
$$= 11 \times 12 \times 13$$
$$= 12(12-1)(12+1)$$
$$= 12(12^2 - 1)$$
$$= 12^3 - 12 \quad (\text{Option 3})$$

Question86

$25^{190} - 19^{190} - 8^{190} + 2^{190}$ is divisible by
[8-Apr-2023 shift 2]

Options:

- A. 34 but not by 14
- B. 14 but not by 34
- C. Both 14 and 34
- D. Neither 14 nor 34

Answer: A

Solution:

$25^{190} - 8^{190}$ is divisible by $25 - 8 = 17$
 $19^{190} - 2^{190}$ is divisible by $19 - 2 = 17$
 $25^{190} - 19^{190}$ is divisible by $25 - 19 = 6$
 $8^{190} - 2^{190}$ is divisible by $8 - 2 = 6$
 L.C.M. of $1746 = 34$
 $\therefore$ divisible by 34 but not by 14

Question 87

If the coefficient of x^7 in $\left(ax - \frac{1}{bx^2}\right)^{13}$ and the coefficient of x^{-5} in $\left(ax + \frac{1}{bx^2}\right)^{13}$ are equal, then $a^4 b^4$ is equal to :
[10-Apr-2023 shift 1]

Options:

- A. 22
- B. 44
- C. 11
- D. 33

Answer: A

Solution:

$$\left(ax - \frac{1}{bx^2}\right)^{13}$$

We have,

$$T_{r+1} = {}^{13}C_r (p)^{13-r} (q)^r$$

$$\begin{aligned}
 T_{r+1} &= {}^{13}C_r (ax)^{13-r} \left(-\frac{1}{bx^2}\right)^r \\
 &= {}^{13}C_r (a)^{13-r} \left(-\frac{1}{b}\right)^r (x)^{13-r} \cdot (x)^{-2r} \\
 &= {}^{13}C_r (a)^{13-r} \left(-\frac{1}{b}\right)^r (x)^{13-3r} \dots (1)
 \end{aligned}$$

Coefficient of x^7

$$\Rightarrow 13 - 3r = 7$$

$$r = 2$$

r in equation (1)

$$\begin{aligned}
 T_3 &= {}^{13}C_2 (a)^{13-2} \left(-\frac{1}{b}\right)^2 (x)^{13-6} \\
 &= {}^{13}C_2 (a)^{11} \left(\frac{1}{b}\right)^2 (x)^7
 \end{aligned}$$

$$\text{Coefficient of } x^7 \text{ is } {}^{13}C_2 \frac{(a)^{11}}{b^2}$$

$$\text{Now, } \left(ax + \frac{1}{bx^2}\right)^{13}$$

$$\begin{aligned}
 T_{r+1} &= {}^{13}C_r (ax)^{13-r} \left(\frac{1}{bx^2}\right)^r \\
 &= {}^{13}C_r (a)^{13-r} \left(\frac{1}{b}\right)^r (x)^{13-r} (x)^{-2r} \\
 &= {}^{13}C_r (a)^{13-r} \left(\frac{1}{b}\right)^r (x)^{13-3r} \dots (2)
 \end{aligned}$$

Coefficient of x^{-5}

$$\Rightarrow 13 - 3r = -5$$

$$r = 6$$

r in equation

$$\begin{aligned}
 T_7 &= {}^{13}C_6 (a)^{13-6} \left(\frac{1}{b}\right)^6 (x)^{13-18} \\
 T_7 &= {}^{13}C_6 (a)^7 \left(\frac{1}{b}\right)^6 (x)^{-5}
 \end{aligned}$$

$$\text{Coefficient of } x^{-5} \text{ is } {}^{13}C_6 (a)^7 \left(\frac{1}{b}\right)^6$$

ATQ

$$\text{Coefficient of } x^7 = \text{coefficient of } x^{-5}$$

$$T_3 = T_7$$

$${}^{13}C_2 \left(\frac{a^{11}}{b^2} \right) = {}^{13}C_6 (a)^7 \left(\frac{1}{b} \right)^6$$

$$a^4 \cdot b^4 = \frac{{}^{13}C_6}{{}^{13}C_2}$$

$$= \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 1}{13 \times 12 \times 6 \times 5 \times 4 \times 3} = 22$$

Question 88

The coefficient of x^7 in $(1 - x + 2x^3)^{10}$ is _____.
[10-Apr-2023 shift 1]

Answer: 960

Solution:

$$(1 - x + 2x^3)^{10}$$

$$T_n = \frac{10!}{a!b!c!} (-2x)^b (x^3)^c$$

$$= \frac{10!}{a!b!c!} (-2)^b x^{b+3c}$$

$$\Rightarrow b + 3c = 7, a + b + c = 10$$

$$\therefore \text{Coefficient of } x^7 = \frac{10!}{3!7!0!} (-1)^7 + \frac{10!}{5!4!1!} (-1)^4 (2)$$

$$+ \frac{10!}{7!1!2!} (-1)^1 (2)^2$$

$$= -120 + 2520 - 1440 = 960$$

Question 89

Let the number $(22)^{2022} + (2022)^{22}$ leave the remainder α when divided by 3 and β when divided by 7.

Then $(\alpha^2 + \beta^2)$ is equal to
[10-Apr-2023 shift 2]

Options:

- A. 13
- B. 20
- C. 10
- D. 5

Answer: D

Solution:

$$(22)^{2022} + (2022)^{22}$$

divided by 3

$$(21+1)^{2022} + (2022)^{22}$$

$$= 3k + 1$$

$$(\alpha = 1)$$

Divided by 7

$$(21+1)^{2022} + (2023-1)^{22}$$

$$7k + 1 + 1 \quad (\beta = 2)$$

$$7k + 2$$

So $\alpha^2 + \beta^2 \Rightarrow 5$

Question90

If the coefficients of x and x^2 in $(1+x)^p(1-x)^q$ are 4 and -5 respectively, then $2p+3q$ is equal to
[10-Apr-2023 shift 2]

Options:

- A. 60
- B. 63
- C. 66
- D. 69

Answer: B

Solution:

$$\begin{aligned} & (1+x)^p(1-x)^q \\ & \left(1+px+\frac{p(p-1)}{2!}x^2+\dots\right) \\ & \left(1-qx+\frac{q(q-1)}{2!}x^2-\dots\right) \\ & p-q=4 \\ & \frac{p(p-1)}{2}+\frac{q(q-1)}{2}-pq=-5 \\ & p^2+q^2-p-q-2pq=-10 \\ & (q+4)^2+q^2-(q+4)-q-2(4+q)q=-10 \\ & q^2+8q+16-q^2-q-4-q-8q-2q^2=-10 \\ & -2q=-22 \\ & q=11 \\ & p=15 \\ & 2(15)+3(11) \\ & 30+33=63 \end{aligned}$$

Question91

The number of integral terms in the expansion of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to :
[11-Apr-2023 shift 1]

Answer: 171

Solution:

The number of integral term in the expression of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to

$$\begin{aligned} \text{General term} &= {}^{680}C_r \left(3^{\frac{1}{2}}\right)^{680-r} \left(5^{\frac{1}{4}}\right)^r \\ &= {}^{680}C_r 3^{\frac{680-r}{2}} 5^{\frac{r}{4}} \end{aligned}$$

Values' s of r , where $\frac{r}{4}$ goes to integer

$$r = 0, 4, 8, 12, \dots, 680$$

All value of r are accepted for $\frac{680-r}{2}$ as well so

No of integral terms = 171.

Question92

The mean of the coefficients of $x, x^2, \dots, x^7$ in the binomial expansion of $(2+x)^9$ is _____.

Answer: 2736

Solution:

$$\text{Coefficient of } x = {}^9C_1 \cdot 2^8$$

$$\text{Coef. } x^2 = {}^9C_2 \cdot 2^7$$

$$\text{Coef. } x^7 = {}^9C_7 \cdot 2^2$$

$$\text{Mean} = \frac{{}^9C_1 \cdot 2^8 + {}^9C_2 \cdot 2^7 + \dots + {}^9C_7 \cdot 2^2}{7}$$

$$= \frac{(1+2)^9 - {}^9C_0 \cdot 2^9 - {}^9C_8 \cdot 2^1 - {}^9C_9}{7}$$

$$= \frac{3^9 - 2^9 - 18 - 1}{7}$$

$$= \frac{19152}{7} = 2736$$

Question93

If the 1011th term from the end in the binominal expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^{2022}$ is 1024 times 1011th term from the beginning, the $|x|$ is equal to

[11-Apr-2023 shift 2]

Options:

A. 8

B. 12

C. 10

D. 15

Answer: C

Solution:

$$T_{1011} \text{ from beginning} = T_{1010+1}$$

$$= {}^{2022}C_{1010} \left(\frac{4x}{5}\right)^{1012} \left(\frac{-5}{2x}\right)^{1010}$$

$$T_{1011} \text{ from end}$$

$$= {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1012} \left(\frac{4x}{5}\right)^{1010}$$

$$\text{Given: } = {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1012} \left(\frac{4x}{5}\right)^{1010}$$

$$= 2^{10} \cdot {}^{2022}C_{1010} \left(\frac{-5}{2x}\right)^{1010} \left(\frac{4x}{5}\right)^{1012}$$

$$\left(\frac{-5}{2x}\right)^2 = 2^{10} \left(\frac{4x}{5}\right)^2$$

$$x^4 = \frac{5^4}{2^{16}}$$

$$|x| = \frac{5}{16}$$

Question94

The sum of the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+2}$, which are in the ratio 1 : 3 : 5, is equal to
[11-Apr-2023 shift 2]

Options:

- A. 63
- B. 92
- C. 25
- D. 41

Answer: A

Solution:

$$\begin{aligned} {}^{n+2}C_{r-1} : {}^{n+2}C_r : {}^{n+2}C_{r+1} &:: 1 : 3 : 5 \\ \frac{(n+2)!}{(r-1)!(n-r+3)!} \times \frac{r!(n+2-r)!}{(n+2)!} &= \frac{1}{3} \\ \frac{r}{(n-r+3)} &= \frac{1}{3} \Rightarrow n-r+3 = 3r \\ n &= 4r-3-0 \\ \frac{(n+1)!}{r!(n+2-r)!} \times \frac{(r+1)!(n-r+1)!}{(n+2)!} &= \frac{3}{5} \\ \frac{r+1}{n+2-r} &= \frac{3}{5} \\ 8r-1 &= 3n \dots (2) \\ \text{By equation 1 and 2} \\ \frac{8r-1}{3} &= 4r-3 \quad n = 4r-3 \\ r = 2 \quad n &= 4 \times 2 - 3 \\ n &= 5 \\ \text{Sum: } {}^7C_1 + {}^7C_2 + {}^7C_3 &= 7 + 21 + 35 = 63 \end{aligned}$$

Question95

If $\frac{1}{n+1} {}^nC_n + \frac{1}{n} {}^nC_{n-1} + \dots + \frac{1}{2} {}^nC_1 + {}^nC_0 = \frac{1023}{10}$ then n is equal to
[12-Apr-2023 shift 1]

Options:

- A. 7
- B. 9
- C. 6
- D. 8

Answer: B

Solution:

$$\begin{aligned} \sum_{r=0}^n \frac{{}^nC_r}{r+1} &= \frac{1}{n+1} \sum_{r=0}^{n+1} {}^nC_{r+1} \\ &= \frac{1}{n+1} (2^{n+1} - 1) = \frac{1023}{10} \\ n+1 &= 10 \Rightarrow n = 9 \end{aligned}$$

Question96

The sum, of the coefficients of the first 50 terms in the binomial expansion of $(1-x)^{100}$, is equal to
[12-Apr-2023 shift 1]

Options:

A. $-^{101}C_{50}$

B. $^{99}C_{49}$

C. $^{101}C_{50}$

D. $-^{99}C_{49}$

Answer: D

Solution:

$$\begin{aligned}(1-x)^{100} &= C_0 - C_1x + C_2x^2 - C_3x^3 + \dots + C_{99}x^{99} + C_{100}x^{100} \\ \Rightarrow C_0 - C_1 + C_2 - C_3 + \dots - C_{99} + C_{100} &= 0 \\ C_0 - C_1 + C_2 + \dots - C_{99} &= -\frac{1}{2}C_{100} \\ -\frac{1}{2} \frac{100!}{50!50!} &= -\frac{1}{2} \times \frac{100 \times 99!}{50!50!} = -^{99}C_{49}\end{aligned}$$

Question97

Fractional part of the number is $\frac{4^{2022}}{15}$ equal to
[13-Apr-2023 shift 1]

Options:

A. $\frac{4}{15}$

B. $\frac{8}{15}$

C. $\frac{1}{15}$

D. $\frac{14}{15}$

Answer: C

Solution:

$$\left\{ \frac{4^{2022}}{15} \right\} = \left\{ \frac{2^{4044}}{15} \right\} = \left\{ \frac{(1+15)^{1011}}{15} \right\} = \frac{1}{15}$$

Question98

Let a be the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{6}{x^2} \right)^n$, $n \leq 15$. If the sum of the

coefficients of the remaining terms in the expansion is 649 and the coefficient of x^{-n} is λa , then λ is equal to _____.

[13-Apr-2023 shift 1]

Answer: 36

Solution:

$$T_{k+1} = {}^nC_k(x)^{\frac{n-k}{2}}(-6)^k(x)^{\frac{-3}{2}k}$$
$$\frac{n-k}{2} - \frac{3}{2}k = 0$$
$$n - 4k = 0$$
$$(-5)^n - \left({}^nC_n(-6)^{\frac{n}{4}}\right) = 649$$

By observation $(625 + 24 = 649)$, we get $n = 4$

$\therefore n = 4$ & $k = 1$

Required is coefficient of

$$x^{-4} \text{ is } \left(\sqrt{4} - \frac{6}{x^{\frac{3}{2}}}\right)^4$$

$${}^4C_1(-6)^3$$

By calculating we will get $\lambda = 36$

Question 99

Let for $x \in \mathbb{R}$, $S_0(x) = x$, $S_k(x) = C_k x + k \int_0^x S_{k-1}(t) dt$ where $C_0 = 1$, $C_k = 1 - \int_0^1 S_{k-1}(x) dx$, $k = 1, 2, 3, \dots$

Then $S_2(3) + 6C_3$ is equal to _____.

[13-Apr-2023 shift 1]

Answer: 18

Solution:

$$\text{Given, } S_k(x) = C_k x + k \int_0^x S_{k-1}(t) dt$$

put $k = 2$ and $x = 3$

$$S_2(3) = C_2(3) + 2 \int_0^3 S_1(t) dt$$

$$\text{Also, } S_1(x) = C_1(x) + \int_0^x S_0(t) dt \dots \dots (1)$$

$$= C_1 x + \frac{x^2}{2}$$

$$S_2(3) = 3C_2 + 2 \int_0^3 \left(C_1 t + \frac{t^2}{2}\right) dt$$

$$= 3C_2 + 9C_1 + 9$$

Also,

$$C_1 = 1 - \int_0^1 S_0(x) dx = \frac{1}{2}$$

$$C_2 = 1 - \int_0^1 S_1(x) dx = 0$$

$$C_3 = 1 - \int_0^1 S_2(x) dx$$

$$= 1 - \int_0^1 \left(C_2 x + C_1 x^2 + \frac{x^3}{3}\right) dx = \frac{3}{4}$$

$$S_2(x) = C_2 x + 2 \int_0^x S_1(t) dt$$

$$= C_2 x + C_1 x^2 + \frac{x^3}{3}$$

$$\Rightarrow S_2(3) + 6C_3 = 6C_3 + 3C_2 + 9C_1 + 9 = 18$$

Question 100

The coefficient of x^5 in the expansion of $\left(2x^3 - \frac{1}{3x^2}\right)^5$ is

[13-Apr-2023 shift 2]

Options:

A. $\frac{80}{9}$

B. 8

C. 9

D. $\frac{26}{3}$

Answer: A

Solution:

general term for $\left(2x^3 - \frac{1}{3x^2}\right)^5$

$$T_{r+1} = {}^5C_r \left(-\frac{1}{3x^2}\right)^r (2x^3)^{5-r}$$

$${}^5C_r (-1)^r 3^{-r} 2^{5-r} \cdot x^{15-5r}$$

$$15 - 5r = 5 \Rightarrow r = 2$$

$$\text{Coeff. Of } x^5 = {}^5C_2 (-1)^2 3^{-2} 2^3$$

$$= 10 \times \frac{1}{9} \times 8$$

$$= \frac{80}{9}$$

Question 101

The remainder, when 7^{103} is divided by 17, is _____.

