



Question 1

Sequences and Series

Let $\overbrace{75 \dots 57}^r$ denote the $(r+2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \dots + \overbrace{75 \dots 57}^{98}$. If

$S = \frac{\overbrace{75 \dots 57}^{99} + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m+n$ is

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Options:

Answer: 1219

Solution:

$$\begin{aligned}
 S &= 77 + 757 + 7557 + \dots + \overbrace{75 \dots 57}^{98} \\
 &= 7(10 + 10^2 + \dots + 10^{99}) + 50(1 + 11 + \dots + \overbrace{111 \dots 1}^{98}) + 7 \times 99 \\
 &= 70 \left(\frac{10^{99} - 1}{9} \right) + \frac{50}{9} [(10 - 1) + (10^2 - 1) + \dots + (10^{98} - 1)] + 7 \times 99 \\
 &= 70 \left(\frac{10^{99} - 1}{9} \right) + \frac{50}{9} \left[10 \left(\frac{10^{98} - 1}{9} \right) - 98 \right] + 7 \times 99 \\
 &= \frac{7 \times 10^{100}}{9} - \frac{70}{9} + \frac{50}{9} \left[\frac{10^{99} - 1 - 9}{9} - 98 \right] + 7 \times 99 \\
 &= \frac{7 \times 10^{100}}{9} - \frac{70}{9} + \frac{50}{9} [\overbrace{111 \dots 1}^{99} - 99] + 7 \times 99 \\
 &= \frac{7 \times 10^{100} - 70 + \overbrace{555 \dots 50}^{99}}{9} - 550 + 693
 \end{aligned}$$

$$= \frac{\overbrace{7555 \dots 5}^{99} - 70 + 143 \times 9}{9}$$

$$= \frac{\overbrace{755 \dots 5}^{99} + 1210}{9}$$

$$\therefore m + n = 1219$$

Question2

Let $l_1, l_2, \dots, l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1 , and let $w_1, w_2, \dots, w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2 , where $d_1 d_2 = 10$. For each $i = 1, 2, \dots, 100$, let R_i be a rectangle with length l_i , width w_i and area A_i . If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is _____.

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Options:

Answer: 18900

Solution:

Given,

$l_1, l_2, \dots, l_{100}$ are in A.P with common difference d_1 .

So from property of A.P we can say,

$$l_2 = l_1 + d_1$$

$$l_3 = l_1 + 2d_1$$

$\vdots$

$$l_{100} = l_1 + 99d_1$$

Also given,

$w_1, w_2, \dots, w_{100}$ are in A.P with common difference d_2 .

$\therefore$ From the property of A.P we can say,

$$w_2 = w_1 + d_2$$

$$w_3 = w_1 + 2d_2$$

$\vdots$

$$w_{100} = w_1 + 99d_2$$

Now, also given,

$$d_1 d_2 = 10$$

and R_i is a rectangle whose length is l_i and width is w_i and area A_i .

$\therefore$ We know, area of rectangle

$$A_i = l_i \times w_i$$

$$\therefore A_{51} = l_{51} \times w_{51}$$

$$\text{and } A_{50} = l_{50} \times w_{50}$$

$$\text{Given, } A_{51} - A_{50} = 1000$$

$$\Rightarrow (l_1 + 50d_1)(w_1 + 50d_2) - (l_1 + 49d_1)(w_1 + 49d_2) = 1000$$

$$\Rightarrow [l_1w_1 + 2500d_1d_2 + 50(l_1d_2 + d_1w_1)] - [l_1w_1 + 49 \times 49d_1d_2 + 49(l_1d_2 + w_1d_1)] = 1000$$

$$\Rightarrow [(50)^2d_1d_2 - (49)^2d_1d_2] + (50 - 49)(l_1d_2 + d_1w_1) = 1000$$

