



Application of Derivatives

Question1

Consider the function $f : (0, \infty) \rightarrow (-\infty, \infty)$ given by

$$f(x) = \sqrt{x} \log_e(x) - x + 1.$$

Then which one of the following statements is TRUE?

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Options:

A.

The derivative of the function f is decreasing in the interval $(0, 1)$

B.

The function f has a local maximum at some point $a \in (0, \infty)$

C.

The function f has a local minimum at some point $b \in (0, \infty)$

D.

The function f has **NEITHER** a point of local maximum **NOR** a point of local minimum in the interval $(0, \infty)$

Answer: D

Solution:

We are given the function:

$$f(x) = \sqrt{x} \ln x - x + 1$$

To find where the function increases or decreases, we take the derivative:

$$f'(x) = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} - 1$$

$$f'(x) = \frac{2 + \ln x - 2\sqrt{x}}{2\sqrt{x}}$$

$$f'(x) = \frac{\ln x + 2}{2\sqrt{x}} - 1$$

The second derivative tells us how the slope is changing:

$$f''(x) = \frac{\left(\frac{1}{x}\right)(2\sqrt{x}) - (\ln x + 2)\left(\frac{1}{\sqrt{x}}\right)}{4x}$$

$$f''(x) = -\frac{\ln x}{4x\sqrt{x}}$$

For $x \in (0, 1)$, $\ln x < 0$. Therefore,

$$f''(x) > 0$$

This means $f'(x)$ is increasing in $(0, 1)$.

$$\frac{\ln x + 2}{2\sqrt{x}} - 1 = 0$$

$$\Rightarrow \ln x + 2 = 2\sqrt{x}$$

Rearranging, we define:

$$g(x) = 2\sqrt{x} - \ln x - 2$$

$$g'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x} = \frac{\sqrt{x} - 1}{x}$$

So, $g'(x) = 0$ when $x = 1$.

For $x < 1$, $g'(x) < 0$ (decreasing), and for $x > 1$, $g'(x) > 0$ (increasing).

Thus, minimum value of $g(x)$ occurs at $x = 1$.

$$\text{At } x = 1, g(1) = 2(1) - \ln(1) - 2 = 0.$$

Hence, $g(x) \geq 0$ for all $x > 0$.

$$f'(x) = -\frac{g(x)}{2\sqrt{x}}$$

Since $g(x) \geq 0$, we get $f'(x) \leq 0$ for all $x > 0$.

$$f'(x) = 0 \text{ only when } x = 1.$$

The derivative $f'(x)$ is negative everywhere except at $x = 1$ where it is zero. However, the sign of the derivative does not change around $x = 1$.

Therefore, the function $f(x)$ is strictly decreasing throughout its domain, and the tangent at $x = 1$ is horizontal but not a turning point.

Hence, the function has no local maximum or minimum.

Question2

Let $\mathbb{R}$ denote the set of all real numbers. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{6x + \sin x}{2x + \sin x}, & \text{if } x \neq 0, \\ \frac{7}{3}, & \text{if } x = 0. \end{cases}$$

Then which of the following statements is (are) TRUE?

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Options:

A.

The point $x = 0$ is a point of local maxima of f

B.

The point $x = 0$ is a point of local minima of f

C.

Number of points of local maxima of f in the interval $[\pi, 6\pi]$ is 3

D.

Number of points of local minima of f in the interval $[2\pi, 4\pi]$ is 1

Answer: B

Solution:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 3$$

$$\text{since } 3 > \frac{7}{3} \Rightarrow 3 > f(0)$$

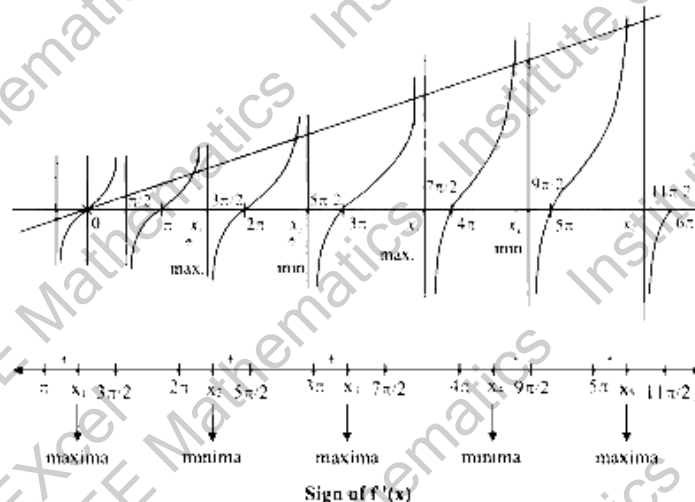
$\Rightarrow x = 0$ is local minima ...option (B) is correct