[13-Apr-2023 shift 2]

Answer: 12

Solution:

$$7^{103} = 7 \cdot 7^{102}$$

$$= 7(7^{51})$$

$$= 7(51 - 2)^{51} \rightarrow \text{remainder} = 7(-2)^{51}$$

$$= 7(2^3)(16)^{12} = -56(17 - 1)^{12} \rightarrow \text{remainder} = -56(-1)^{12}$$

$$\text{Remainder} = -56 + 17k$$

$$= -56 + 68$$

$$= 12$$

Question 102

Let $(a + bx + cx^2)^{10} = \sum_{i=0}^{20} p_i x^i$, $a, b, c \in \mathbb{N}$. If $p_1 = 20$ and $p_2 = 210$, then $2(a + b + c)$ is equal to

[15-Apr-2023 shift 1]

Options:

A. 8

B. 12

C. 6

D. 15

Answer: B

Solution:

$$(a + bx + cx^2)^{10} = \sum_{i=0}^{20} p_i x^i$$

Coefficient of $x^1 = 20$

$$20 = \frac{10!}{9!1!} \times a^9 \times b^1$$

$$a^9 \cdot b = 2$$

$$a = 2, b = 2$$

Coefficient of $x^2 = 210$

$$210 = \frac{10!}{9!1!} \times a^9 \times c^1 + \frac{10!}{8!2!} \times a^8 b^2$$

$$210 = 10 \cdot c + 45 \times 4$$

$$10c = 30$$

$$c = 3$$

$$2(a + b + c) = 12$$

Question 103

The remainder when 3^{2022} is divided by 5 is :
[24-Jun-2022-Shift-1]

Options:

A. 1

B. 2

C. 3

D. 4

Answer: D

Solution:

$$3^{2022}$$

$$= (3^2)^{1011}$$

$$= (9)^{1011}$$

$$= (10 - 1)^{1011}$$

$$= {}^{1011}C_0 (10)^{1011} + \dots + {}^{1011}C_{1010} \cdot (10)^1 - {}^{1011}C_{1011}$$

$$= 10[{}^{1011}C_0 (10)^{1010} + \dots + {}^{1011}C_{1010}] - 1$$

$$= 10K - 1$$

$$[\text{As } 10[{}^{1011}C_0 \cdot (10)^{1010} + \dots + {}^{1011}C_{1010}] \text{ is multiple of } 10]$$

$$= 10K + 5 - 5 - 1$$

$$= 10K - 5 + 5 - 1$$

$$= 5(2K - 1) + 4$$

$\therefore$ Unit digit = 4 when divided by 5.

Question 104

The remainder on dividing $1 + 3 + 3^2 + 3^3 + \dots + 3^{2021}$ by 50 is__
[24-Jun-2022-Shift-2]

Answer: 4

Solution:

Given,

$$1 + 3 + 3^2 + 3^3 + \dots + 3^{2021}$$
$$= 3^0 + 3^1 + 3^2 + 3^3 + \dots + 3^{2021}$$

This is a G.P with common ratio = 3

$$\begin{aligned}\therefore \text{Sum} &= \frac{1(3^{2022} - 1)}{3 - 1} \\&= \frac{3^{2022} - 1}{2} \\&= \frac{(3^2)^{1011} - 1}{2} \\&= \frac{(10 - 1)^{1011} - 1}{2} \\&= \frac{[{}^{1011}C_0 \cdot 10^{1011} - {}^{1011}C_1 \cdot 10^{1010} + \dots - {}^{1011}C_{1009} \cdot (10)^2 + {}^{1011}C_{1010} \cdot 10 - {}^{1011}C_{1011}] - 1}{2} \\&= \frac{10^2 [{}^{1011}C_0 \cdot (10)^{1009} - {}^{1011}C_1 \cdot (1008) + \dots - {}^{1011}C_{1009}] + 10110 - 1 - 1}{2} \\&= \frac{100k + 10110 - 2}{2} \\&= \frac{100k + 10108}{2} \\&= 50k + 5054 \\&= 50k + 50 \times 101 + 4 \\&= 50[k + 101] + 4 \\&= 50k \cdot k\end{aligned}$$

$\therefore$ By dividing 50 we get remainder as 4 .

Question 105

Let C_r denote the binomial coefficient of x^r in the expansion of $(1+x)^{10}$. If for

$\alpha, \beta \in \mathbb{R}$, $C_1 + 3.2C_2 + 5.3C_3 + \dots$ upto 10 terms $= \frac{\alpha \times 2^{11}}{2^\beta - 1} \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots$ upto 10 terms) then the value of $\alpha + \beta$ is equal to
[25-Jun-2022-Shift-1]

Answer: 286

Solution:

$$(1+x)^{10} = C_0 + C_1x + C_2x^2 + \dots + C_{10}x^{10}$$

Differentiating

$$10(1+x)^9 = C_1 + 2C_2x + 3C_3x^2 + \dots + 10C_{10}x^9$$

replace $x \rightarrow x^2$

$$10(1+x^2)^9 = C_1 + 2C_2x^2 + 3C_3x^4 + \dots + 10C_{10}x^{18}$$

$$10 \cdot x(1+x^2)^9 = C_1x + 2C_2x^3 + 3C_3x^5 + \dots + 10C_{10}x^{19}$$

Differentiating

$$10((1+x^2)^9 \cdot 1 + x \cdot 9(1+x^2)^8 \cdot 2x)$$

$$= C_1x + 2C_2 \cdot 3x^3 + 3 \cdot 5 \cdot C_3x^5 + \dots + 10 \cdot 19C_{10}x^{18}$$

putting $x = 1$

$$10(2^9 + 18 \cdot 2^8)$$

$$= C_1 + 3 \cdot 2 \cdot C_2 + 5 \cdot 3 \cdot C_3 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$C_1 + 3 \cdot 2 \cdot C_2 + \dots + 19 \cdot 10 \cdot C_{10}$$

$$= 10 \cdot 2^9 \cdot 10 = 100 \cdot 2^9$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} + \frac{C_{10}}{11} = \frac{2^{11}-1}{11}$$

10th term 11th term

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_9}{11} = \frac{2^{11}-2}{11}$$

$$\text{Now, } 100 \cdot 2^9 = \frac{\alpha \cdot 2^{11}}{2^\beta - 1} \left(\frac{2^{11}-2}{11} \right)$$

Eqn. of form $y = k(2^x - 1)$

It has infinite solutions even if we take $x, y \in \mathbb{N}$.

Question 106

The coefficient of x^{101} in the expression $(5+x)^{500} + x(5+x)^{499} + x^2(5+x)^{498} + \dots + x^{500}$, $x > 0$, is
[25-Jun-2022-Shift-2]

Options:

A. $^{501}C_{101}(5)^{399}$

B. $^{501}C_{101}(5)^{400}$

C. $^{501}C_{100}(5)^{400}$

D. $^{500}C_{101}(5)^{399}$

Answer: A

Solution:

Given,
 $(5+x)^{500} + x(5+x)^{499} + x^2(5+x)^{498} + \dots + x^{500}$

This is a G.P. with first term $(5+x)^{500}$

Common ratio = $\frac{x(5+x)^{499}}{(5+x)^{500}} = \frac{x}{5+x}$ and 501 terms present.

$$\therefore \text{Sum} = \frac{(5+x)^{500} \left(\left(\frac{x}{5+x} \right)^{501} - 1 \right)}{\frac{x}{5+x} - 1}$$

$$= \frac{(5+x)^{500} \left(\frac{x^{501} - (5+x)^{501}}{(5+x)^{501}} \right)}{\frac{x-5-x}{5+x}}$$

$$= \frac{\frac{x^{501} - (5+x)^{501}}{5+x}}{\frac{-5}{5+x}}$$

$$= \frac{1}{5} ((5+x)^{501} - x^{501})$$

Coefficient of x^{101} in $(5+x)^{501}$ is $= ^{501}C_{101} \cdot 5^{400}$

$$\therefore \ln \frac{1}{5} ((5+x)^{500} - x^{501}) \text{ coefficient of } x^{101} \text{ is } = \frac{1}{5} \cdot ^{501}C_{101} \cdot 5^{400}$$
$$= ^{501}C_{101} \cdot 5^{399}$$

Question 107

If the sum of the co-efficient of all the positive even powers of x in the binomial expansion of $\left(2x^3 + \frac{3}{x} \right)^{10}$ is $5^{10} - \beta \cdot 3^9$, then β is equal to _____
[25-Jun-2022-Shift-2]

Answer: 83

Solution:

Given, Binomial Expansion

$$\left(2x^3 + \frac{3}{x}\right)^{10}$$

General term

$$\begin{aligned}T_{r+1} &= {}^{10}C_r \cdot (2x^3)^{10-r} \cdot \left(\frac{3}{x}\right)^r \\&= {}^{10}C_r \cdot 2^{10-r} \cdot 3^r \cdot x^{30-3r} \cdot x^{-r} \\&= {}^{10}C_r \cdot 2^{10-r} \cdot 3^r \cdot x^{30-4r}\end{aligned}$$

For positive even power of x , $30 - 4r$ should be even and positive.

For $r = 0$, $30 - 4 \times 0 = 30$ (even and positive)

For $r = 1$, $30 - 4 \times 1 = 26$ (even and positive)

For $r = 2$, $30 - 4 \times 2 = 22$ (even and positive)

For $r = 3$, $30 - 4 \times 3 = 18$ (even and positive)

For $r = 4$, $30 - 4 \times 4 = 14$ (even and positive)

For $r = 5$, $30 - 4 \times 5 = 10$ (even and positive)

For $r = 6$, $30 - 4 \times 6 = 6$ (even and positive)

For $r = 7$, $30 - 4 \times 7 = 2$ (even and positive)

For $r = 8$, $30 - 4 \times 8 = -2$ (even but not positive)

So, for $r = 1, 2, 3, 4, 5, 6$ and 7 we can get positive even power of x .

$\therefore$ Sum of coefficient for positive even power of

$$\begin{aligned}x &= {}^{10}C_0 \cdot 2^{10} \cdot 3^0 + {}^{10}C_{1,2} \cdot 2^9 \cdot 3^1 + {}^{10}C_2 \cdot 2^8 \cdot 3^2 + {}^{10}C_3 \cdot 2^7 \cdot 3^3 + {}^{10}C_4 \cdot 2^6 \cdot 3^4 + {}^{10}C_5 \cdot 2^5 \cdot 3^5 + {}^{10}C_6 \cdot 2^4 \cdot 3^6 + {}^{10}C_7 \cdot 2^3 \cdot 3^7 \\&= {}^{10}C_{10} \cdot 2^{10} \cdot 3^0 + {}^{10}C_{1,2} \cdot 2^9 \cdot 3^1 + \dots + {}^{10}C_{10} \cdot 2^0 \cdot 3^{10} - [{}^{10}C_8 \cdot 2^2 \cdot 3^8 + {}^{10}C_9 \cdot 2 \cdot 3^9 + {}^{10}C_{10} \cdot 2^0 \cdot 3^{10}] = (2+3)^{10} - [45.4.3^8 + 10.2.3^9 + 1.1.3^{10}] \\&= 5^{10} - [60 \times 3^9 + 20 \cdot 3^9 + 3 \cdot 3^9] \\&= 5^{10} - (60 + 20 + 3)3^9 \\&= 5^{10} - 83.3^9 \\&\therefore \beta = 83\end{aligned}$$

Question 108

The remainder when $(2021)^{2023}$ is divided by 7 is :
[26-Jun-2022-Shift-1]

Options:

- A. 1
- B. 2
- C. 5
- D. 6

Answer: C

Solution:

$$(2021)^{2023}$$

$$= (2016 + 5)^{2023} \text{ [here 2016 is divisible by 7]}$$

$$= {}^{2023}C_0(2016)^{2023} + \dots + {}^{2023}C_{2022}(2016)(5)^{2022} + {}^{2023}C_{2023}(5)^{2023}$$

$$= 2016[{}^{2023}C_0 \cdot (2016)^{2022} + \dots + {}^{2023}C_{2022} \cdot (5)^{2022}] + (5)^{2023}$$

$$= 2016\lambda + (5)^{2023}$$

$$= 7 \times 288\lambda + (5)^{2023}$$

$$= 7K + (5)^{2023} \dots (1)$$

$$\text{Now, } (5)^{2023}$$

$$= (5)^{2022} \cdot 5$$

$$= (5^3)^{674} \cdot 5$$

$$= (125)^{674} \cdot 5$$

$$= (126 - 1)^{674} \cdot 5$$

$$= (126 - 1)^{674} \cdot 5$$

$$= 5[{}^{674}C_0(126)^{674} + \dots - {}^{674}C_{673}(126) + {}^{674}C_{674}]$$

$$= 5 \times 126[{}^{674}C_0(126)^{673} + \dots - {}^{674}C_{673}] + 5$$

$$= 5 \cdot 7 \cdot 18[{}^{674}C_0(126)^{673} + \dots - {}^{674}C_{673}] + 5$$

$$= 7\lambda + 5$$

Replacing $(5)^{2023}$ in equation (1) with $7\lambda + 5$, we get,

$$(2021)^{2023} = 7K + 7\lambda + 5$$

$$= 7(K + \lambda) + 5$$

$$\therefore \text{Remainder} = 5$$

Question109

If $({}^{40}C_0) + ({}^{41}C_1) + ({}^{42}C_2) + \dots + ({}^{60}C_{20}) = \frac{m}{n} {}^{60}C_{20}$, m and n are coprime, then $m + n$ is equal to ____
[26-Jun-2022-Shift-2]

Answer: 102

Solution:

Here property used is

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$$

$$\text{Given, } {}^{40}C_0 + {}^{41}C_1 + {}^{42}C_2 + \dots + {}^{60}C_{20} = \frac{m}{n} {}^{60}C_{20}$$

$$\text{As } {}^{40}C_0 = {}^{41}C_0 = 1$$

So, we replace ${}^{40}C_0$ with ${}^{41}C_0$.

$$\Rightarrow {}^{41}C_0 + {}^{41}C_1 + {}^{42}C_2 + \dots + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow {}^{42}C_1 + {}^{42}C_2 + \dots + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow {}^{43}C_2 + {}^{43}C_3 + \dots + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow {}^{44}C_3 + {}^{44}C_4 + \dots + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow {}^{45}C_4 + {}^{45}C_5 + \dots + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$\vdots$

$$\Rightarrow {}^{60}C_{19} + {}^{60}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow {}^{61}C_{20} = \frac{m}{n} \cdot {}^{60}C_{20}$$

$$\Rightarrow \frac{61!}{20!41!} = \frac{m}{n} \cdot \frac{60!}{20!40!}$$

$$\Rightarrow \frac{61}{41} = \frac{m}{n}$$

$$\therefore m = 61 \text{ and } n = 41$$

$$\therefore m + n = 61 + 41 = 102$$

Question110

If the coefficient of x^{10} in the binomial expansion of $\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}} + \frac{\sqrt{5}}{x^{\frac{1}{3}}} \right)^{60}$ is $5^k \cdot I$, where $I, k \in \mathbb{N}$ and I is co-prime to 5, then k is equal to
[27-Jun-2022-Shift-1]

Answer: 5

Solution:

Given Binomial Expansion

$$= \left(\frac{\sqrt{x}}{5^{\frac{1}{4}}} + \frac{\sqrt{x}}{5^{\frac{1}{3}}} \right)^{60}$$

∴ General term

$$T_{r+1} = {}^{60}C_r \cdot \left(\frac{x}{5^{1/4}} \right)^{60-r} \cdot \left(\frac{5^{1/2}}{x^{1/3}} \right)^r$$

$$= {}^{60}C_r \cdot 5^{\left(\frac{r}{4} - 15 + \frac{r}{2} \right)} \cdot x^{\left(30 - \frac{r}{2} - \frac{r}{3} \right)}$$

$$= {}^{60}C_r \cdot 5^{\left(\frac{3r-60}{4} \right)} \cdot x^{\left(\frac{180-5r}{6} \right)}$$

For x^{10} term,

$$\frac{180-5r}{6} = 10$$

$$\Rightarrow 5r = 120$$

$$\Rightarrow r = 24$$

$$\therefore \text{Coefficient of } x^{10} = {}^{60}C_{24} \cdot 5^{\left(\frac{3 \times 24 - 60}{4} \right)}$$

$$= {}^{60}C_{24} \cdot 5^3$$

$$= \frac{60!}{24!36!} \cdot 5^3$$

It is given that,

$$\frac{60!}{24!36!} \cdot 5^3 = 5^k \cdot 1 \dots \dots (1)$$

Also given that, 1 is coprime to 5 means 1 can't be multiple of 5. So we have to find all the factors of 5 in 60!, 24! and 36!

[Note : Formula for exponent or degree of prime number in n!]

$$\text{Exponent of } p \text{ in } n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots \text{ until } 0 \text{ comes}$$

here p is a prime number.]