$$\Rightarrow (99 \times 1)d_1d_2 + l_1d_2 + d_1w_1 = 1000$$

$$\Rightarrow 99 \times 10 + l_1d_2 + w_1d_1 = 1000$$

$$\Rightarrow l_1d_2 + w_1d_1 = 10$$

Now,

$$A_{100} - A_{90}$$

$$= l_{100} \cdot w_{100} - l_{90} \cdot w_{90}$$

$$= (l_1 + 99d_1)(w_1 + 99d_2) - (l_1 + 89d_1)(w_1 + 89d_2)$$

$$= [l_1w_1 + (99)^2d_1d_2 + 99(l_1d_2 + w_1d_1)] - [l_1w_1 + (89)^2d_1d_2 + 89(l_1d_2 + w_1d_1)]$$

$$= [(99)^2 - 89^2]d_1d_2 + 10(l_1d_2 + w_1d_1)$$

$$= 188 \times 10 \times d_1d_2 + 10 \times 10$$

$$= 188 \times 10 \times 10 + 100$$

$$= 18800 + 100$$

$$= 18900$$

Question3

Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is

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Options:

Answer: 8

Solution:

For real numbers y_1, y_2, y_3 , the quantities $3^{y_1}, 3^{y_2}$ and 3^{y_3} are positive real numbers, so according to the AM-GM inequality, we have

$$\frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq (3^{y_1} \cdot 3^{y_2} \cdot 3^{y_3})^{\frac{1}{3}}$$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3(3^{y_1} \cdot 3^{y_2} \cdot 3^{y_3})^{\frac{1}{3}}$$

On applying logarithm with base '3', we get

$$\log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq \left[1 + \frac{1}{3}(y_1 + y_2 + y_3)\right]$$

$$= 1 + 3$$

$$= 4$$

$$\{\because y_1 + y_2 + y_3 = 9\}$$

$$\therefore m = 4$$

Now, for positive real numbers x_1, x_2 and x_3 according to AM-GM inequality, we have

$$\frac{x_1 + x_2 + x_3}{3} \geq (x_1 x_2 x_3)^{\frac{1}{3}}$$

On applying logarithm with base '3', we get

$$\log_3 \left(\frac{x_1 + x_2 + x_3}{3} \right) \geq \frac{1}{3} (\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$$

$$\Rightarrow 1 \geq \frac{1}{3} (\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$$

$$\{\because x_1 + x_2 + x_3 = 9\}$$

$$\therefore M = 3$$

$$\text{Now, } \log_2(m^3) + \log_3(M^2)$$

$$= 3\log_2(4) + 2\log_2(3) = 6 + 2 = 8$$

Question4

Let $a_1, a_2, a_3, \dots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \dots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c , for which the equality $2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$ holds for some positive integer n , is

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Options:

Answer: 1

Solution:

Given arithmetic progression of positive integers terms $a_1, a_2, a_3, \dots$ having common difference '2' and geometric progression of positive integers terms $b_1, b_2, b_3, \dots$ having common ratio '2' with $a_1 = b_1 = c$, such that

$$2(a_1 + a_2 + a_3 + \dots + a_n) = b_1 + b_2 + b_3 + \dots + b_n$$

$$\Rightarrow 2 \times \frac{n}{2} [2C + (n-1)2] = C \left(\frac{2^n - 1}{2 - 1} \right)$$

$$\Rightarrow 2nC + 2n^2 - 2n = 2^n \cdot C - C$$

$$\Rightarrow C[2^n - 2n - 1] = 2n^2 - 2n$$

$$\because C \in N \Rightarrow 2n^2 - 2n \geq 2^n - 2n - 1$$

$$\Rightarrow 2n^2 + 1 \geq 2^n \Rightarrow n \leq 6$$

$$\text{and, also } C > 0 \Rightarrow n > 2$$

$\therefore$ The possible values of n are 3, 4, 5, 6

$$\text{So, at } n = 3, C = \frac{(2 \times 9) - 6}{8 - 6 - 1} = 12$$

$$\text{at, } n = 4, C = \frac{32 - 8}{16 - 8 - 1} = \frac{24}{9} = \frac{8}{3} \notin N$$

$$\text{at, } n = 5, C = \frac{50 - 10}{32 - 10 - 1} = \frac{40}{21} \notin N$$

$$\text{and at, } n = 6, C = \frac{72 - 12}{64 - 12 - 1} = \frac{60}{51} \notin N$$

$\therefore$ The required value of $C = 12$ for $n = 3$

so number of possible value of C is 1