Now,

$$f(x) = \frac{6x + \sin x}{2x + \sin x} = 1 + \frac{4x}{2x + \sin x}$$

$$f'(x) = \frac{4[(2x + \sin x) \cdot 1 - x(2 + \cos x)]}{(2x + \sin x)^2} = \frac{4(\sin x - x \cos x)}{(2x + \sin x)^2}$$

$$= \frac{4 \cos x (\tan x - x)}{(2x + \sin x)^2}$$



Options (C) & (D) are also correct.

Question3

Let

$$\alpha = \sum_{k=1}^{\infty} \sin^{2k} \left(\frac{\pi}{6} \right)$$

Let $g : [0, 1] \rightarrow \mathbb{R}$ be the function defined by

$$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}.$$

Then, which of the following statements is/are TRUE ?

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Options:

- A. The minimum value of $g(x)$ is $2^{\frac{7}{6}}$
- B. The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$
- C. The function $g(x)$ attains its maximum at more than one point
- D. The function $g(x)$ attains its minimum at more than one point

Answer: A

Solution:

Given,

$$\begin{aligned} \alpha &= \sum_{k=1}^{\infty} \sin^{2k} \left(\frac{\pi}{6} \right) \\ &= \sin^2 \left(\frac{\pi}{6} \right) + \sin^4 \left(\frac{\pi}{6} \right) + \sin^6 \left(\frac{\pi}{6} \right) + \dots \end{aligned}$$

This is an infinite G.P.

$$\begin{aligned} \therefore \alpha &= \frac{\sin^2 \left(\frac{\pi}{6} \right)}{1 - \sin^2 \left(\frac{\pi}{6} \right)} \\ &= \frac{\frac{1}{4}}{1 - \frac{1}{4}} \\ &= \frac{1}{4} \times \frac{4}{3} \\ &= \frac{1}{3} \end{aligned}$$

$$\therefore g(x) = 2^{\frac{x}{3}} + 2^{\frac{1}{3}(1-x)}$$

Given $x \in [0, 1]$

We know,

$$AM \geq GM$$

For $2^{\frac{x}{3}}$ and $2^{\frac{1}{3}(1-x)}$ both are positive in $x \in [0, 1]$

$$\frac{x^{\frac{x}{3}} + 2^{\frac{1}{3}(1-x)}}{2} \geq \left(2^{\frac{x}{3}} \cdot 2^{\frac{1}{3}(1-x)}\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{2^{\frac{x}{3}} + 2^{\frac{1}{3}(1-x)}}{2} \geq \left(2^{\frac{1}{3}}\right)^{\frac{1}{2}}$$

$$\Rightarrow 2^{\frac{x}{3}} + 2^{\frac{1}{3}(1-x)} \geq 2 \cdot 2^{\frac{1}{6}}$$

$$\Rightarrow g(x) \geq 2^{\frac{7}{6}}$$

We know, $AM = GM$ when terms are equal.

$$\therefore g(x) = 2^{\frac{7}{6}} \text{ when}$$

$$2^{\frac{x}{3}} = 2^{\frac{1}{3}(1-x)}$$

$$\Rightarrow \frac{x}{3} = \frac{1-x}{3}$$

$$\Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

$$\therefore \text{At } x = \frac{1}{2}, [g(x)]_{\min} = 2^{\frac{7}{6}}$$

$\therefore$ Option (A) is correct.