∴ Exponent of 5 in 60 !

$$= \left\lfloor \frac{60}{5} \right\rfloor + \left\lfloor \frac{60}{5^2} \right\rfloor + \left\lfloor \frac{60}{5^3} \right\rfloor + \dots$$

$$= 12 + 2 + 0 + \dots$$

$$= 14$$

Exponent of 5 in 24 !

$$= \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{5^2} \right\rfloor + \left\lfloor \frac{24}{5^3} \right\rfloor + \dots$$

$$= 4 + 0 + 0 \dots$$

$$= 4$$

Exponent of 5 in 36 !

$$= \left\lfloor \frac{36}{5} \right\rfloor + \left\lfloor \frac{36}{5^2} \right\rfloor + \left\lfloor \frac{36}{5^3} \right\rfloor + \dots$$

$$= 7 + 1 + 0 \dots$$

$$= 8$$

∴ From equation (1), exponent of 5 overall

$$\frac{5^{14}}{5^4 \cdot 5^8} \cdot 5^3 = 5^k$$

$$\Rightarrow 5^3 = 5^k$$

$$\Rightarrow k = 3$$

Question 111

If the sum of the coefficients of all the positive powers of x, in the Binomial expansion of $\left(x^n + \frac{2}{x^3} \right)^7$ is

939, then the sum of all the possible integral values of n is
[27-Jun-2022-Shift-2]

Answer: 57

Solution:

Given, Binomial expression is

$$= \left(x^n + \frac{2}{x^5}\right)^7$$

$\therefore$ General term

$$T_{r+1} = {}^7C_r \cdot (x^n)^{7-r} \cdot \left(\frac{2}{x^5}\right)^r$$
$$= {}^7C_r \cdot x^{7n-nr-5r} \cdot 2^r$$

For positive power of x,

$$7n - nr - 5r > 0$$

$$\Rightarrow 7n > r(n+5)$$

$$\Rightarrow r < \frac{7n}{n+5}$$

As r represent term of binomial expression so r is always integer.

Given that sum of coefficient is 939.

When $r=0$,

$$\text{sum of coefficient} = {}^7C_0 \cdot 2^0 = 1$$

when $r=1$,

$$\text{sum of coefficient} = {}^7C_0 \cdot 2^0 + {}^7C_1 \cdot 2^1 = 1 + 14 = 15$$

when $r=2$,

sum of coefficient

$$= {}^7C_0 \cdot 2^0 + {}^7C_1 \cdot 2^1 + {}^7C_2 \cdot 2^2$$

$$= 1 + 14 + 84$$

$$= 99$$

when $r=3$,

sum of coefficient

$$= {}^7C_0 \cdot 2^0 + {}^7C_1 \cdot 2^1 + {}^7C_2 \cdot 2^2 + {}^7C_3 \cdot 2^3$$

$$= 1 + 14 + 84 + 280$$

$$= 379$$

when $r=4$,

sum of coefficient

$$= {}^7C_0 \cdot 2^0 + {}^7C_1 \cdot 2^1 + {}^7C_2 \cdot 2^2 + {}^7C_3 \cdot 2^3 + {}^7C_4 \cdot 2^4$$

$$= 1 + 14 + 84 + 280 + 560$$

$$= 939$$

To get value of $r=4$, value of $\frac{7n}{n+5}$ should be between 4 and 5.

$$\therefore 4 < \frac{7n}{n+5} < 5$$

$$\Rightarrow 4n + 20 < 7n < 5n + 25$$

$$\therefore 4n + 20 < 7n$$

$$\Rightarrow 3n > 20$$

$$\Rightarrow n > \frac{20}{3}$$

$$\Rightarrow n > 6.66$$

and

$$7n < 5n + 25$$

$$\Rightarrow 2n < 25$$

$$\Rightarrow n < 12.5$$

$$\therefore 6.66 < n < 12.5$$

$\therefore$ Possible integer values of $n = 7, 8, 9, 10, 11, 12$

$\therefore$ Sum of values of $n = 7 + 8 + 9 + 10 + 11 + 12 = 57$

Question 112

If

$$\sum_{k=1}^{31} ({}^{31}C_k)({}^{31}C_{k-1}) - \sum_{k=1}^{30} ({}^{31}C_k)({}^{31}C_{k-1}) = \frac{\alpha(60!)}{(30!)(31!)}$$

where $\alpha \in \mathbb{R}$, then the value of 16α is equal to

[28-Jun-2022-Shift-1]

Options:

A. 1411

B. 1320

C. 1615

D. 1855

Answer: A

Solution:

Given,

$$\sum_{k=1}^{31} ({}^{31}C_k)({}^{31}C_{k-1}) - \sum_{k=1}^{30} ({}^{30}C_k)({}^{30}C_{k-1}) = \frac{\alpha(60!)}{(30!)(31!)}$$

Now,

$$\sum_{k=1}^{31} ({}^{31}C_k)({}^{31}C_{k-1})$$

$$= ({}^{31}C_1 \cdot {}^{31}C_0 + {}^{31}C_2 \cdot {}^{31}C_1 + {}^{31}C_3 \cdot {}^{31}C_2 + \dots + {}^{31}C_{31} \cdot {}^{31}C_{30})$$

$$= ({}^{31}C_0 \cdot {}^{31}C_{31-1} + {}^{31}C_1 \cdot {}^{31}C_{31-2} + \dots + {}^{31}C_{30} \cdot {}^{31}C_{31-30})$$

$$[\text{using } {}^nC_r = {}^nC_{n-r}]$$

$$= ({}^{31}C_0 \cdot {}^{31}C_{30} + {}^{31}C_1 \cdot {}^{31}C_{29} + \dots + {}^{31}C_{30} \cdot {}^{31}C_0)$$

$$= {}^{62}C_{30}$$

$$\text{Now, } \sum_{k=1}^{30} {}^{30}C_k \cdot {}^{30}C_{k-1}$$

$$= ({}^{30}C_1 \cdot {}^{30}C_0 + {}^{30}C_2 \cdot {}^{30}C_1 + \dots + {}^{30}C_{30} \cdot {}^{30}C_{29})$$

$$= ({}^{30}C_0 \cdot {}^{30}C_{29} + {}^{30}C_1 \cdot {}^{30}C_{28} + \dots + {}^{30}C_{29} \cdot {}^{30}C_0)$$

$$= {}^{60}C_{29}$$

$$\therefore {}^{60}C_{30} - {}^{60}C_{29} = \frac{\alpha(60!)}{30!31!}$$

$$\Rightarrow \frac{62 \cdot 61 \cdot 60!}{30!32!} - \frac{60!}{29!31!} = \frac{\alpha(60!)}{30!31!}$$

$$\Rightarrow \frac{62 \cdot 61 \cdot 60!}{30!32!} - \frac{60!}{\frac{30!}{30} \cdot 31!} = \frac{\alpha(60!)}{30!31!}$$

$$\Rightarrow \frac{60!}{30!31!} \left(\frac{62 \cdot 61}{32} - 30 \right) = \frac{\alpha(60!)}{30!31!}$$

$$\Rightarrow \alpha = \frac{62 \cdot 61}{32} - 30$$

$$\Rightarrow 16\alpha = \frac{62 \cdot 61 - 30 \times 32}{2}$$

$$\Rightarrow 16\alpha = \frac{2822}{2} = 1411$$

Question 113

The number of positive integers k such that the constant term in the binomial expansion of

$$\left(2x^3 + \frac{3}{x^k} \right)^{12}, x \neq 0 \text{ is } 2^8 \cdot I, \text{ where } I \text{ is an odd integer, is } \underline{\hspace{2cm}}$$

[28-Jun-2022-Shift-1]

Answer: 2

Solution:

$$\left(2x^3 + \frac{3}{x^k} \right)^{12}$$

$$t_{r+1} = {}^{12}C_r (2x^3)^r \left(\frac{3}{x^k} \right)^{12-r}$$

$$x^{3r - (12-r)k} \rightarrow \text{constant}$$

$$\therefore 3r - 12k + rk = 0$$

$$\Rightarrow k = \frac{3r}{12-r}$$

$\therefore$ possible values of r are 3, 6, 8, 9, 10 and corresponding values of k are 1, 3, 6, 9, 15

Now ${}^{12}C_r = 220, 924, 495, 220, 66$

$\therefore$ possible values of k for which we will get 2^8 are 3, 6

Question 114

The term independent of x in the expansion of $(1 - x^2 + 3x^3) \left(\frac{5}{2}x^3 - \frac{1}{5x^2} \right)^{11}$, $x \neq 0$ is :

[28-Jun-2022-Shift-2]

Options:

A. $\frac{7}{40}$

B. $\frac{33}{200}$

C. $\frac{39}{200}$

D. $\frac{11}{50}$

Answer: B

Solution:

General term of Binomial expansion $\left(\frac{5}{2}x^3 - \frac{1}{5x^2} \right)^{11}$ is

$$T_{r+1} = {}^{11}C_r \cdot \left(\frac{5}{2}x^3 \right)^{11-r} \cdot \left(-\frac{1}{5x^2} \right)^r$$
$$= {}^{11}C_r \cdot \left(\frac{5}{2} \right)^{11-r} \cdot \left(-\frac{1}{5} \right)^r \cdot x^{33-5r}$$

In the term,

$$(1 - x^2 + 3x^3) \left(\frac{5}{2}x^3 - \frac{1}{5x^2} \right)^{11}$$

Term independent of x is when

$$(1) \quad 33 - 5r = 0$$

$$\Rightarrow r = \frac{33}{5} \notin \text{integer}$$

$$(2) \quad 33 - 5r = -2$$

$$\Rightarrow 5r = 35$$

$$\Rightarrow r = 7 \in \text{integer}$$

$$(3) \quad 33 - 5r = -3$$

$$\Rightarrow 5r = 36$$

$$\Rightarrow r = \frac{36}{5} \notin \text{integer}$$

$\therefore$ Only for $r = 7$ independent of x term possible.

$\therefore$ Independent of x term

$$= - \left({}^{11}C_7 \left(\frac{5}{2} \right)^4 \cdot \left(-\frac{1}{5} \right)^7 \right)$$

$$= - \left(\frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{5^4}{2^4} \cdot -\frac{1}{5^7} \right)$$

$$= \frac{11 \cdot 10 \cdot 3}{2^4 \cdot 5^3}$$

$$= \frac{11 \cdot 3}{2^3 \cdot 5^2}$$

$$= \frac{33}{200}$$

Question 115

If the constant term in the expansion of

$\left(3x^3 - 2x^2 + \frac{5}{x} \right)^{10}$ is $2^k \cdot I$, where I is an odd integer, then the value of k is equal to;

[29-Jun-2022-Shift-1]

Options:

A. 6

- B. 7
C. 8
D. 9

Answer: D

Solution:

Note : Multinomial Theorem :

The general term of $(x_1 + x_2 + \dots + x_n)^n$ the expansion is

$$\frac{n!}{n_1!n_2!\dots n_n!} x_1^{n_1} x_2^{n_2} \dots x_n^{n_n}$$

where $n_1 + n_2 + \dots + n_n = n$

Given,

$$\left(3x^2 - 2x^2 + \frac{5}{x}\right)^{10} \\ = \frac{(3x^8 - 2x^7 + 5)^{10}}{x^{50}}$$

Now constant term in $\left(3x^3 - 2x^2 + \frac{5}{x}\right)^{10} = x^{50}$ term in $(3x^8 - 2x^7 + 5)^{10}$

General term in $(3x^8 - 2x^7 + 5)^{10}$ is

$$= \frac{10!}{n_1!n_2!n_3!} (3x^8)^{n_1} (-2x^7)^{n_2} (5)^{n_3} \\ = \frac{10!}{n_1!n_2!n_3!} (3)^{n_1} (-2)^{n_2} (5)^{n_3} \cdot x^{8n_1 + 7n_2}$$

$\therefore$ Coefficient of $x^{8n_1 + 7n_2}$ is

$$= \frac{10!}{n_1!n_2!n_3!} (3)^{n_1} (-2)^{n_2} (5)^{n_3}$$

where $n_1 + n_2 + n_3 = 0$

For coefficient of x^{50} :

$$8n_1 + 7n_2 = 50$$

$\therefore$ Possible values of n_1, n_2 and n_3 are

n_1	n_2	n_3
1	6	3

$\therefore$ Coefficient of x^{50}

$$= \frac{10!}{1!6!3!} (3)^1 (-2)^6 (5)^3 \\ = \frac{10 \times 9 \times 8 \times 7 \times 3 \times 3 \times 3 \times 2^6}{6} \\ = 5 \times 3 \times 8 \times 7 \times 3 \times 5^3 \times 2^6 \\ = 7 \times 5^4 \times 3^2 \times 2^9 \\ = 2^k \cdot 1 \\ \therefore 1 = 7 \times 5^4 \times 3^2 = \text{An odd integer} \\ \text{and } 2^k = 2^9 \\ \Rightarrow k = 9$$

Question 116

Let $n \geq 5$ be an integer. If $9^n - 8n - 1 = 64\alpha$ and $6^n - 5n - 1 = 25\beta$, then $\alpha - \beta$ is equal to
[29-Jun-2022-Shift-2]

Options:

- A. $1 + {}^nC_2(8-5) + {}^nC_3(8^2-5^2) + \dots + {}^nC_n(8^{n-1}-5^{n-1})$
B. $1 + {}^nC_3(8-5) + {}^nC_4(8^2-5^2) + \dots + {}^nC_n(8^{n-2}-5^{n-2})$
C. ${}^nC_3(8-5) + {}^nC_4(8^2-5^2) + \dots + {}^nC_n(8^{n-2}-5^{n-2})$
D. ${}^nC_4(8-5) + {}^nC_5(8^2-5^2) + \dots + {}^nC_n(8^{n-3}-5^{n-3})$

Answer: C

Solution:

Given,

$$9^n - 8n - 1 = 64\alpha$$

$$\Rightarrow \alpha = \frac{(1+8)^n - 8n - 1}{64}$$

$$= \frac{{}^nC_0 \cdot 1 + {}^nC_1 \cdot 8^1 + {}^nC_2 \cdot 8^2 + \dots + {}^nC_n \cdot 8^n - 8n - 1}{64}$$

$$= \frac{1 + 8n + {}^nC_2 \cdot 8^2 + \dots + {}^nC_n \cdot 8^n - 8n - 1}{8^2}$$

$$= \frac{{}^nC_2 \cdot 8^2 + {}^nC_3 \cdot 8^3 + \dots + {}^nC_n \cdot 8^n}{8^2}$$

$$= {}^nC_2 + {}^nC_3 \cdot 8 + {}^nC_4 \cdot 8^2 + \dots + {}^nC_n \cdot 8^{n-2}$$

Also given,

$$6^n - 5n - 1 = 25\beta$$

$$\Rightarrow \beta = \frac{(1+5)^n - 5n - 1}{25}$$

$$= \frac{{}^nC_0 \cdot 1 + {}^nC_1 \cdot 5 + {}^nC_2 \cdot 5^2 + \dots + {}^nC_n \cdot 5^n - 5n - 1}{5^2}$$

$$= \frac{1 + 5n + {}^nC_2 \cdot 5^2 + {}^nC_3 \cdot 5^3 + \dots + {}^nC_n \cdot 5^n - 5n - 1}{5^2}$$

$$= \frac{{}^nC_2 \cdot 5^2 + {}^nC_3 \cdot 5^3 + {}^nC_4 \cdot 5^4 + \dots + {}^nC_n \cdot 5^n}{5^2}$$

$$= {}^nC_2 + {}^nC_3 \cdot 5 + {}^nC_4 \cdot 5^2 + \dots + {}^nC_n \cdot 5^{n-2}$$

$$\therefore \alpha - \beta$$

$$= ({}^nC_2 + {}^nC_3 \cdot 8 + {}^nC_4 \cdot 8^2 + \dots + {}^nC_n \cdot 8^{n-2}) - ({}^nC_2 + {}^nC_3 \cdot 5 + {}^nC_4 \cdot 5^2 + \dots + {}^nC_n \cdot 5^{n-2})$$

$$= {}^nC_3 \cdot (8 - 5) + {}^nC_4 \cdot (8^2 - 5^2) + \dots + {}^nC_n (8^{n-2} - 5^{n-2})$$

Question 117

Let the coefficients of x^{-1} and x^{-3} in the expansion of $\left(2x^{\frac{1}{5}} - \frac{1}{x^{\frac{1}{5}}}\right)^{15}$, $x > 0$, be m and n respectively. If

r is a positive integer such that $mn^2 = {}^{15}C_r \cdot 2^r$, then the value of r is equal to _____

[29-Jun-2022-Shift-2]

Answer: 5

Solution:

$$T_{r+1} = (-1)^r \cdot {}^{15}C_r \cdot 2^{15-r} \times \frac{15-2r}{5}$$

$$m = {}^{15}C_{10} 2^5$$

$$n = -1$$

$$\text{so } mn^2 = {}^{15}C_5 2^5$$

Question 118

If the maximum value of the term independent of t in the expansion of $\left(t^2 x^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{15}$, $x \geq 0$ and $t \neq 0$, is K, then 8K is equal to

