

Properties of Triangles

Question1

For a triangle ABC, the value of $\cos 2A + \cos 2B + \cos 2C$ is least. If its inradius is 3 and incentre is M, then which of the following is NOT correct?

[1-Feb-2023 Shift 1]

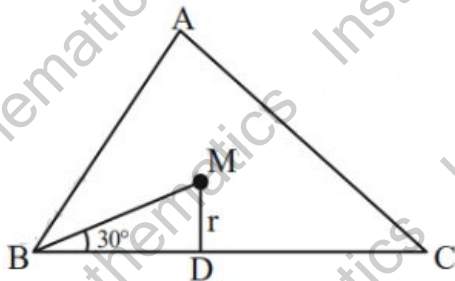
Options:

- A. Perimeter of $\triangle ABC$ is $18\sqrt{3}$
- B. $\sin 2A + \sin 2B + \sin 2C = \sin A + \sin B + \sin C$
- C. $\vec{MA} \cdot \vec{MB} = -18$
- D. area of $\triangle ABC$ is $\frac{27\sqrt{3}}{2}$

Answer: D

Solution:

Solution:



If $\cos 2A + \cos 2B + \cos 2C$ is minimum then $A = B = C = 60^\circ$

So $\triangle ABC$ is equilateral

Now in-radius $r = 3$

So in $\triangle MBD$ we have

$$\tan 30^\circ = \frac{MD}{BD} = \frac{r}{a/2} = \frac{6}{a}$$

$$1/\sqrt{3} = \frac{1}{a} = a = 6\sqrt{3}$$

$$\text{Perimeter of } \triangle ABC = 18\sqrt{3}$$

$$\text{Area of } \triangle ABC = \frac{\sqrt{3}}{4}a^2 = 27\sqrt{3}$$

Question2

Let $C(\alpha, \beta)$ be the circumcenter of the triangle formed by the lines

$$4x + 3y = 69$$

$$4y - 3x = 17 \text{ and}$$

$$x + 7y = 61$$

Then $(\alpha - \beta)^2 + \alpha + \beta$ is equal to
[8-Apr-2023 shift 1]

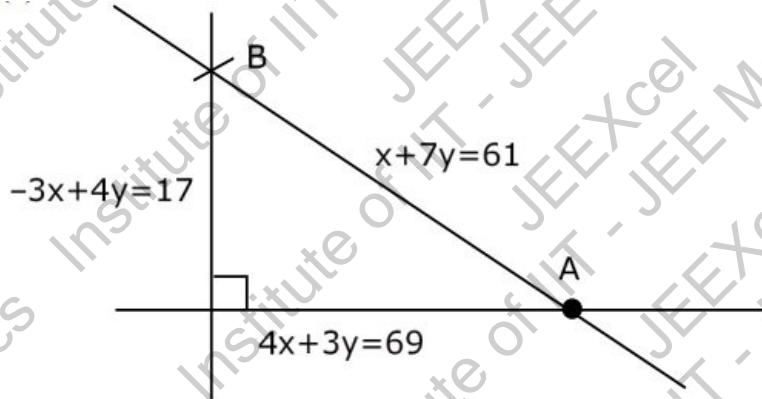
Options:

- A. 18
- B. 15
- C. 16
- D. 17

Answer: D

Solution:

Solution:



$$4x + 28y = 244$$

$$4x + 3y = 69$$

$$- -$$

$$25y = 175$$

$$y = 7, x = 12$$

$$A(12, 7)$$

$$- 3x + 4y = 17$$

$$3x + 21y = 183$$

$$25y = 200$$

$$y = 8, x = 5$$

$$B(5, 8)$$

$\therefore$ Circumcenter

$$\alpha = \frac{17}{2}, \beta = \frac{15}{2}$$

$$\left(\frac{17}{2}, \frac{15}{2} \right)$$

$$(\alpha - \beta)^2 + \alpha + \beta$$

$$1 + 16 = 17$$

Question3

The ratio of intensities at two points P and Q on the screen in a Young's double slit experiment where phase difference between two waves of same amplitude are $\pi / 3$ and $\pi / 2$, respectively are
[10-Apr-2023 shift 2]

Options:

- A. 3 : 2
- B. 3 : 1
- C. 2 : 3
- D. 1 : 3

Answer: A

Solution:

Solution:

$$I_{\text{res}} = 4I_0 \cos^2\left(\frac{\theta}{2}\right)$$

$$\text{If } \theta = \frac{\pi}{3}, I_{\text{res}} = 4I_0 \cdot \cos^2\left[\frac{\pi}{6}\right]$$

$$= 4I_0 \cdot \left(\frac{\sqrt{3}}{2}\right)^2$$

$$I_1 = (4I_0) \left(\frac{3}{4}\right) = 3I_0$$

$$\text{If } \theta = \frac{\pi}{2}, I_{\text{res}} = 4I_0 \cdot \cos^2\left(\frac{\pi}{2}\right)$$

$$= 4I_0 \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= (4I_0) \left(\frac{1}{2}\right)$$

$$I_2 = 2I_0$$

$$\frac{I_1}{I_2} = \frac{3}{2}$$

Question4

In a triangle ABC, if $\cos A + 2 \cos B + \cos C = 2$ and the lengths of the sides opposite to the angles A and C are 3 and 7 respectively, then $\cos A - \cos C$ is equal to [12-Apr-2023 shift 1]

