



Differential Equations

Question1

Let $y = f(x)$ be the real valued function defined on the interval $(0, \infty)$, satisfying $y(1) = 0$ and the differential equation

$$x \frac{dy}{dx} = y - x^3.$$

Then which of the following statements is (are) TRUE?

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Options:

A.

The function f has a local minimum at $x = \frac{1}{\sqrt{3}}$

B.

The function f has a local maximum at $x = \frac{1}{\sqrt{3}}$

C.

The function f is increasing in the interval $(1, 2)$

D.

If $g(x) = 4x^3 - 5x^2 + \frac{3}{2}x$ for $x > 0$, then the number of elements in the set $\{x \in (0, \infty) : f(x) = g(x)\}$ is 2

Answer: B

Solution:

We start with the differential equation:

$$x dy - y dx = -x^3 dx$$

Divide both sides by x^2 to simplify:

$$\frac{x dy - y dx}{x^2} = -x dx$$

The left side can be written as the derivative of a quotient:

$$d\left(\frac{y}{x}\right) = -x dx$$

Now, integrate both sides:

$$\frac{y}{x} = -\frac{x^2}{2} + c$$

Multiply by x to find y :

$$y = -\frac{x^3}{2} + cx$$

We are given $y(1) = 0$. Substitute $x = 1$ and $y = 0$:

$$0 = -\frac{1}{2} + c \Rightarrow c = \frac{1}{2}$$

Hence, the function becomes:

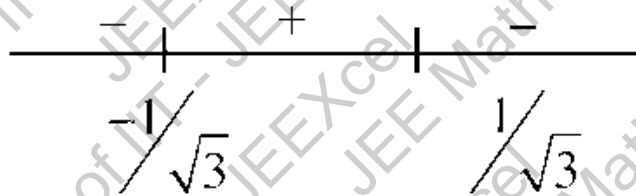
$$y = -\frac{x^3}{2} + \frac{x}{2}$$

Differentiate y with respect to x :

$$\frac{dy}{dx} = -\frac{3x^2}{2} + \frac{1}{2}$$

We can factorize this as:

$$\frac{dy}{dx} = -\frac{3}{2} \left(x - \frac{1}{\sqrt{3}} \right) \left(x + \frac{1}{\sqrt{3}} \right)$$



At $x = \frac{1}{\sqrt{3}}$, the derivative changes from positive to negative, so there is a local maximum.

At $x = -\frac{1}{\sqrt{3}}$, the derivative changes from negative to positive, so there is a local minimum.

For $g(x) = f(x)$:

$$4x^3 - 5x^2 + \frac{3}{2}x = -\frac{x^3}{2} + \frac{x}{2}$$

Multiply through to remove fractions and simplify:

$$9x^3 - 10x^2 + 2x = 0$$

Take x common:

$$x(9x^2 - 10x + 2) = 0$$

So, $x = 0$ or $9x^2 - 10x + 2 = 0$.

Using the quadratic formula:

$$x = \frac{5 \pm \sqrt{7}}{9}$$

Both these solutions are positive.

Question2

Let $y : (-\infty, \infty) \rightarrow (0, \infty)$ be the solution of the differential equation

$$\frac{dy}{dx} = \frac{e^{5x}y^3 + y^3}{e^x + e^x y^4},$$

satisfying $y(0) = \frac{1}{\sqrt{2}}$. Then the value of $y(\log_e 2)$ is

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Options:

A.

$$\sqrt{\frac{5+\sqrt{35}}{2}}$$

B.

$$\sqrt{\frac{7+\sqrt{53}}{2}}$$

C.

$$\frac{7+\sqrt{53}}{2}$$

D.

$$\frac{5+\sqrt{35}}{2}$$

Answer: B

Solution:

$$I = \int_0^2 \frac{1}{3+3^x} dx \quad \dots (1)$$

$$I = \int_0^2 \frac{1}{3+3^{2-x}} dx \quad \dots (2)$$

Adding (1) and (2)

$$2I = \int_0^2 \left(\frac{1}{3+3^x} + \frac{3^x}{3(3+3^x)} \right) dx$$

$$2I = \int_0^2 \left(\frac{3+3^x}{3(3+3^x)} \right) dx$$

$$2I = \int_0^2 \frac{1}{3} dx$$

$$2I = \frac{1}{3} [x]_0^2$$

$$2I = \frac{2}{3}$$

$$\Rightarrow I = \frac{1}{3}$$

$\Rightarrow$ Option (B) is correct.

Question3

Let $y(x)$ be the solution of the differential equation

$$x^2 \frac{dy}{dx} + xy = x^2 + y^2, \quad x > \frac{1}{e}$$

satisfying $y(1) = 0$. Then the value of $2 \frac{(y(e))^2}{y(e^2)}$ is _____.