[25-Jul-2022-Shift-1]

Answer: 6006

Solution:

General term of $\left(t^2x\frac{1}{5} + \frac{(1-x)\frac{1}{10}}{t}\right)^{15}$ is

$$T_{r+1} = {}^{15}C_r \cdot \left(t^2x\frac{1}{5}\right)^{15-r} \cdot \left(\frac{(1-x)\frac{1}{10}}{t}\right)^r$$

$$= {}^{15}C_r \cdot t^{30-2r} \cdot x^{\frac{15-r}{5}} \cdot (1-x)^{\frac{r}{10}} \cdot t^{-r}$$

$$= {}^{15}C_r \cdot t^{30-3r} \cdot x^{\frac{15-r}{5}} \cdot (1-x)^{\frac{r}{10}}$$

Term will be independent of t when $30 - 3r = 0 \Rightarrow r = 10$

$\therefore T_{10+1} = T_{11}$ will be independent of t

$$\therefore T_{11} = {}^{15}C_{10} \cdot x^{\frac{15-10}{5}} \cdot (1-x)^{\frac{10}{10}}$$

$$= {}^{15}C_{10} \cdot x^1 \cdot (1-x)^1$$

T_{11} will be maximum when $x(1-x)$ is maximum.

$$\text{Let } f(x) = x(1-x) = x - x^2$$

$f(x)$ is maximum or minimum when $f'(x) = 0$

$$\therefore f'(x) = 1 - 2x$$

For maximum/minimum $f'(x) = 0$

$$\therefore 1 - 2x = 0$$

$$\Rightarrow x = \frac{1}{2}$$

Now, $f''(x) = -2 < 0$

$\therefore$ At $x = \frac{1}{2}$, $f(x)$ maximum

$\therefore$ Maximum value of T_{11} is

$$= {}^{15}C_{10} \cdot \frac{1}{2} \left(1 - \frac{1}{2}\right)$$

$$= {}^{15}C_{10} \cdot \frac{1}{4}$$

$$\text{Given } K = {}^{15}C_{10} \cdot \frac{1}{4}$$

$$\text{Now, } 8K = 2({}^{15}C_{10})$$

$$= 6006$$

Question 119

The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is

[25-Jul-2022-Shift-2]

Options:

A. 1

B. 4

C. 6

D. 8

Answer: D

Solution:

$$\operatorname{Re}\left(\frac{(11)^{1011} + (1011)^{11}}{9}\right) = \operatorname{Re}\left(\frac{2^{1011} + 3^{11}}{9}\right)$$

$$\text{For } \operatorname{Re}\left(\frac{2^{1011}}{9}\right)$$

$$2^{1011} = (9-1)^{337} = {}^{337}C_0 9^{337} (-1)^0 + {}^{337}C_1 9^{336} (-1)^1 + {}^{337}C_2 9^{335} (-1)^2 + \dots + {}^{337}C_{337} 9^0 (-1)^{337}$$

So, remainder is 8

$$\text{and } \operatorname{Re}\left(\frac{3^{11}}{9}\right) = 0$$

So, remainder is 8

Question 120

If the coefficients of x and x^2 in the expansion of $(1+x)^p(1-x)^q$, $p, q \leq 15$, are -3 and -5 respectively, then the coefficient of x^3 is equal to _____.

[26-Jul-2022-Shift-1]

Answer: 23

Solution:

Coefficient of x in $(1+x)^p(1-x)^q$

$${}^pC_0 {}^qC_1 + {}^pC_1 {}^qC_0 = -3 \Rightarrow p - q = -3$$

Coefficient of x^2 in $(1+x)^p(1-x)^q$

$${}^pC_0 {}^qC_2 - {}^pC_1 {}^qC_1 + {}^pC_2 {}^qC_0 = -5$$

$$\frac{q(q-1)}{2} - pq + \frac{p(p-1)}{2} = -5$$

$$\frac{q^2 - q}{2} - (q-3)q + \frac{(q-3)(q-4)}{2} = -5$$

$$\Rightarrow q = 11, p = 8$$

Coefficient of x^3 in $(1+x)^8(1-x)^{11}$

$$= -{}^{11}C_3 + {}^8C_1 {}^{11}C_2 - {}^8C_2 {}^{11}C_1 + {}^8C_3 = 23$$

Question 121

$\sum_{\substack{i,j=0 \\ i \neq j}}^n {}^nC_i {}^nC_j$ is equal to

[26-Jul-2022-Shift-2]

Options:

A. $2^{2n} - 2^n {}^nC_n$

B. $2^{2n-1} - 2^{n-1} {}^nC_{n-1}$

C. $2^{2n} - \frac{1}{2} 2^n {}^nC_n$

D. $2^{2n-1} + 2^{n-1} {}^nC_n$

Answer: B

Solution:

$$\begin{aligned} \sum_{\substack{i,j=0 \\ i \neq j}}^n {}^nC_i {}^nC_j &= \sum_{\substack{i,j=0 \\ i \neq j}}^n {}^nC_i {}^nC_j - \sum_{i=j}^n {}^nC_i {}^nC_j \\ &= \sum_{j=0}^n {}^nC_i \sum_{j=0}^n {}^nC_j - \sum_{i=0}^n {}^nC_i {}^nC_i \end{aligned}$$

$$= 2^n \cdot 2^n - {}^{2n}C_n$$

$$= 2^{2n} - {}^{2n}C_n$$

Question 122

The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is
[27-Jul-2022-Shift-1]

Options:

- A. 0
- B. 1
- C. 2
- D. 6

Answer: A

Solution:

$$(2021)^{2022} + (2022)^{2021}$$

$$= (7k_1 - 2)^{2022} + (7k_1 - 1)^{2021}$$

$$= [(7k_1 - 2)^3]^{674} + (7k_1)^{2021} - 2021(7k_1)^{2020} + \dots - 1$$

$$= (7k_2 - 1)^{674} + (7m - 1)$$

$$= (7n + 1) + (7m - 1) = 7(m + n) \text{ (multiple of 7)}$$

$$\therefore \text{Remainder} = 0$$

Question 123

Let for the 9th term in the binomial expansion of $(3 + 6x)^n$, in the increasing powers of $6x$, to be the greatest for $x = \frac{3}{2}$, the least value of n is n_0 . If k is the ratio of the coefficient of x^6 to the coefficient of x^3 , then $k + n_0$ is equal to:
[27-Jul-2022-Shift-2]

Answer: 24

Solution:

$$(3 + 6x)^n = 3^n (1 + 2x)^n$$

If T_9 is numerically greatest term

$$\therefore T_8 \leq T_9 \leq T_{10}$$

$${}^nC_7 3^{n-7} (6x)^7 \leq {}^nC_8 3^{n-8} (6x)^8 \geq {}^nC_9 3^{n-9} (6x)^9$$

$$\Rightarrow \frac{n!}{(n-7)!7!} 9 \leq \frac{n!}{(n-8)!8!} 3 \cdot (6x) \geq \frac{n!}{(n-9)!9!} (6x)^2$$

$$\Rightarrow \frac{9}{(n-7)(n-8)} \leq \frac{18 \left(\frac{3}{2}\right)}{(n-8)8} \geq \frac{36 \cdot 9}{9 \cdot 8 \cdot 4}$$

$$72 \leq 27(n-7) \text{ and } 27 \geq 9(n-8)$$

$$\frac{29}{3} \leq n \text{ and } n \leq 11$$

$$\therefore n_0 = 10$$

For $(3 + 6x)^{10}$

$$T_{r+1} = {}^{10}C_r$$

$$3^{10-r} (6x)^r$$

For coeff. of x^6

$$r = 6 \Rightarrow {}^{10}C_6 3^4 \cdot 6^6$$

For coeff. of x^3

$$r = 3 \Rightarrow {}^{10}C_3 3^7 \cdot 6^3$$

$$\therefore k = \frac{{}^{10}C_6 \cdot 3^4 \cdot 6^6}{{}^{10}C_3 \cdot 3^7 \cdot 6^3} = \frac{10!7!3!}{6!4!10!} \cdot 8$$

$$\Rightarrow k = 14$$

$$\therefore k + n_0 = 24$$

Question 124

The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is :
[28-Jul-2022-Shift-1]

Options:

A. 0

B. 2

C. 3

D. 4

Answer: C

Solution:

$$\begin{aligned} 7^{2022} + 3^{2022} &= (49)^{1011} + (9)^{1011} \\ &= (50 - 1)^{1011} + (10 - 1)^{1011} \\ &= 5\lambda - 1 + 5\mu - 1 \\ &= 5m - 2 \\ \text{Remainder} &= 5 - 2 = 3 \end{aligned}$$

Question 125

Let the coefficients of the middle terms in the expansion of $\left(\frac{1}{\sqrt{6}} + \beta x\right)^4$, $(1 - 3\beta x)^2$ and $\left(1 - \frac{\beta}{2}x\right)^6$, $\beta > 0$, respectively form the first three terms of an A.P. If d is the common difference of this A.P., then $50 - \frac{2d}{\beta^2}$ is equal to _____.
[28-Jul-2022-Shift-2]

Answer: 57

Solution:

Coefficients of middle terms of given expansions are ${}^4C_2 \cdot \frac{1}{\sqrt{6}}\beta^2$, ${}^2C_1(-3\beta)$, ${}^6C_3\left(\frac{-\beta}{2}\right)^3$ form an A.P.

$$\therefore 2 \cdot 2(-3\beta) = \beta^2 - \frac{5\beta^3}{2}$$

$$\Rightarrow -24 = 2\beta - 5\beta^2$$

$$\Rightarrow 5\beta^2 - 2\beta - 24 = 0$$

$$\Rightarrow 5\beta^2 - 12\beta + 10\beta - 24 = 0$$

$$\Rightarrow \beta(5\beta - 12) + 2(5\beta - 12) = 0$$

$$\beta = \frac{12}{5}$$

$$d = -6\beta - \beta^2$$

$$\therefore 50 - \frac{2d}{\beta^2} = 50 - 2 \frac{(-6\beta - \beta^2)}{\beta^2} = 50 + \frac{12}{\beta} + 2 = 57$$

Question 126

If $1 + ({}^{49}C_1 + {}^{49}C_2 + \dots + {}^{49}C_{49})({}^{50}C_2 + {}^{50}C_4 + \dots + {}^{50}C_{50})$ is equal to $2^n \cdot m$, where m is odd, then $n + m$ is equal to _____.
[28-Jul-2022-Shift-2]

Answer: 99

Solution:

$$1 = 1 + ({}^{49}C_0 + {}^{49}C_1 + \dots + {}^{49}C_{49})({}^{50}C_2 + {}^{50}C_4 + \dots + {}^{50}C_{50})$$

$$\text{As } {}^{49}C_0 + {}^{49}C_1 + \dots + {}^{49}C_{49} = 2^{49}$$

$$\text{and } {}^{50}C_0 + {}^{50}C_2 + \dots + {}^{50}C_{50} = 2^{49}$$

$$\Rightarrow {}^{50}C_2 + {}^{50}C_4 + \dots + {}^{50}C_{50} = 2^{49} - 1$$

$$\therefore 1 = 1 + (2^{49} + 1)(2^{49} - 1)$$

$$= 2^{98}$$

$$\therefore m = 1 \text{ and } n = 98$$

$$m + n = 99$$

Question 127

Let the ratio of the fifth term from the beginning to the fifth term from the end in the binomial expansion of $\left(4\sqrt{2} + \frac{1}{4\sqrt{3}}\right)^n$, in the increasing powers of $\frac{1}{4\sqrt{3}}$ be $4\sqrt{6} : 1$. If the sixth term from the beginning is $\frac{\alpha}{4\sqrt{3}}$, then α is equal to _____.
[29-Jul-2022-Shift-1]

Answer: 84

Solution:

$$\text{Fifth term from beginning} = {}^nC_4 \left(2\sqrt{2}\right)^{n-4} \left(\frac{-1}{4}\right)^4$$

$$\text{Fifth term from end} = (n-5+1)^{\text{th}} \text{ term from begin} = {}^nC_{n-4} \left(2\sqrt{2}\right)^3 \left(\frac{-1}{4}\right)^{n-4}$$

$$\text{Given } \frac{{}^nC_4 2^{\frac{n-4}{2}} \cdot 3^{-1}}{{}^nC_{n-4} 2^{\frac{3}{2}} \cdot 3^{\frac{n-4}{2}}} = 6\sqrt{4}$$

$$\Rightarrow \frac{{}^nC_4 2^{\frac{n-4}{2}} \cdot 3^{-1}}{{}^nC_{n-4} 2^{\frac{3}{2}} \cdot 3^{\frac{n-4}{2}}} = 6\sqrt{4}$$

$$\Rightarrow \frac{n-8}{4} = \frac{1}{4} \Rightarrow n = 9$$

$$\Rightarrow \frac{n-8}{4} = \frac{1}{4} \Rightarrow n = 9$$

$$T_6 = T_{5+1} = {}^9C_5 \left(2\sqrt{2}\right)^4 \left(\frac{-1}{4}\right)^5$$

$$= \frac{{}^9C_3 \cdot 2}{\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{3}} = \frac{84}{\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{3}} = \frac{\alpha}{\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{3}}$$

$$\Rightarrow \alpha = 84$$

Question 128

If $\sum_{k=1}^{10} k^2 ({}^{10}C_k)^2 = 22000L$, then L is equal to _____.

[29-Jul-2022-Shift-2]

Answer: 221

Solution:

Given,

$$\sum_{k=1}^{10} k^2 ({}^{10}C_k)^2 = 22000L$$

$$\Rightarrow \sum_{k=1}^{10} (k \cdot {}^{10}C_k)^2 = 22000L$$

$$\Rightarrow \sum_{k=1}^{10} \left(k \cdot \frac{10}{k} \cdot {}^{10}C_{k-1} \right)^2 = 22000L$$

$$\Rightarrow \sum_{k=1}^{10} (10 \cdot {}^9C_{k-1})^2 = 22000L$$

$$\Rightarrow 100 \cdot \sum_{k=1}^{10} ({}^9C_{k-1})^2 = 22000L$$

$$\Rightarrow 100({}^9C_0)^2 + ({}^9C_1)^2 + \dots + ({}^9C_9)^2 = 22000L$$

$$\Rightarrow 100({}^{18}C_9) = 22000L$$

$$[\text{Note : } ({}^nC_1)^2 + ({}^nC_2)^2 + \dots + ({}^nC_n)^2 = {}^{2n}C_n]$$

$$\Rightarrow 100 \times \frac{18!}{9!9!} = 22000L$$

$$\Rightarrow L = 221$$

Question 129

If $n \geq 2$ is a positive integer, then the sum of the series

$${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2) \text{ is}$$

[2021, 24 Feb. Shift-II]

Options:

A. $\frac{n(n-1)(2n+1)}{6}$

B. $\frac{n(n+1)(2n+1)}{6}$

C. $\frac{n(2n+1)(3n+1)}{6}$

D. $\frac{n(n+1)^2(n+2)}{12}$

Answer: B

Solution:

Given, $n \geq 2$

$$\text{Let } S = {}^2C_2 + {}^3C_2 + \dots + {}^nC_2 = {}^{n+1}C_3$$

$$\text{Now, } {}^{n+1}C_2 + 2 \times ({}^2C_2 + {}^3C_2 + \dots + {}^nC_2)$$

$$\begin{aligned}
&= {}^{n+1}C_2 + 2 \times {}^{n+1}C_3 \\
&= ({}^{n+1}C_2 + {}^{n+1}C_3) + {}^{n+1}C_3 \\
&= {}^{n+2}C_3 + {}^{n+1}C_3 = \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} \\
&= \frac{(n+2)(n+1)n(n-1)!}{3 \times 2 \times 1 \times (n-1)!} \\
&+ \frac{(n+1) \times n \times (n-1) \times (n-2)!}{3 \times 2 \times 1 \times (n-2)!} \\
&= \frac{n(n+1)}{6} [n+2+n-1] \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Question 130

For integers n and r , let $\binom{n}{r} = \begin{cases} {}^nC_r & \text{if } n \geq r \geq 0 \\ 0 & \text{otherwise} \end{cases}$. The maximum value of k for which the sum,

$$\sum_{i=0}^k \binom{10}{i} \binom{5}{k-i} + \sum_{i=0}^{k+1} \binom{12}{i} \binom{13}{k+1-i}$$

exists, is equal to
[2021, 24 Feb Shift-II]

Answer: 1

Solution:

$$\text{Given, } \binom{n}{r} = \begin{cases} {}^nC_r & \text{if } n \geq r \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \sum_{i=0}^k \binom{10}{i} \binom{15}{k-i} + \sum_{i=0}^{k+1} \binom{12}{i} \binom{13}{k+1-i}$$

$$\therefore (1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \dots + {}^{10}C_{10}x^{10}$$

$$\text{and } (1+x)^{15} = {}^{15}C_0 + {}^{15}C_1x + {}^{15}C_2x^2 + \dots + {}^{15}C_{15}x^{15}$$

$$\therefore \sum_{i=0}^k ({}^{10}C_i)({}^{15}C_{k-i}) = {}^{10}C_0 \cdot {}^{15}C_k$$

$$+ {}^{15}C_1 \cdot {}^{15}C_{k-1} + \dots + {}^{10}C_k \cdot {}^{15}C_0$$

$$\Rightarrow \text{Coefficient of } x^k \text{ in } (1+x)^{25} = {}^{25}C_k$$

$$\text{Also, } \sum_{i=0}^{k+1} ({}^{12}C_i)({}^{13}C_{k+1-i}) = {}^{12}C_0 \cdot {}^{13}C_{k+1}$$

$$+ {}^{12}C_1 \cdot {}^{13}C_k + \dots + {}^{12}C_{k+1} \cdot {}^{13}C_0$$

$$\Rightarrow \text{Coefficient of } x^{k+1} \text{ in } (1+x)^{25} = {}^{25}C_{k+1}$$

$$\Rightarrow {}^{25}C_k + {}^{25}C_{k+1} = {}^{26}C_{k+1}$$

As, we know by the definition of ${}^nC_r$, the maximum value of ${}^{26}C_{k+1}$ is possible for any possible large value of k .