And option (D) is wrong as at only at a single point $x = \frac{1}{2}$, $g(x)$ is minimum.

$$\text{Now, } g(x) = 2^{\frac{x}{3}} + 2^{\frac{1}{3}(1-x)}$$

$$= 2^{\frac{x}{3}} + \frac{2^{\frac{1}{3}}}{2^{\frac{x}{3}}}$$

$$= 2^{\frac{x}{3}} + 2^{\frac{1}{3}} \cdot 2^{-\frac{x}{3}}$$

$$\therefore g'(x) = 2^{\frac{x}{3}} \cdot \log 2 \cdot \frac{1}{3} + 2^{\frac{1}{3}} \cdot 2^{-\frac{x}{3}} \cdot \log 2 \cdot \left(-\frac{1}{3}\right)$$

$$= \frac{1}{3} \log 2 \left[2^{\frac{x}{3}} - \frac{2^{\frac{1}{3}}}{2^{\frac{x}{3}}}\right]$$

$$= \frac{1}{3} \log 2 \left[\frac{2^{\frac{2x}{3}} + 2^{\frac{1}{3}}}{2^{\frac{x}{3}}}\right]$$

We already found that at $x = \frac{1}{2}$ $g(x)$ is minimum.

$$\therefore g'(x) = 0 \text{ at } x = \frac{1}{2}$$

Now, $g'(x) > 0$ when

$$\frac{1}{3} \log 2 \left[\frac{2^{\frac{2x}{3}} + 2^{\frac{1}{3}}}{2^{\frac{x}{3}}}\right] > 0$$

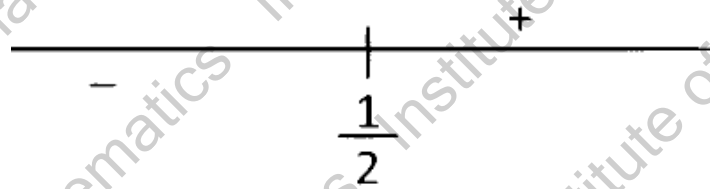
$$\Rightarrow 2^{\frac{2x}{3}} > 2^{\frac{1}{3}}$$

$$\Rightarrow \frac{2x}{3} > \frac{1}{3}$$

$$\Rightarrow x > \frac{1}{2}$$

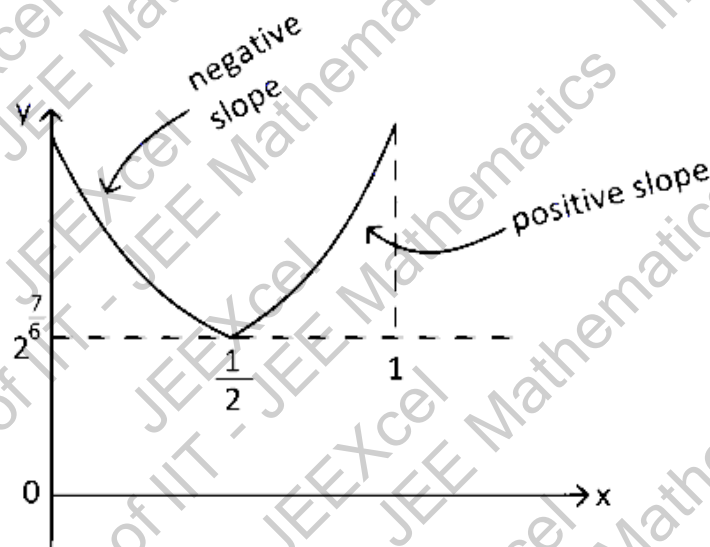
Similarly, $g'(x) < 0$ when $x < \frac{1}{2}$

If we put it in number line we get this



We know $g'(x)$ represent slope of curve $g(x)$ and it is negative when $x < \frac{1}{2}$ and positive when $x > \frac{1}{2}$ and zero when $x = \frac{1}{2}$.

$\therefore$ Graph of $g(x)$ is



From graph you can see value of $g(x)$ is maximum either at $x = 0$ or $x = 1$ in the range $x \in [0, 1]$.

$$\therefore g(0) = 2^0 + 2^{\frac{1}{2}} = 1 + 2^{\frac{1}{2}}$$

$$g(1) = 2^{\frac{1}{2}} + 2^0 = 1 + 2^{\frac{1}{2}}$$

$\therefore$ We get maximum value at $x = 0$ and $x = 1$ both.

$\therefore$ B and C options are correct.