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Answer: 0.75

Solution:

$$\begin{aligned} \frac{dy}{dx} + \frac{y}{x} &= \frac{x^2}{x^2} + \frac{y^2}{x^2} \\ \frac{dy}{dx} + \frac{y}{x} &= 1 + \left(\frac{y}{x}\right)^2 \\ \text{Let } \frac{y}{x} &= t \\ y &= xt \\ \frac{dy}{dx} &= x \frac{dt}{dx} + t \\ \therefore x \frac{dt}{dx} + t + t &= 1 + t^2 \\ \Rightarrow x \frac{dt}{dx} + 2t &= 1 + t^2 \\ \Rightarrow x \frac{dt}{dx} + t^2 + 1 - 2t & \\ \Rightarrow x \frac{dt}{dx} &= (t-1)^2 \\ \Rightarrow \frac{dt}{(t-1)^2} &= \frac{dx}{x} \end{aligned}$$

Integrating both sides

$$\begin{aligned} \Rightarrow \int \frac{dt}{(t-1)^2} &= \int \frac{dx}{x} \\ \Rightarrow \frac{-1}{(t-1)} &= \ln x + C \\ \Rightarrow \frac{-1}{\frac{y}{x} - 1} &= \ln x + C \\ \Rightarrow \frac{-x}{y-x} &= \ln x + C \end{aligned}$$

Given $y(1) = 0$

$$\begin{aligned} 1 &= C \\ \Rightarrow \frac{-x}{y-x} &= \ln x + 1 \dots\dots\dots (i) \end{aligned}$$

$\therefore$ Put $x = e$

$$\frac{-e}{y-e} = 1 + 1$$

$$\Rightarrow -e = 2(y - e)$$

$$\Rightarrow e = 2(e - y)$$

$$\Rightarrow \frac{e}{2} = e - y$$

$$\Rightarrow y = e - \frac{e}{2} \Rightarrow \frac{e}{2} \Rightarrow y = \frac{e}{2}$$

Put $x = e^2$ in (i)

$$\Rightarrow \frac{-e^2}{y - e^2} = 2 + 1$$

$$\Rightarrow -e^2 = 3(y - e^2)$$

$$\Rightarrow -e^2 = 3y - 3e^2$$

$$\Rightarrow 2e^2 = 3y$$

$$\Rightarrow y = \frac{2}{3}e^2$$

$$\therefore \frac{2(y(e))^2}{y(e^2)} = 2 \frac{\left(\frac{e}{2}\right)^2}{\frac{2}{3}e^2}$$

$$\Rightarrow \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4} = 0.75$$

Question4

For all $x > 0$, let $y_1(x)$, $y_2(x)$, and $y_3(x)$ be the functions satisfying

$$\frac{dy_1}{dx} - (\sin x)^2 y_1 = 0, \quad y_1(1) = 5,$$

$$\frac{dy_2}{dx} - (\cos x)^2 y_2 = 0, \quad y_2(1) = \frac{1}{3},$$

$$\frac{dy_3}{dx} - \frac{(2-x^3)}{x^3} y_3 = 0, \quad y_3(1) = \frac{3}{5e},$$

respectively. Then

$$\lim_{x \rightarrow 0^+} \frac{y_1(x)y_2(x)y_3(x) + 2x}{e^{3x} \sin x}$$

is equal to _____.

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Answer: 2

Solution:

Concept

Each is a first-order linear ODE of the type $y' = p(x)y$ with solution

$$y(x) = C e^{\int p(x) dx}$$

Steps

1) Solve the ODEs

- $y_1' = (\sin x)^2 y_1 \Rightarrow y_1(x) = 5 e^{\int_1^x (\sin t)^2 dt} = 5 e^{\frac{x-1}{2} - \frac{\sin 2x - \sin 2}{4}}$
- $y_2' = (\cos x)^2 y_2 \Rightarrow y_2(x) = \frac{1}{3} e^{\int_1^x (\cos t)^2 dt} = \frac{1}{3} e^{\frac{x-1}{2} + \frac{\sin 2x - \sin 2}{4}}$

Multiplying,

$$y_1(x)y_2(x) = \frac{5}{3} e^{x-1}.$$

- $y_3' = \left(\frac{2}{x^3} - 1\right) y_3 \Rightarrow y_3(x) = C e^{\int \left(\frac{2}{x^3} - 1\right) dx} = C e^{-x^{-2} - x}.$
Use $y_3(1) = \frac{3e}{5e}$ to get $C = \frac{3e}{5}$; hence

$$y_3(x) = \frac{3e}{5} e^{-x^{-2} - x}.$$

Therefore

$$y_1 y_2 y_3 = \left(\frac{5}{3} e^{x-1}\right) \left(\frac{3e}{5} e^{-x^{-2} - x}\right) = e^{-x^{-2}}.$$