Hence, k can have any large value.

Question 131

The value of $-{}^{15}C_1 + 2 \cdot {}^{15}C_2 - 3 \cdot {}^{15}C_3 + \dots - 15 \cdot {}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \dots + {}^{14}C_{11}$ is

[2021, 24 Feb. Shift-I]

Options:

A. $2^{16} - 1$

B. $2^{13} - 14$

C. $2^{13} - 13$

D. 2^{14}

Answer: B

Solution:

Given,

$$(-^{15}C_1 + 2 \cdot ^{15}C_2 - 3 \cdot ^{15}C_3 + \dots - 15 \cdot ^{15}C_{15}) + (^{14}C_1 + ^{14}C_3 + \dots + ^{14}C_{11})$$

Let

$$S_1 = -^{15}C_1 + 2 \cdot ^{15}C_2 - 3 \cdot ^{15}C_3 + \dots - 15 \cdot ^{15}C_{15}$$

$$= \sum_{r=1}^{15} (-1)^r \cdot r \cdot ^{15}C_r = 15 \sum_{r=1}^{15} (-1)^r \cdot ^{14}C_{r-1}$$

$$= 15(-^{14}C_0 + ^{14}C_1 - ^{14}C_2 + \dots - ^{14}C_{14})$$

$$= 15(0) = 0$$

$$S_2 = ^{14}C_1 + ^{14}C_3 + ^{14}C_5 + \dots + ^{14}C_{11}$$

$$= (^{14}C_1 + ^{14}C_3 + \dots + ^{14}C_{13}) - ^{14}C_{13}$$

$$= 2^{13} - 14$$

Now, the required value is

$$(-^{15}C_1 + 2 \cdot ^{15}C_2 - 3 \cdot ^{15}C_3 + \dots - 15 \cdot ^{15}C_{15})$$

$$+ (^{14}C_1 + ^{14}C_3 + \dots + ^{14}C_{11})$$

$$= S_1 + S_2$$

$$= 0 + 2^{13} - 14$$

$$= 2^{13} - 14$$

Question 132

If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is [2021, 25 Feb. Shift-II]

Answer: 1

Solution:

Given, when x is divided by 4, the remainder is 3.

Let $x = 4p + 3$, then

$$(2020 + x)^{2022} = (2020 + 4p + 3)^{2022}$$

$$= (2024 + 4p - 1)^{2022}$$

$$= (4k - 1)^{2022}$$

$$(\because 2024 \text{ is divisible by } 4) = {}^{2022}C_0(4K)^{2022}(-1)^0 + {}^{2022}C_1(4K)^{2021}(-1)^1 + \dots + {}^{2022}C_{2022}(4A)^0(-1)^{2022}$$

On expansion $(2020 + x)^{2022}$, we get the form of $8\lambda + 1$. Since, each terms have 2022 and $4k$, so if we take 2 common from 2022 we get 8. Thus, each term have 8 in common. Hence, remainder is 1.

Question 133

Let $m, n \in \mathbb{N}$ and $\gcd(2, n) = 1$. If

$$30 \binom{30}{0} + 29 \binom{30}{28} + \dots + 2 \binom{30}{28} + 1 \binom{30}{29} = n \cdot 2^m, \text{ then } n + m \text{ is } \dots, \text{ (Here, } \binom{n}{k} = {}^nC_k)$$

[2021, 26 Feb. Shift-I]

Answer: 45

Solution:

Given,

$$30 \cdot {}^{30}C_0 + 29 \cdot {}^{30}C_1 + \dots + 2 \cdot {}^{30}C_{28} + {}^{30}C_{29} \\ = n \cdot 2^m$$

This can be written as,

$$\sum_{r=0}^{29} (30-r) \cdot {}^{30}C_r = n \cdot 2^m$$

$$\text{or } \sum_{r=0}^{30} (30-r) \cdot {}^{30}C_r = n \cdot 2^m$$

$$\Rightarrow \sum_{r=0}^{30} 30 \cdot {}^{30}C_r - \sum_{r=0}^{30} r \cdot {}^{30}C_r = n \cdot 2^m$$

$$\Rightarrow 30 \sum_{r=0}^{30} {}^{30}C_r - \sum_{r=0}^{30} r \cdot {}^{30}C_r = n \cdot 2^m \text{ Using combination properties,}$$

$$\Rightarrow 30 \cdot (2^{30} - 30 \cdot (2)^{29}) = n \cdot 2^m$$

$$\Rightarrow 30 \cdot (2)^{29} (2-1) = n \cdot 2^m$$

$$\Rightarrow 2 \cdot 15 \cdot (2)^{29} = n \cdot 2^m$$

$$\Rightarrow 15 \cdot (2)^{30} = n \cdot 2^m \text{ Comparing both sides,}$$

$$n = 15 \text{ and } m = 30$$

$$\Rightarrow n + m = 15 + 30 = 45$$

Question 134

The maximum value of the term independent of 't' in the expansion of $\left(tx^{1/5} + \frac{(1-x)^{1/10}}{t} \right)^{10}$, where

$x \in (0, 1)$ is

[2021, 26 Feb. Shift-I]

Options:

A. $\frac{10!}{\sqrt{3}(5!)^2}$

B. $\frac{2 \cdot 10!}{3\sqrt{3}(5!)^2}$

C. $\frac{2 \cdot 10!}{3(5!)^2}$

D. $\frac{10!}{3(5!)^2}$

Answer: B

Solution:

Using Binomial expansion, its $(r+1)$ th term be,

$$T_{r+1} = {}^{10}C_r (tx^{1/5})^{10-r} \left\{ \frac{(1-x)^{1/10}}{t} \right\}^r$$

$$= {}^{10}C_r \frac{(t)^{10-r}}{(t)^r} (x)^{1/5 \cdot 10-r} (1-x)^{r/10}$$

$$= {}^{10}C_r (t)^{10-2r} (x)^{\frac{10-r}{5}} (1-x)^{r/10}$$

If this term is independent of 't', then we have $10 - 2r = 0$ gives, $r = 5$

$$\therefore T_6 = {}^{10}C_5 (x)^1 (1-x)^{1/2}$$

Let $f(x) = x(1-x)^{1/2}$, to obtain its maximum value, we have to differentiate it and equate it to 0.

$$\text{i.e. } f'(x) = 0 \Rightarrow \frac{x}{2\sqrt{1-x}} (-1) + \sqrt{1-x} = 0$$

$$\Rightarrow -x + 2(1-x) = 0$$

$$\Rightarrow -3x + 2 = 0$$

$$\Rightarrow x = 2/3$$

(Maximum value)

Thus, greatest term will be

$$T_6 = {}^{10}C_5 \left(\frac{2}{3} \right) \left(1 - \frac{2}{3} \right)^{1/2}$$

$$= {}^{10}C_5 \frac{2}{3\sqrt{3}} = \frac{10! \cdot 2}{(5!)^2 (3\sqrt{3})}$$

Question 135

The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right]^{10}$, $x \neq 1$, is equal to
[2021, 18 March Shift-II]

Answer: 210

Solution:

$$\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right]^{10}$$

$$= \left[\frac{(x^{1/3})^3 + (1)^3}{x^{2/3} - x^{1/3} + 1} - \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x}(\sqrt{x} - 1)} \right]^{10}$$

$$= \left[(x^{1/3} + 1) - \left(\frac{\sqrt{x} + 1}{\sqrt{x}} \right) \right]^{10}$$

[use $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$]

$$= (x^{1/3} - x^{-1/2})^{10}$$

General term, $T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{1/2})^r$

$$= {}^{10}C_r (x)^{\frac{10-r}{3} - \frac{r}{2}} \cdot (-x)^{\frac{r}{2}}$$

For term independent of x , we must put

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$$\Rightarrow 20 - 2r - 3r = 0$$

$$\Rightarrow r = 4$$

$$\therefore$$

$$T_{4+1} = T_5 = {}^{10}C_4 = \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!}$$

$$= 210$$

Question 136

Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1+x)^n$.

If $\sum_{k=0}^{10} (2^2 + 3k) {}^nC_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$

$\alpha, \beta \in \mathbb{R}$, then $\alpha + \beta$ is equal to
[2021, 18 March Shift-II]

Answer: 19

Solution:

$$\sum_{k=0}^{10} (2^2 + 3k) \cdot {}^nC_k = 4 \sum_{k=0}^{10} {}^nC_k + 3 \cdot \sum_{k=0}^{10} k \cdot {}^nC_k$$

$$= 4 \times 2^n + 3 \times n \sum_{k=1}^{n-1} C_{k-1}$$

$$\begin{aligned}
 &= 4 \times 2^n + 3n \times 2^{n-1} \left[\because {}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} \right] \\
 &= 2^n \left(4 + \frac{3n}{2} \right) \\
 &= \left(4 + \frac{3n}{2} \right) \cdot 2^n + 0 \times 3^n \sum_{k=0}^{10} (2^2 + 3k) \cdot {}^nC_k = 4 \sum_{k=0}^{10} {}^nC_k + 3 \cdot \sum_{k=0}^{10} k \cdot {}^nC_k \\
 &= 4 \times 2^n + 3 \times n \sum_{k=1}^{n-1} {}^{n-1}C_{k-1} \\
 &= 4 \times 2^n + 3n \times 2^{n-1} \left[\because {}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} \right] \\
 &= 2^n \left(4 + \frac{3n}{2} \right) \\
 &= \left(4 + \frac{3n}{2} \right) \cdot 2^n + 0 \times 3^n
 \end{aligned}$$

On comparing, $\left[0 \times 3^n + \left(4 + \frac{3n}{2} \right) \cdot 2^n \right] + 0$

$[\alpha \cdot 3^{10} + \beta \cdot 2^{10}]$,

we get $n = 10, \alpha = 0, \beta = 19$

$\therefore \alpha + \beta = 0 + 19 = 19$

Question 137

If the fourth term in the expansion of $(x + x^{\log_2 x})^7$ is 4480, then the value of x, where $x \in \mathbb{N}$ is equal to
[2021, 17 March Shift-I]

Options:

- A. 2
- B. 4
- C. 3
- D. 1

Answer: A

Solution:

$$\begin{aligned}
 (x + x^{\log_2 x})^7 \\
 T_4 &= 4480 \\
 T_r &= {}^nC_r (x^{\log_2 x})^r x^{(n-r)} \\
 T_4 &= {}^7C_4 (x^{\log_2 x})^4 x^3 = 35x^{4\log_2 x} x^3 \\
 T_4 &= 4480 \\
 35x^{3+4\log_2 x} &= 4480 \\
 x^{3+4\log_2 x} &= 128 = 2^7
 \end{aligned}$$

Taking log on both sides,

$$\begin{aligned}
 \log_2 x^{3+4\log_2 x} + 4\log_2 x &= \log_2 2^7 \\
 \Rightarrow (3 + 4\log_2 x)(\log_2 x) &= 7 \\
 \Rightarrow 4(\log_2 x)^2 + 3\log_2 x - 7 &= 0 \\
 \Rightarrow (\log_2 x - 1)(4\log_2 x + 7) &= 0 \\
 \Rightarrow \log_2 x = 1 \quad (\because x \in \mathbb{N}) \\
 \therefore x &= 2
 \end{aligned}$$

Question 138

If $(2021)^{3762}$ is divided by 17, then the remainder is
[2021, 17 March Shift-I]

Answer: 4

Solution:

$$\begin{aligned}(2021)^{3762} \\ 2021 &= (17 \times 119 - 2) \Rightarrow (17\lambda - 2) \\ (2021)^{3762} &= (17\lambda - 2)^{3762} = {}^{3762}C_0(17\lambda)^{3762} \\ &- {}^{3762}C_1(17\lambda)^{3761}2^1 + \dots + {}^{3762}C_n 2^{3762}\end{aligned}$$

Now, $(2021)^{3762}$ will be divisible by 17 all the terms except the last one for last one.

$$\begin{aligned}\therefore (2021)^{3762} &= 17\mu - 2^{3762} \\ &= 17\mu - 2^2(2^{3760}) \\ &= 17\mu - 4(16)^{235} \\ &= 17\mu - 4 \cdot (17 - 1)^{235} \\ (17 - 1)^{235} &= (-1)(1 - 17)^{235} \\ &= -(C_0 - C_1 17 + C_2 17^2 - \dots) \\ &= -C_0 + 17\gamma = -1 + 17\gamma \\ 17\mu - 4(17 - 1)^{235} &= 17\mu - 4[-1 + 17\gamma] \\ &= 17(\mu - 4\gamma) + 4 \\ \therefore (2021)^{3762} &= 17k + 4 \\ \text{Hence, 4 is the remainder.}\end{aligned}$$

Question 139

Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio

12 : 8 : 3. Then, the term independent of x in the expansion, is equal to
[2021, 17 March Shift-II]

Answer: 4

Solution:

$$\begin{aligned}\text{General term, } T_{r+1} &= {}^nC_r \cdot x^{n-r} \cdot \left(\frac{a}{x^2}\right)^r \\ &= {}^nC_r \cdot a^r \cdot x^{n-3r} \\ \therefore T_3 &= {}^nC_2 \cdot a^2 \cdot x^{n-6} \\ T_4 &= {}^nC_3 \cdot a^3 \cdot x^{n-9} \\ T_5 &= {}^nC_4 \cdot a^4 \cdot x^{n-12}\end{aligned}$$

$$\text{Now, } \frac{\text{coefficient of } T_3}{\text{coefficient of } T_4} = \frac{{}^nC_2 \cdot a^2}{{}^nC_3 \cdot a^3}$$

$$\begin{aligned}&= \frac{3}{a(n-2)} = \frac{3}{2} \\ \Rightarrow a(n-2) &= 2 \dots\dots\dots (i)\end{aligned}$$

$$\text{Also, } \frac{\text{coefficient of } T_4}{\text{coefficient of } T_5} = \frac{{}^nC_3 \cdot a^3}{{}^nC_4 \cdot a^4}$$

$$\begin{aligned}&= \frac{4}{a(n-3)} = \frac{8}{3} \\ \Rightarrow a(n-3) &= \frac{3}{2} \dots\dots\dots (ii)\end{aligned}$$

$$\text{From Eqs. (i) and (ii), } n = 6, a = \frac{1}{2}$$

For the term independent of 'x'

$$n - 3r = 0$$

$$\Rightarrow r = \frac{n}{3}$$

$$\Rightarrow r = \frac{6}{3}$$

$$\Rightarrow r = 2$$

$\therefore$ Independent term is T_3 .

$$\begin{aligned}\text{Now, } T_3 &= {}^6C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot (x)^0 \\ &= \frac{15}{4} = 3.75 \approx 4\end{aligned}$$

Question 140

If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then $(n - 1)$ is divisible by [2021, 16 Mar Shift-I]

Options:

- A. 26
- B. 30
- C. 8
- D. 7

Answer: A

Solution:

$$\begin{aligned}&(3^{1/4} + 5^{1/8})^{60} \\ \text{By using Binomial expansion, } (r+1) \text{ th term,} \\ T_{r+1} &= {}^{60}C_r (3^{1/4})^r (5^{1/8})^{60-r} \\ &= {}^{60}C_r 3^{r/4} 5^{\frac{60-r}{8}}\end{aligned}$$

For this term to be a rational number, r should be a multiple of 4 and $(60 - r)$ should be a multiple of 8.

Let A be a set when r is the multiple of 4.

$$A = \{4, 8, 12, \dots, 56, 60\}$$

$$n(A) = 15$$

Let B be a set of r , when $(60 - r)$ is the multiple of 8.

$$B = \{4, 12, 20, 28, 36, 44, 52, 60\}$$

$$n(B) = 8$$

$$\text{Now, } n(A \cap B) = 8$$

So, there are only 8 terms out of 61 terms which will be rational numbers.

53 terms will be irrational.

$$\text{So, } n = 53$$

and $n - 1 = 52$ which is divisible only by 26 among the given options.

Question 141

Let $[x]$ denote greatest integer less than or equal to x . If for $n \in \mathbb{N}$, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then

$$\left[\frac{3n}{2} \right] \sum_{j=0}^{\left[\frac{3n}{2} \right]} a_{2j} + 4 \sum_{j=0}^{\left[\frac{3n-1}{2} \right]} a_{2j} + 1 \text{ is equal to}$$

[2021, 16 March Shift-I]

Options:

- A. 2
- B. 2^{n-1}
- C. 1
- D. n

Answer: C

Solution:

Given, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j \times j$

$$(1 - x + x^3)^n = a_0 + a_1x + a_2x^2 + \dots + a_{3n}x^{3n}$$

Putting $x = 1$,

$$(1 - 1 + 1)^n = a_0 + a_1 + a_2 + \dots + a_{3n}$$

$$1 = a_0 + a_1 + a_2 + \dots + a_{3n} \dots (i)$$

Putting $x = -1$,

$$(1 + 1 - 1)^n = a_0 - a_1 + a_2 - a_3 + \dots + (-1)^{3n} a_{3n}$$

$$1 = a_0 - a_1 + a_2 - a_3 + \dots + (-1)^{3n} a_{3n} \dots (ii)$$

Adding Eqs. (i) and (ii), we get

$$2 = 2(a_0 + a_2 + a_4 + a_6 \dots)$$

$$a_0 + a_2 + a_4 + \dots = 1$$

On subtracting Eq. (ii) from Eq. (i), we get

$$0 = 2(a_1 + a_3 + a_5 + \dots)$$

$$a_1 + a_3 + a_5 + \dots = 0$$

$$\left[\frac{3n}{2} \right] \left[\frac{3n-1}{2} \right]$$

$$\text{Now, } \sum_{j=0}^{3n} a_{2j} + 4 \sum_{j=0}^{3n} a_{2j} + 1 = [a_0 + a_2 + a_4 + \dots] + 4$$

$$= 1 + 4 \times 0$$

$$= 1$$

Question 142

The value of $\sum_{r=0}^6 ({}^6C_r \cdot {}^6C_{6-r})$ is equal to
[2021, 17 March Shift-II]

Options:

A. 1124

B. 1324

C. 1024

D. 924

Answer: D

Solution:

Method (1) (Proper Method)

$$\begin{aligned} \sum_{r=0}^6 {}^6C_r \cdot {}^6C_{6-r} \\ = {}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + {}^6C_2 \cdot {}^6C_4 + \\ {}^6C_3 \cdot {}^6C_3 + {}^6C_4 \cdot {}^6C_2 + {}^6C_5 \cdot {}^6C_1 + {}^6C_6 \cdot {}^6C_0 \end{aligned}$$

Now,

$$\begin{aligned} (1+x)^6 \cdot (1+x)^6 &= ({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + \\ &{}^6C_3x^3 + \dots + {}^6C_6x^6)({}^6C_0 + {}^6C_1x + {}^6C_2x^2 \\ &+ {}^6C_3x^3 + \dots + {}^6C_6x^6) \end{aligned}$$

On comparing the coefficients of x^6 from both sides, we have

$$\begin{aligned} {}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + {}^6C_2 \cdot {}^6C_4 + \dots + \\ = \frac{{}^6C_6 \cdot {}^6C_0 = {}^{12}C_6}{6!(12-6)!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6! \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)} \\ = 924 \end{aligned}$$

Method (2) (Short-cut Method)

As, we know that,

$$\begin{aligned} r^n C_0^2 + {}^nC_1^2 + {}^nC_2^2 + {}^nC_3^2 + \dots + {}^nC_n^2 = 2^n C_n \\ \Rightarrow {}^nC_0 \cdot {}^nC_0 + {}^nC_1 \cdot {}^nC_1 + {}^nC_2 \cdot {}^nC_2 + \dots + \\ {}^nC_n \cdot {}^nC_n = 2^n C_n \end{aligned}$$

$$\Rightarrow {}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2}$$

$$+ \dots + {}^nC_n \cdot {}^nC_0 = 2^n {}^nC_n (\because {}^nC_r = {}^nC_{n-r})$$

Putting $n = 6$, we get,

$${}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + {}^6C_2 \cdot {}^6C_4 + \dots +$$

$${}^6C_6 \cdot {}^6C_0 = {}^{12}C_6$$

Question 143

Let n be a positive integer. Let

$$A = \sum_k (-1)^k {}^nC_k \left[\begin{aligned} &\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k \\ &+ \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k \end{aligned} \right]$$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to

[2021, 16 March Shift-II]

Answer: 6

Solution:

$$\text{Given, } A = \sum_{k=0}^n (-1)^k \cdot {}^nC_k$$

$$\left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k \right]$$

$$\Rightarrow A = \sum_{k=0}^n (-1)^k$$

$$\left[\begin{aligned} &{}^nC_k \left(\frac{1}{2}\right)^k + {}^nC_k \left(\frac{3}{4}\right)^k + {}^nC_k \left(\frac{7}{8}\right)^k \\ &+ {}^nC_k \left(\frac{15}{16}\right)^k + {}^nC_k \left(\frac{31}{32}\right)^k \end{aligned} \right]$$

$$\Rightarrow (1-x)^n = \sum_{r=0}^n (-1)^r \cdot C_r x^r$$

$$x = \frac{1}{2}$$

$$\Rightarrow \left(1 - \frac{1}{2}\right)^n = C_0 - C_1 \left(\frac{1}{2}\right) + C_2 \left(\frac{1}{2}\right)^2 \dots$$

$$\Rightarrow x = \frac{3}{4}$$

$$\left(1 - \frac{3}{4}\right)^n = C_0 - C_1 \left(\frac{3}{4}\right) + C_2 \left(\frac{3}{4}\right)^2 \dots$$

Similarly, we will get

$$A = \sum_{k=0}^n (-1)^k \left[\begin{aligned} &{}^nC_k \left(\frac{1}{2}\right)^k + {}^nC_k \left(\frac{3}{4}\right)^k + {}^nC_k \left(\frac{7}{8}\right)^k \\ &+ {}^nC_k \left(\frac{15}{16}\right)^k + {}^nC_k \left(\frac{31}{32}\right)^k \end{aligned} \right]$$

$$\Rightarrow A = \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n$$

$$+ \left(1 - \frac{15}{16}\right)^n + \left(1 - \frac{31}{32}\right)^n$$

$$\Rightarrow A = \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \frac{1}{2^{5n}} \Rightarrow A = \frac{1}{2^n} \cdot \left[\frac{1 - \frac{1}{(2^n)^5}}{1 - \frac{1}{2^n}} \right]$$

$$\Rightarrow A = \left(\frac{1}{2^n}\right) \left(\frac{2^{5n}-1}{2^n-1}\right) \left(\frac{2^n}{2^{5n}}\right)$$

$$\Rightarrow A = \left(\frac{2^{5n}-1}{2^n-1}\right) \left(\frac{1}{2^{5n}}\right)$$

$$\because 63A = 1 - \frac{1}{2^{30}} \Rightarrow 63 \frac{(2^{5n}-1)}{2^{5n}(2^n-1)} = \frac{2^{30}-1}{2^{30}}$$

$$\Rightarrow \left(\frac{63}{2^n - 1} \right) \left(1 - \frac{1}{2^{5n}} \right) = \left(1 - \frac{1}{2^{30}} \right)$$

$n = 6$, satisfies the equation.

Question 144

The term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right)^{10}$, where $x \neq 0, 1$ is equal to

[2021, 2 July Shift I]

Answer: 210

Solution:

$$\begin{aligned} & \left[\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} \right) - \left(\frac{x-1}{x - x^{1/2}} \right) \right]^{10} \\ &= \left(x^{1/3} + 1 - \frac{x^{1/2} + 1}{x^{1/2}} \right)^{10} \\ &= \left(x^{1/3} - \frac{1}{x^{1/2}} \right)^{10} \end{aligned}$$

General term,

$$\Rightarrow T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} \left(\frac{-1}{x^{1/2}} \right)^r$$

For independent term,

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$$\Rightarrow r = 4$$

$$\therefore T_5 = {}^{10}C_4 = 210$$

Question 145

If b is very small as compared to the value of a , so that the cube and other higher powers of $\frac{b}{a}$ can be neglected in the identity

$$\frac{1}{a-b} + \frac{1}{a-2b} + \frac{1}{a-3b} + \dots + \frac{1}{a-nb} = \alpha n + \beta n^2 + \gamma n^3, \text{ then the value of } \gamma \text{ is}$$

[2021, 25 July Shift-I]

Options:

A. $\frac{a^2 + b}{3a^3}$

B. $\frac{a+b}{3a^2}$

C. $\frac{b^2}{3a^3}$

D. $\frac{a+b^2}{3a^3}$

Answer: C

Solution:

$$(a-b)^{-n} = a^{-n} \left(1 - \frac{b}{a}\right)^{-n}$$

$$\left(1 - \frac{b}{a}\right)^{-n} = \left[1 + n\left(\frac{b}{a}\right) + \frac{n(n+1)}{2}\left(\frac{b}{a}\right)^2\right]$$

As, we can ignore the powers greater than or equal to 3.

$$\begin{aligned} a^{-n} \left(1 - \frac{b}{a}\right)^{-n} &= \frac{1}{a^n} + \frac{n \cdot b}{a^{n+1}} + \frac{n(n+1)}{2} \cdot \frac{b^2}{a^{n+2}} \\ &= \frac{1}{a^n} + \frac{(a-b)^{-n}}{a^{n+1}} + \frac{n(n+1)}{2} \cdot \frac{b^2}{a^{n+2}} \end{aligned}$$

When $n=1$

$$c(a-b)^{-1} = \frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} b \rightarrow rb$$

$$\sum_{r=b}^n (a-rb)^{-1} = \sum \left[\frac{1}{a} + r\left(\frac{b}{a^2}\right) + r^2\left(\frac{b^2}{a^3}\right) \right]$$

$$S = \frac{n}{a} + \frac{n(n+1)b}{2a^2} + \frac{n(n+1)(2n+1)b^2}{6a^3}$$

$$\text{Coefficient of } n^3 = \frac{2b^2}{6a^3} = \frac{b^2}{3a^3}$$

Question 146

The sum of all those terms which are rational numbers in the expansion of $(2^{1/3} + 3^{1/4})^{12}$ is
[2021, 25 July Shift-II]

Options:

- A. 89
- B. 27
- C. 35
- D. 43

Answer: D

Solution:

In the given expansion of $\left(2^{1/3} + 3^{1/4}\right)^{12}$

$$\text{General term} \Rightarrow T_{r+1} = {}^{12}C_r (2^{1/3})^r (3^{1/4})^{12-r} \left(2^{1/3}\right)^r$$

will be a rational number when $r = 0, 3, 6, 9, 12$ and $\left(3^{1/4}\right)^{12-r}$

will be rational number when $r = 0, 4, 8, 12 \Rightarrow r = 0, 12$

If $r = 0$, then

$$T_1 = {}^{12}C_0 \left(2^{1/3}\right)^0 \left(3^{1/4}\right)^{12} = 27$$

$$\text{If } r = 12, \text{ then } T_{13} = {}^{12}C_{12} \left(2^{1/3}\right)^{12} \left(3^{1/4}\right)^0 = 16$$

$$\text{So, } T_1 + T_{13} = 27 + 16 = 43$$

Question 147

If the greatest value of the term independent of x in the expansion of $\left(x \sin \alpha + a \frac{\cos \alpha}{x}\right)^{10}$ is $\frac{10!}{(5!)^2}$, then the value of a is equal to
[2021, 25 July Shift-II]

Options:

- A. -1
- B. 1
- C. -2
- D. 2

Answer: D

Solution:

In the expansion of $\left(x \sin \alpha + a \frac{\cos \alpha}{x}\right)^{10}$

$$T_{r+1} = {}^{10}C_r (x \sin \alpha)^{10-r} \left(\frac{a \cos \alpha}{x}\right)^r$$
$$= {}^{10}C_r (x)^{10-2r} (\sin \alpha)^{10-r} (a \cos \alpha)^r$$

T_{r+1} is independent of x , when $10 - 2r = 0$

$$\therefore 2r = 10$$

$$T_6 = {}^{10}C_5 (\sin \alpha)^5 (a^5) (\cos \alpha)^5$$
$$= {}^{10}C_5 a^5 \cdot \frac{1}{2^5} (\sin 2\alpha)^5$$

For greatest value, $\sin 2\alpha = 1$

$$= {}^{10}C_5 (a)^5 \cdot \frac{1}{2^5}$$

Given, that the greatest value is $\frac{10!}{(5!)^2}$.

$$\text{So, } {}^{10}C_5 \frac{a^5}{2^5} = \frac{10!}{(5!)^2}$$

$$\Rightarrow \frac{10!}{(5!)^2} \cdot \frac{a^5}{2^5} = \frac{10!}{(5!)^2}$$

$$\Rightarrow a = 2$$

Question 148

For the natural numbers m, n , if $(1-y)^m(1+y)^n$

$$= 1 + a_1 y + a_2 y^2 + \dots + a_{m+n} y^{m+n} \text{ and}$$

$a_1 = a_2 = 10$, then the value of

$(m+n)$ is equal to

[2021, 20 July Shift-II]

Options:

- A. 88
- B. 64
- C. 100
- D. 80

Answer: D

Solution:

$$\text{Given, } (1-y)^m(1+y)^n = 1 + a_1 y + a_2 y^2 + \dots + a_{m+n} y^{m+n}$$

Now,

$$(1-y)^m(1+y)^n = (1 - my + {}^mC_2 y^2 + \dots +$$

$$(-1)^m \cdot {}^mC_m y^m)$$

$$\times (1 + ny + {}^nC_2 y^2 + \dots + {}^nC_n y^n)$$

On expanding,

$$(1-y)^m(1+y)^n = 1 + (n-m)y + {}^nC_2 - mn + {}^mC_2 y^2 + \dots$$

$\therefore$ Coefficient of $y = n - m$

$$\text{Coefficient of } y^2 = {}^nC_2 - mn + {}^mC_2$$

Given expression have

$$\text{Coefficient of } y = a_1 = 10$$

$$\text{Coefficient of } y^2 = a_2 = 10$$

$$\therefore n - m = 10 \quad \dots\dots (i)$$

$$\text{and } {}^nC_2 + {}^mC_2 - mn = 10 \quad \dots\dots (ii)$$

From Eq. (ii),

$$\frac{n!}{2!(n-2)!} + \frac{m!}{2!(m-2)!} - mn = 10$$

$$\frac{n(n-1)}{2} + \frac{m(m-1)}{2} - mn = 10$$

$$\Rightarrow n(n-1) + m(m-1) - 2mn = 20$$

$$\Rightarrow n^2 - n + m^2 - m - 2mn = 20$$

$$\Rightarrow (m^2 + n^2 - 2mn) - (m + n) = 20$$

$$\Rightarrow (m - n)^2 - (m + n) = 20$$

$$\Rightarrow (-10)^2 - (m + n) = 20 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow 100 - (m + n) = 20$$

$$\Rightarrow m + n = 100 - 20 = 80$$

Question 149

The coefficient of x^{256} in the expansion of $(1-x)^{101}(x^2+x+1)^{100}$ is
[2021, 20 July Shift-I]

Options:

A. ${}^{100}C_{16}$

B. ${}^{100}C_{15}$

C. $-{}^{100}C_{16}$

D. $-{}^{100}C_{15}$

Answer: B

Solution:

$$(1-x)^{101}(x^2+x+1)^{100}$$

Coefficient of x^{256}

$$= [(1-x)(1+x+x^2)]^{100}(1-x) =$$

$$(1-x^3)^{100}(1-x)$$

$$\Rightarrow ({}^{100}C_0 - {}^{100}C_1x^3 + {}^{100}C_2x^6 - {}^{100}C_3x^9 \dots)$$

$$(1-x)$$

$$\sum (-1)^r {}^{100}C_r x^{3r} (1-x)$$

$$\Rightarrow 3r = 256 \text{ or } 255 \Rightarrow r = \frac{256}{3} \quad (\text{Reject})$$

$$r = 85$$

$$\text{Coefficient} = {}^{100}C_{85} = {}^{100}C_{15}$$

Question 150

The number of rational terms in the binomial expansion of $(4^{1/4} + 5^{1/6})^{120}$ is
[2021, 20 July Shift-I]

Answer: 21

Solution:

$$\left(\frac{1}{4} + 5 \frac{1}{6} \right)^{120}$$

$$\text{General term} = {}^{120}C_r \left(\frac{1}{4} \right)^r \left(5 \frac{1}{6} \right)^{120-r}$$

$$= {}^{120}C_r 4^{\frac{r}{4}} 5^{20-\frac{r}{6}}$$

$$= {}^{120}C_r 2^{\frac{r}{2}} 5^{20-\frac{r}{6}}$$

For this term to be rational, r should be a multiple of 2 and 6 i.e. r should be a multiple of 6.

$$r = \{0, 6, 12, 18, \dots, 120\}$$

$$\text{Number of terms} = 21$$

Question 151

If the constant term, in Binomial expansion of $\left(2x^r + \frac{1}{x^2} \right)^{10}$ is 180, then r is equal to

[2021, 22 July Shift-II]

Answer: 8

Solution:

$$\left(2x^r + \frac{1}{x^2} \right)^{10}$$

$$\text{General term} = {}^{10}C_k (2x^r)^{10-k} x^{-2k}$$

$$= {}^{10}C_k (2^{10-k}) (x^{10r-rk-2k})$$

$$10r - rk - 2k = 0$$

$$\Rightarrow k = \frac{10r}{r+2} \Rightarrow r = \frac{2k}{10-k}$$

$$\Rightarrow r = -2 + \frac{20}{10-k} \Rightarrow k < 10$$

$$\text{If } k = 8 \text{ and } r = -2 + 10 = 8$$

$$rk = 5 \text{ and } r = 2$$

$${}^{10}C_k 2^{10-k} = 180$$

At, r = 8 only this is satisfied.