2) Evaluate the limit

$$\frac{y_1 y_2 y_3 + 2x}{e^{3x} \sin x} = \frac{e^{-x^{-2}} + 2x}{e^{3x} \sin x}.$$

As $x \rightarrow 0^+$: $e^{-x^{-2}} \rightarrow 0$, $e^{3x} \rightarrow 1$, $\sin x \sim x$. Hence

$$\lim_{x \rightarrow 0^+} \frac{e^{-x^{-2}} + 2x}{e^{3x} \sin x} = \lim_{x \rightarrow 0^+} \frac{2x}{x} = 2.$$

Final Answer

2

Question 5

For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation

$$(x^2 - 5) \frac{dy}{dx} - 2xy = -2x(x^2 - 5)^2 \text{ such that } y(2) = 7.$$

Then the maximum value of the function $y(x)$ is :

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Answer: 16

Solution:

$$(x^2 - 5) \frac{dy}{dx} - 2xy = -2x(x^2 - 5)^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{2x}{(x^2 - 5)} y = -2x(x^2 - 5)$$

$$\text{I.F.} = e^{-\int \frac{2x}{x^2-5} dx} = \frac{1}{(x^2-5)}$$

Now

$$\begin{aligned} y \cdot \frac{1}{(x^2 - 5)} &= -\int 2x dx \\ &= -x^2 + c \end{aligned}$$

$$\Rightarrow y = c(x^2 - 5) - x^2(x^2 - 5)$$

$$y(2) = 7$$

$$\Rightarrow 7 = -c + 4$$

$$\Rightarrow c = -3$$

So,

$$y = (x^2 - 5)(-x^2 - 3) \dots \dots \dots (1)$$

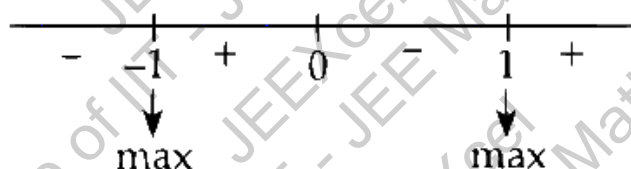
$$\frac{dy}{dx} = (x^2 - 5)(-2x) + (-x^2 - 3)(2x)$$

$$= 2x(-x + 5 - x^2 - 3)$$

$$= 2x(-2x^2 + 2)$$

For maxima and minima, put $\frac{dy}{dx} = 0$

$$\Rightarrow x = 0, \pm 1$$



From (1),

$$y_{\max} = 16$$

Question6

If $y(x)$ is the solution of the differential equation

$$x dy - (y^2 - 4y) dx = 0 \text{ for } x > 0, y(1) = 2,$$

and the slope of the curve $y = y(x)$ is never zero, then the value of $10y(\sqrt{2})$ is

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Answer: 8

Solution:

$$x dy = (y^2 - 4y) dx = 0$$

$$\Rightarrow x dy = (y^2 - 4y) dx$$

$$\Rightarrow \frac{dy}{(y^2 - 4y)} = \frac{dx}{x}$$

$$\Rightarrow \frac{dy}{y(y-4)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{4} \times \frac{y-(y-4)}{y(y-4)} dy = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{4} \left(\frac{1}{y-4} - \frac{1}{y} \right) dy = \frac{dx}{x}$$

Integrating both side, we get

$$\Rightarrow \frac{1}{4} \int \left(\frac{1}{y-4} - \frac{1}{y} \right) dy = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{4} \left(\ln \left| \frac{y-4}{y} \right| \right) = \ln |x| + \frac{1}{4} \ln C \text{ (Cont.)}$$

$$\Rightarrow \frac{1}{4} \ln \left| \frac{y-4}{y} \right| = \ln x + \frac{1}{4} \ln C \text{ [As } x > 0, \text{ so } |x| = x]$$

$$\Rightarrow \ln \left| \frac{y-4}{y} \right| = 4 \ln x + \ln C$$

$$\Rightarrow \ln \left| \frac{y-4}{y} \right| = \ln(x^4 C)$$

$$\Rightarrow \frac{y-4}{y} = \pm C x^4$$

$$\Rightarrow \frac{y-4}{y} = \lambda x^4 \text{ [Assume } \pm C = \lambda]$$

$$\text{Given, } y(1) = 2$$

$$\therefore \frac{2-4}{2} = \lambda(1)^4$$

$$\Rightarrow \lambda = -1$$

$$\therefore \frac{y-4}{y} = -x^4$$

$$\text{Now, } y(\sqrt{2}) = -(\sqrt{2})^4 = \frac{y-4}{y}$$

$$\Rightarrow \frac{y-4}{y} = -4$$

$$\Rightarrow y - 4 = -4y$$

$$\Rightarrow 5y = 4$$

$$\Rightarrow y = \frac{4}{5}$$

$$\therefore 10y(\sqrt{2}) = 10 \times \frac{4}{5} = 8$$

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$, then the value of

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