Question 152

The number of elements in the set $\{n \in \{1, 2, 3, \dots, 100\} : (11)^n > (10)^n + (9)^n\}$ is

[2021, 22 July Shift-II]

Answer: 96

Solution:

$$11^n > 10^n + 9^n$$

$$\Rightarrow 11^n - 9^n > 10^n$$

$$\Rightarrow (10+1)^n - (10-1)^n > 10^n$$

$${}^nC_0 10^n + {}^nC_1 10^{n-1} + {}^nC_2 10^{n-2} + \dots$$

$-\{C_0^n C_0 10^n - {}^nC_1 10^{n-1}\}$
 $\Rightarrow 2({}^nC_1 10^{n-1} + {}^nC_3 10^{n-3}) > 10^n$
 For, $n = 1 \Rightarrow 2.1 \times 10$
 For, $n = 2 \Rightarrow 2(2) \times 100$
 For, $n = 3 \Rightarrow 2(3 \cdot 10^2 + 1) \times 1000$
 For, $n = 5 \Rightarrow$
 $2({}^5C_1 10^n + {}^5C_3 10^2 + {}^5C_5) \times 10^5$
 Hence, $n \in \{5, 6, 7, \dots, 100\}$
 $\therefore$ Number of elements = 96

Question 153

The lowest integer which is greater than $\left(1 + \frac{1}{10^{100}}\right)^{10^{100}}$ is

[2021, 25 July Shift-11]

Options:

- A. 3
- B. 4
- C. 2
- D. 1

Answer: A

Solution:

$$\text{Let } P = \left(1 + \frac{1}{10^{100}}\right)^{10^{100}}$$

$$\text{Let } x = 10^{100} \Rightarrow P = \left(1 + \frac{1}{x}\right)^x$$

$$P = {}^x C_0 + {}^x C_1 + {}^x C_2 \left(\frac{1}{x}\right) + {}^x C_3 \left(\frac{1}{x}\right)^2 + \dots \text{ up to } (10^{100} + 1) \text{ term}$$

$$\Rightarrow P = 1 + x \left(\frac{1}{x}\right) + \frac{x(x-1)}{2!} \cdot \frac{1}{x^2}$$

$$+ \frac{x(x-1)(x-2)}{3!} \cdot \frac{1}{x^3}$$

$$+ \dots \text{ upto } (10^{100} + 1) \text{ terms}$$

$$\Rightarrow P = 1 + 1 + \left[\left(\frac{1}{2!} - \frac{1}{2!x^2}\right) + \left(\frac{1}{3!} + \dots\right) + \dots \right]$$

$$\Rightarrow P = 2 +$$

$$(\text{Positive value less than } \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots)$$

Now

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$\Rightarrow \frac{1}{2!} + \frac{1}{3!} + \dots = e - 2$$

$$\Rightarrow P = 2 + [\text{Positive value less than } (e - 2)]$$

$$\Rightarrow P \in (2, 3)$$

So, lowest integer which is greater than P is 3.

Question 154

If the co-efficient of x^7 and x^8 in the expansion of $\left(2 + \frac{x}{3}\right)^n$ are equal, then the value of n is equal to

.....

[2021, 25 July Shift-II]

Answer: 55

Solution:

The coefficient of x^7 in the expansion of $\left(2 + \frac{x}{3}\right)^7 = {}^nC_7 2^{n-7} \frac{1}{3^7}$

and the coefficient of $x^8 = {}^nC_8 2^{n-8} \frac{1}{3^8}$

According to the question,

$${}^nC_7 2^{n-7} \cdot \frac{1}{3^7} = {}^nC_8 2^{n-8} \frac{1}{3^8}$$

$$\frac{n!}{7!(n-7)!} \cdot 2 \cdot 2^{n-8} \frac{1}{3^7} = \frac{n!}{8!(n-8)!} 2^{n-8} \times \frac{1}{3^8}$$

$$\Rightarrow \frac{1}{7!(n-7)(n-8)!} \times 2 = \frac{1}{8 \cdot 7!(n-8)!} \times \frac{1}{3}$$

$$\Rightarrow \frac{2}{n-7} = \frac{1}{24}$$

$$\Rightarrow n-7=48 \Rightarrow n=55$$

Question 155

$\sum_{k=0}^{20} ({}^{20}C_k)^2$ is equal to

[2021, 27 Aug. Shift-I]

Options:

A. ${}^{40}C_{21}$

B. ${}^{40}C_{19}$

C. ${}^{40}C_{20}$

D. ${}^{41}C_{20}$

Answer: C

Solution:

$$\sum_{k=0}^{20} ({}^{20}C_k)^2 = ({}^{20}C_0)^2 + ({}^{20}C_1)^2$$

$$+ ({}^{20}C_2)^2 + \dots + ({}^{20}C_{20})^2$$

$$\left[\because C_0^2 + C_1^2 + \dots + C_n^2 = \frac{(2n)!}{(n!)^2} \right]$$

$$= \frac{40!}{(20!)^2} = {}^{40}C_{20}$$

Question 156

If ${}^{20}C_r$ is the coefficient of x^r in the expansion of $(1+x)^{20}$, then the value of $\sum_{r=0}^{20} r {}^{20}C_r$ is equal to

[2021, 26 Aug. Shift-I]

Options:

A. 420×2^{19}

B. 380×2^{18}

C. 380×2^{19}

D. 420×2^{18}

Answer: D

Solution:

$${}^nC_r = \binom{n}{r} {}^{n-1}C_{r-1}$$

$$r {}^nC_r = n {}^{n-1}C_{r-1}$$

Similarly, $(r-1) {}^{n-1}C_{r-1} = (n-1) {}^{n-2}C_{r-2}$ (i)

Multiplying Eq. (i) with $(r-1)$

$$r(r-1) {}^nC_r = n(r-1) {}^{n-1}C_{r-1}$$

$$\Rightarrow r(r-1) {}^nC_r = n \cdot (n-1) {}^{n-2}C_{r-2}$$

$$r {}^{2n}C_r = [r(r-1) + r] {}^nC_r$$

$$= r(r-1) {}^nC_r + r {}^nC_r$$

$$= n(n-1) {}^{n-2}C_{r-2} + n \cdot {}^{n-1}C_{r-1}$$

$$\sum r {}^{2n}C_r = n(n-1) \sum_{r=2}^{n-2} {}^{n-2}C_{r-2} + n \sum_{r=1}^{n-1} {}^{n-1}C_{r-1}$$

Now, when $n = 20$

$$\sum r {}^{2n}C_r = (20 \times 19) \sum {}^{18}C_r + 20 \sum {}^{19}C_r$$

$$= (20 \times 19) 2^{18} + 20 \cdot 2^{19} = 420 \cdot 2^{18}$$

Question 157

Let $\binom{n}{k}$ denotes ${}^nC_k$ and

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \begin{cases} \binom{n}{k} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

If $A_k = \sum_{i=0}^9 \binom{9}{i} \left[\begin{matrix} 12 \\ 12-k+i \end{matrix} \right]$

$+ \sum_{i=0}^8 \binom{8}{i} \left[\begin{matrix} 13 \\ 13-k+i \end{matrix} \right]$

and $A_4 - A_3 = 190p$, then p is equal to
[2021, 26 Aug. Shift-II]

Answer: 49

Solution:

Given, $a_k = \sum_{i=0}^9 ({}^9C_i \times {}^{12}C_{12-k+i}) + \sum_{i=0}^8 ({}^8C_i \times {}^{13}C_{13-k+i})$

$$\Rightarrow A_k = \sum_{i=0}^9 {}^9C_i {}^{12}C_{k-i} + \sum_{i=0}^8 {}^8C_i {}^{13}C_{k-i}$$

$${}^9C_0 {}^{12}C_k + {}^9C_1 {}^{12}C_{k-1} + {}^9C_2 {}^{12}C_{k-2} + \dots +$$

$${}^9C_9 {}^{12}C_{k-9} = {}^{21}C_k \left[\because \sum_{r=0}^a {}^nC_r \times {}^mC_{a-r} = {}^{m+n}C_a \right]$$

Similarly,

$$\sum_{i=0}^8 {}^8C_i {}^{13}C_{k-i} = {}^{21}C_k$$

$$\begin{aligned}
 A_k &= {}^{21}C_k + {}^{21}C_k {}^{21}C_k \\
 A_4 - A_3 &= 2 \cdot ({}^{21}C_4 - {}^{21}C_3) \\
 &= 2(5985 - 1330) \\
 &\Rightarrow 190p = 9310 \\
 p &= 49
 \end{aligned}$$

Question 158

If the coefficients of x^7 in $\left(x^2 + \frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(x - \frac{1}{bx^2}\right)^{11}$, $b \neq 0$, are equal, then the value of b is equal to
[2021, 27 July Shift-1]

Options:

- A. 2
- B. -1
- C. 1
- D. -2

Answer: C

Solution:

Coefficient of x^7 in $\left(x^2 + \frac{1}{bx}\right)^{11}$

General term = ${}^{11}C_r (x^2)^r \left(\frac{1}{bx}\right)^{11-r}$

$$= {}^{11}C_r x^{3r-11} b^{r-11}$$

$$\Rightarrow 3r - 11 = 7 \Rightarrow r = 6$$

$$\therefore \text{Coefficient of } x^7 = {}^{11}C_6 b^{-5}$$

Now, coefficient of x^{-7} in $\left(x - \frac{1}{bx^2}\right)^{11}$

General term = ${}^{11}C_r x^r \left(-\frac{1}{bx^2}\right)^{11-r}$

$$= {}^{11}C_r \left(-\frac{1}{b}\right)^{11-r} x^r \cdot \frac{1}{x^{22-2r}}$$

$$= {}^{11}C_r \left(-\frac{1}{b}\right)^{11-r} x^{3r-22}$$

$$\text{Irr} \Rightarrow 3r - 22 = -7$$

$$\Rightarrow r = 5$$

$$\text{Coefficient} = {}^{11}C_5 \left(-\frac{1}{b}\right)^6 = {}^{11}C_5 b^{-6}$$

Now, according to the question,

$${}^{11}C_6 b^{-5} = {}^{11}C_5 b^{-6}$$

$$b = 1$$

Question 159

A possible value of 'x', for which the ninth term in the expansion of

$$\left\{ 3^{\log_3 \sqrt{25^{x-1} + 7}} + 3 \left(-\frac{1}{8}\right)^{\log_3 (5^{x-1} + 1)} \right\}^{10}$$

is equal to 180, is

[2021, 27 July Shift-II]

Options:

- A. 0
B. -1
C. 2
D. 1

Answer: D

Solution:

We have,

$$\left\{ 3^{\log_3 \sqrt{25^{x-1}+7}} + 3^{-\frac{1}{8} \cdot \log_3 (5^{x-1}+1)} \right\}^{10}$$

$$= \left\{ \sqrt{25^{x-1}+7} + (5^{x-1}+1)^{-\frac{1}{8}} \right\}^{10}$$

Ninth term in the expansion is 180.

$$\text{c So, } {}^{10}C_8 \left(\sqrt{25^{x-1}+7} \right)^{10-8} \left[(5^{x-1}+1)^{-\frac{1}{8}} \right]^8$$

$$= 180 \because (r+1) \text{ th term or expansion } (x+a)^n,$$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$\Rightarrow {}^{10}C_8 (25^{x-1}+7)(5^{x-1}+1)^{-1} = 180$$

$$\Rightarrow 45(25^{x-1}+7)(5^{x-1}+1)^{-1} = 180$$

$$\Rightarrow \frac{25^{x-1}+7}{5^{x-1}+1} = 4$$

$$\text{Let } 5^{x-1} = t$$

$$\Rightarrow \frac{t^2+7}{t+1} = 4$$

$$\Rightarrow t^2+7 = 4t+4$$

$$\Rightarrow t^2-4t+3 = 0$$

$$\Rightarrow (t-3)(t-1) = 0$$

$$\Rightarrow t = 3 \text{ or } t = 1$$

$$\text{When } t = 3$$

$$5^{x-1} = 3$$

$$5^x = 15$$

$$x = \log_5 15$$

$$\text{When, } t = 1$$

$$\text{then } 5^{x-1} = 1 [t = 5^{x-1}]$$

$$\Rightarrow x-1 = 0$$

$$\Rightarrow x = 1$$

Question 160

The ratio of the coefficient of the middle term in the expansion of $(1+x)^{20}$ and the sum of the coefficients of two middle terms in expansion of $(1+x)^{19}$ is

[2021, 25 July Shift-I]

Answer: 1

Solution:

$$\text{Coefficient of middle term of } (1+x)^{20} \text{ is } {}^{20}C_{10}$$

$$\text{Coefficient of middle term of } (1+x)^{19} \text{ is } {}^{19}C_9 \text{ and } {}^{19}C_{10}$$

According to the question

$$\left(\frac{{}^{20}C_{10}}{{}^{19}C_9 + {}^{19}C_{10}} \right) = \frac{{}^{20}C_{10}}{{}^{20}C_{10}} = 1$$

Question 161

If $\left(\frac{3^6}{4^4}\right)^k$ is the term, independent of x , in the binomial expansion of $\left(\frac{x}{4} - \frac{12}{x^2}\right)^{12}$, then k is equal to
[2021, 31 Aug. Shift-I]

Answer: 55

Solution:

$$\begin{aligned} &\left(\frac{x}{4} - \frac{12}{x^2}\right)^{12} \\ T_{r+1} &= (-1)^r {}^{12}C_r \left(\frac{x}{4}\right)^{12-r} \left(\frac{12}{x^2}\right)^r \\ &= (-1)^r \left(\frac{{}^{12}C_r \cdot 12^r}{4^{12-r}}\right) x^{(12-r-2r)} \\ \text{Term independent of } x & \\ 12 - 3r &= 0 \\ \Rightarrow r &= 4 \\ T_5 &= (-1)^4 \left(\frac{{}^{12}C_4 \cdot 12^4}{4^8}\right) = \frac{3^6}{4^4} k \\ k &= 55 \end{aligned}$$

Question 162

$3 \times 7^{22} + 2 \times 10^{22} - 44$ when divided by 18 leaves the remainder
[2021, 27 Aug. Shift-II]

Answer: 15

Solution:

$$\begin{aligned} &3 \times 7^{22} + 2 \times 10^{22} - 44 \\ &= 3 \times (6+1)^{22} + 2 \times (9+1)^{22} - 44 \\ \text{Now,} & \\ (1+6)^{22} &= 1 + {}^{22}C_1 6 + {}^{22}C_2 6^2 + \dots + {}^{22}C_{22} 6^{22} \\ &= (1+6\lambda) \\ (1+9)^{22} &= 1 + {}^{22}C_1 9 + {}^{22}C_2 9^2 + \dots + {}^{22}C_{22} 9^{22} \\ &= (1+9\mu) \\ \therefore &= 3(1+6\lambda) + 2(1+9\mu) - 44 \\ &= 18\lambda + 3 + 18\mu + 2 - 44 \\ &= 18\delta - 39 = 18\alpha + 15 \\ 3 \times 7^{22} + 2 \times 10^{22} - 44, &\text{ when divided by 18} \\ &\text{leaves remainder 15.} \end{aligned}$$

Question 163

If the coefficient of $a^7 b^8$ in the expansion of $(a + 2b + 4ab)^{10}$ is $k \cdot 2^{16}$, then k is equal to
[2021, 31 Aug. Shift-II]

Answer: 315

Solution:

$$(a + 2b + 4ab)^{10} = a^{10}b^{10} \left(\frac{1}{b} + \frac{2}{a} + 4 \right)^{10}$$

General term

$$= a^{10}b^{10} \frac{10! \left(\frac{1}{b} \right)^{r_1} \left(\frac{2}{a} \right)^{r_2} 4^{10-r_1-r_2}}{r_1! \cdot r_2! (10-r_1-r_2)!}$$

$$\text{So, } r_1 = 2, r_2 = 3$$

$$\text{Coefficient of } a^7b^8 = \frac{10! \cdot 2^3 \cdot 4^{10-2-3}}{2!3!(10-2-3)!}$$

$$= \frac{2^{13} \cdot 10!}{2!3!5!} = 2^{16} \cdot 315$$

$$\therefore k = 315$$

Question 164

If the sum of the coefficients of all even powers of x in the product

$(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$ is 61, then n is equal to _____.

[NA Jan. 7, 2020 (I)]

Answer: 30

Solution:

$$\text{Let } (1 - x + x^2 + \dots + x^{2n})(1 + x + x^2 + \dots + x^{2n})$$

$$= a_0 + a_1x + a_2x^2 + \dots$$

$$\text{put } x = 1$$

$$1(2n+1) = a_0 + a_1 + a_2 + \dots + a_{2n} \dots (i)$$

$$\text{put } x = -1$$

$$(2n+1) \times 1 = a_0 - a_1 + a_2 + \dots + a_{2n} \dots (ii)$$

Adding (i) and (ii), we get,

$$4n+2 = 2(a_0 + a_2 + \dots) = 2 \times 61$$

$$\Rightarrow 2n+1 = 61 \Rightarrow n = 30$$

Question 165

The coefficient of x^4 in the expansion of $(1 + x + x^2)^{10}$ is _____.

[NA Jan. 9, 2020 (I)]

Answer: 615

Solution:

$$\text{General term of the expansion} = \frac{10!}{\alpha!\beta!\gamma!} x^{\beta+2\gamma}$$

$$\text{For coefficient of } x^4; \beta + 2\gamma = 4$$

Here, three cases arise

Case-1 : When $\gamma = 0, \beta = 4, \alpha = 6$

$$\Rightarrow \frac{10!}{\alpha!\beta!\gamma!} x^{\beta+2\gamma}$$

Case-2 : When $\gamma = 1, \beta = 2, \alpha = 7$

$$\Rightarrow \frac{10!}{7!2!1!} = 360$$

Case-3 : When $\gamma = 2, \beta = 0, \alpha = 8$

$$\Rightarrow \frac{10!}{8!0!2!} = 45$$

Hence, total = 615

Question166

If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $(x + \sqrt{x^2 - 1})^6 + (x - \sqrt{x^2 - 1})^6$, then:

[Jan. 8, 2020 (II)]

Options:

A. $\alpha + \beta = 60$

B. $\alpha + \beta = -30$

C. $\alpha - \beta = 60$

D. $\alpha - \beta = -132$

Answer: D

Solution:

Using Binomial expansion

$$(x+a)^n + (x-a)^n = 2(T_1 + T_3 + T_5 + T_7 \dots)$$

$$\therefore (x + \sqrt{x^2 - 1})^6 + (x - \sqrt{x^2 - 1})^6 = 2(T_1 + T_3 + T_5 + T_7)$$

$$2[{}^6C_0 x^6 + {}^6C_2 x^4 (x^2 - 1) + {}^6C_4 x^2 (x^2 - 1)^2 + {}^6C_6 (x^2 - 1)^3]$$

$$= 2[x^6 + 15x^4(x^2 - 1) + 15x^2(x^4 - 2x^2 + 1) + (-1 + 3x^2 - 3x^4 + x^6)]$$

$$= 2(32x^6 - 48x^4 + 18x^2 - 1)$$

$$\alpha = -96 \text{ and } \beta = 36$$

$$\therefore \alpha - \beta = -132$$

Question167

In the expansion of $\left(\frac{x}{\cos \theta} + \frac{1}{x \sin \theta}\right)^{16}$, if l_1 is the least value of the term independent of x when $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$

and l_2 is the least value of the term independent of x when $\frac{\pi}{16} \leq \theta \leq \frac{\pi}{8}$, then the ratio $l_2 : l_1$ is equal to :

[Jan. 9, 2020 (II)]

Options:

A. 1: 8

B. 16: 1

C. 8: 1

D. 1: 16

Answer: B

Solution:

General term of the given expansion

$$T_{r+1} = {}^{16}C_r \left(\frac{x}{\sin \theta}\right)^{16-r} \left(\frac{1}{x \cos \theta}\right)^r$$

For $r = 8$ term is free from 'x'

$$T_9 = {}^{16}C_8 \frac{1}{\sin^8 \theta \cos^8 \theta}$$

$$T_9 = {}^{16}C_8 \frac{2^8}{(\sin 2\theta)^8}$$

When $\theta \in \left[\frac{\pi}{8}, \frac{\pi}{4} \right]$, then least value of the term independent of x

$$I_1 = {}^{16}C_8 2^8 \quad [\because \text{min. value of } I_1 \text{ at } \theta = \pi/4]$$

When $\theta \in \left[\frac{\pi}{16}, \frac{\pi}{8} \right]$, then least value of the term independent of x ,

$$I_2 = {}^{16}C_8 = \frac{2^8}{\left(\frac{1}{\sqrt{2}} \right)^8} = {}^{16}C_8 \cdot 2^8 \cdot 2^4$$

$[\because \text{min. value of } I_2 \text{ at } \theta = \pi/8]$

$$\text{Now, } \frac{I_2}{I_1} = \frac{{}^{16}C_8 \cdot 2^8 \cdot 2^4}{{}^{16}C_8 \cdot 2^8} = 16$$

Question 168

If $\{p\}$ denotes the fractional part of the number p , then $\left\{ \frac{3^{200}}{8} \right\}$, is equal to :

[Sep. 06, 2020 (I)]

Options:

A. $\frac{5}{8}$

B. $\frac{7}{8}$

C. $\frac{3}{8}$

D. $\frac{1}{8}$

Answer: D

Solution:

$$\frac{3^{200}}{8} = \frac{1}{8} (9^{100})$$

$$= \frac{1}{8} (1+8)^{100} = \frac{1}{8} \left[1 + n \cdot 8 + \frac{n(n-1)}{2} \cdot 8^2 + \dots \right]$$

$$= \frac{1}{8} + \text{Integer}$$

$$\therefore \left\{ \frac{3^{200}}{8} \right\} = \left\{ \frac{1}{8} + \text{integer} \right\} = \frac{1}{8}$$

Question 169

The natural number m , for which the coefficient of x in the binomial expansion of $\left(x^m + \frac{1}{x^2} \right)^{22}$ is 1540, is _____.

[NA Sep. 05, 2020 (I)]

Answer: 13

Solution:

$$T_{r+1} = {}^{22}C_r \cdot (x^m)^{22-r} \cdot \left(\frac{1}{x^2}\right)^r$$

$$T_{r+1} = {}^{22}C_r \cdot x^{22m-mr-2r}$$

$$\because 22m - mr - 2r = 1$$

$$\Rightarrow r = \frac{22m-1}{m+2} \Rightarrow r = 22 - \frac{3 \cdot 3 \cdot 5}{m+2}$$

So, possible value of $m = 1, 3, 7, 13, 43$

But ${}^{22}C_r = 1540$

$\therefore$ Only possible value of $m = 13$.

Question 170

The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^6$ in powers of x , is _____.

[NA Sep. 04, 2020 (I)]

Answer: 120

Solution:

$$\text{Coefficient of } x^4 \text{ in } \left(\frac{1-x^4}{1-x}\right)^6 = \text{coefficient of } x^4 \text{ in } (1-6x^4)(1-x)^{-6}$$

$$= \text{coefficient of } x^4 \text{ in } (1-6x^4)[1 + {}^6C_1x + {}^7C_2x^2 + \dots]$$

$$= {}^9C_4 - 6 \cdot 1 = 126 - 6 = 120$$

Question 171

Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{12}}$ is equal to _____.

[NA Sep. 04, 2020 (I)]

Answer: 8

Solution:

The given expression is $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$

$$\text{General term} = \frac{10!}{r_1!r_2!r_3!} (2x^2)^{r_1} (3x)^{r_2} (4)^{r_3}$$

Since, $a_7 = \text{Coeff. of } x^7$

$$2r_1 + r_2 = 7 \text{ and } r_1 + r_2 + r_3 = 10$$

Possibilities are

r_1	r_2	r_3
0	7	3
1	5	4
2	3	5
3	1	6

$$a_7 = \frac{10!3!7!4^3}{7!3!} + \frac{10!(2)(3)^5(4)^4}{5!4!}$$

$$+ \frac{10!(2)^2(3)(4)^5}{2!3!5!} + \frac{10!(2)^3(3)(4)^6}{3!6!}$$

a_{13} = Coeff. of x^{13}

$$2r_1 + r_2 = 13 \text{ and } r_1 + r_2 + r_3 = 10$$

Possibilities are

r_1	r_2	r_3
3	7	0
4	5	1
5	3	2
6	1	3

$$a_{13} = \frac{10!(2^3)(3^7)}{3!7!} + \frac{10!(2^4)(3^5)(4)}{4!5!}$$

$$+ \frac{10!(2^5)(3^3)(4^2)}{5!3!2!} + \frac{10!(2^6)(3)(4^3)}{6!1!3!}$$

$$\therefore \frac{a_7}{a_{13}} = 2^3 = 8$$

Question 172

If the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $|k|$ equals:

[Sep. 06, 2020 (II)]

Options:

- A. 9
- B. 1
- C. 3
- D. 2

Answer: C

Solution:

$$\text{General term} = T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \cdot \left(-\frac{k}{x^2}\right)^r$$

$$= {}^{10}C_r (-k)^r \cdot x^{\frac{10-r}{2} - 2r}$$

$$= {}^{10}C_r (-k)^r \cdot x^{\frac{10-5r}{2}}$$

Since, it is constant term, then

$$\frac{10-5r}{2} = 0 \Rightarrow r = 2$$

$$\therefore {}^{10}C_2 (-k)^2 = 405$$

$$\Rightarrow k^2 = \frac{405 \times 2}{10 \times 9} = \frac{81}{9} = 9$$

$$\therefore |k| = 3$$

Question 173

If for some positive integer n , the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14, then the largest coefficient in this expansion is :

[Sep. 04, 2020 (II)]

Options:

- A. 462
- B. 330
- C. 792
- D. 252

Answer: A

Solution:

Consider the three consecutive coefficients of $(1+x)^{n+5}$ be ${}^{n+5}C_r, {}^{n+5}C_{r+1}, {}^{n+5}C_{r+2}$

$$\therefore \frac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \frac{1}{2} \text{ (Given)}$$

$$\Rightarrow \frac{r+1}{n+5-r} = \frac{1}{2} \Rightarrow 3r = n+3 \dots (i)$$

$$\text{and } \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_{r+2}} = \frac{5}{7}$$

$$\Rightarrow \frac{r+2}{n+4-r} = \frac{5}{7} \Rightarrow 12r = 5n+6 \dots (ii)$$

Solving (i) and (ii) we get $r=4$ and $n=6$

$\therefore$ Largest coefficient in the expansion is ${}^{11}C_6 = 462$.

Question 174

If the number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^n$ is exactly 33, then the least value of n is :

[Sep. 03, 2020 (I)]

Options:

- A. 264
- B. 128
- C. 256
- D. 248

Answer: C

Solution:

$$\text{Here } \left(3^{1/2} + 5^{1/8}\right)^n$$

$$T_{r+1} = {}^nC_r (3)^{\frac{n-r}{2}} (5)^{\frac{r}{8}}$$

$$\therefore \frac{n-r}{2} \text{ and } \frac{r}{8} \text{ are integer}$$

So, r must be 0, 8, 16, 24, ...

$$\text{Now } n = t_{33} = a + (n-1)d = 0 + 32 \times 8 = 256$$

$$\Rightarrow n = 256$$

Question 175

If the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is k , then $18k$ is equal to :

[Sep. 03, 2020 (II)]

Options:

- A. 5
B. 9
C. 7
D. 11

Answer: C

Solution:

$$\text{General term} = T_{r+1} = {}^9C_r \left(\frac{3x^2}{2} \right)^{9-r} \left(-\frac{1}{3x} \right)^r$$

$$= {}^9C_r \left(\frac{3}{2} \right)^{9-r} \left(-\frac{1}{3} \right)^r x^{18-3r}$$

The term is independent of x, then
 $18 - 3r = 0 \Rightarrow r = 6$

$$\therefore T_7 = {}^9C_6 \left(\frac{3}{2} \right)^3 \left(-\frac{1}{3} \right)^6 = {}^9C_3 \left(\frac{1}{6} \right)^3$$

$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \left(\frac{1}{6} \right)^3 = \left(\frac{7}{18} \right)$$

$$\therefore 18k = 18 \times \frac{7}{18} = 7$$

Question 176

Let $\alpha > 0, \beta > 0$ be such that $\alpha^3 + \beta^2 = 4$. If the maximum value of the term independent of x in the binomial expansion of $\left(\alpha x^{\frac{1}{9}} + \beta x^{-\frac{1}{6}} \right)^{10}$ is 10k, then k is equal to :
[Sep. 02, 2020 (I)]

Options:

- A. 336
B. 352
C. 84
D. 176

Answer: A

Solution:

$$\text{General term of } \left(\alpha x^{\frac{1}{9}} + \beta x^{-\frac{1}{6}} \right)^{10} = {}^{10}C_r \left(\alpha x^{\frac{1}{9}} \right)^{10-r} \left(\beta x^{-\frac{1}{6}} \right)^r$$

$$= {}^{10}C_r \alpha^{10-r} \beta^r x^{\frac{10-r}{9} - \frac{r}{6}}$$

Term independent of x if $\frac{10-r}{9} - \frac{r}{6} = 0 \Rightarrow r = 4$

$$\therefore \text{Term independent of x} = {}^{10}C_4 \alpha^6 \beta^4$$

Since $\alpha^3 + \beta^2 = 4$

Then, by AM-GM inequality

$$\frac{\alpha^3 + \beta^2}{2} \geq (\alpha^3 \beta^2)^{\frac{1}{2}}$$

$$\Rightarrow (2)^2 \geq \alpha^3 \beta^2 \Rightarrow \alpha^6 \beta^4 \leq 16$$

$\therefore$ The maximum value of the term independent of x = 10k

$$\therefore 10k = {}^{10}C_4 \cdot 16 \Rightarrow k = 336$$

Question 177

For a positive integer n , $\left(1 + \frac{1}{x}\right)^n$ is expanded in increasing powers of x . If three consecutive coefficients in this expansion are in the ratio, $2 : 5 : 12$, then n is equal to _____.

[NA Sep. 02, 2020 (II)]

Answer: 118

Solution:

According to the question,

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 2 : 5 : 12$$

$$\Rightarrow \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{5}{2} \Rightarrow \frac{n-r+1}{r} = \frac{5}{2}$$

$$\Rightarrow 2n - 7r + 2 = 0 \dots (i)$$

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{12}{5} \Rightarrow \frac{n-r}{r+1} = \frac{12}{5}$$

$$\Rightarrow 5n - 17r - 12 = 0 \dots (ii)$$

Solving eqns. (i) and (ii),

$$n = 118, r = 34$$

Question 178

The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to:

[Sep. 04, 2020 (I)]

Options:

A. ${}^{51}C_7 - {}^{30}C_7$

B. ${}^{50}C_7 - {}^{30}C_7$

C. ${}^{50}C_6 - {}^{30}C_6$

D. ${}^{51}C_7 + {}^{30}C_7$

Answer: A

Solution:

The given series, $\sum_{r=0}^{20} {}^{50-r}C_6$

$$= {}^{50}C_6 + {}^{49}C_6 + {}^{48}C_6 + \dots + {}^{32}C_6 + {}^{31}C_6 + {}^{30}C_6$$

$$= ({}^{30}C_7 + {}^{30}C_6) + {}^{31}C_6 + {}^{32}C_6 + \dots + {}^{49}C_6 + {}^{50}C_6 - {}^{30}C_7$$

$$= ({}^{31}C_7 + {}^{31}C_6) + {}^{32}C_6 + \dots + {}^{49}C_6 + {}^{50}C_6 - {}^{30}C_7$$

$$= ({}^{32}C_7 + {}^{32}C_6) + \dots + {}^{49}C_6 + {}^{50}C_6 - {}^{30}C_7$$

$$\dots$$

$$\dots$$

$$= {}^{51}C_7 - {}^{30}C_7$$