Options:

- A. $\frac{5}{7}$
- B. $\frac{9}{7}$
- C. $\frac{3}{7}$
- D. $\frac{10}{7}$

Answer: D

Solution:

$$\cos A + \cos C = 2(1 - \cos B)$$

$$2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 4 \sin^2 \frac{B}{2}$$

$$\text{as } \cos \left(\frac{A+C}{2} \right) = \sin \frac{B}{2}$$

$$\text{so } \cos \frac{A-C}{2} = 2 \sin \frac{B}{2}$$

$$2 \cos B / 2 \cos \frac{A-C}{2} = 4 \sin B / 2 \cos B / 2$$

$$2 \sin \left(\frac{A+C}{2} \right) \cos \left(\frac{A-C}{2} \right) = 4 \sin B / 2 \cos B / 2$$

$$\sin A + \sin C = 2 \sin B$$

$$a + c = 2b \Rightarrow a = 3, c = 7, b = 5$$

$$\cos A - \cos C = \frac{b^2 + c^2 - a^2}{2bc} - \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{25 + 49 - 9}{70} - \frac{9 + 25 - 49}{30}$$

$$\frac{65}{70} + \frac{1}{2} = \frac{20}{14} = \frac{10}{7}$$

Question5

Let the plane $x + 3y - 2z + 6 = 0$ meet the co-ordinate axes at the points A, B, C. If the orthocenter of the triangle ABC is $\left(\alpha, \beta, \frac{6}{7} \right)$, then $98(\alpha + \beta)^2$ is equal to _____.
[12-Apr-2023 shift 1]

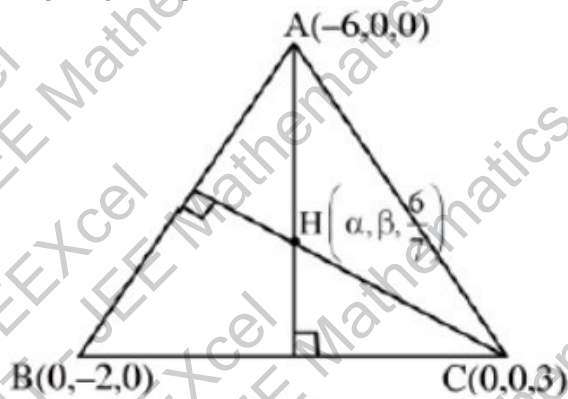
Answer: 288

Solution:

$$A(-6, 0, 0) B(0, -2, 0) C = (0, 0, 3)$$

$$\vec{AB} = 6\hat{i} - 2\hat{j}, \vec{BC} = 2\hat{j} + 3\hat{k}$$

$$\vec{AC} = 6\hat{i} + 3\hat{k}$$



$$\vec{AH} \cdot \vec{BC} = 0$$

$$\left(\alpha + 6, \beta, \frac{6}{7} \right) \cdot (0, 2, 3) = 0$$

$$\beta = \frac{-9}{7}$$

$$\vec{CH} \cdot \vec{AB} = 0$$

$$\left(\alpha, \beta, \frac{-15}{7} \right) \cdot (6, -2, 0) = 0$$

$$6\alpha - 2\beta = 0$$

$$\alpha = \frac{-3}{7}$$

$$98(\alpha + \beta)^2 = (98) \frac{(144)}{49} = 288$$

Question6

Let for a triangle ABC,

$$\vec{AB} = -2\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{CB} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$$

$$\vec{CA} = 4\hat{i} + 3\hat{j} + \delta\hat{k}$$

If $\delta > 0$ and the area of the triangle ABC is $5\sqrt{6}$, then $\vec{CB} \cdot \vec{CA}$ is equal to [13-Apr-2023 shift 2]

Options:

A. 108

B. 60

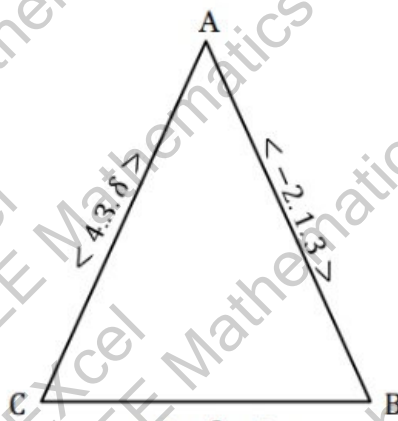
C. 54

D. 120

Answer: B

Solution:

Solution:



$\langle \alpha, \beta, \gamma \rangle$

$$\vec{CA} + \vec{AB} = \vec{CB}$$

$$\langle 4, 3, \delta \rangle + \langle -2, 1, 3 \rangle = \vec{CB}$$

$$\Rightarrow \vec{CB} = \langle 2, 4, 3 + \delta \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 3 \\ -4 & -3 & -\delta \end{vmatrix}$$

$$= \hat{i}(9 - \delta) - \hat{j}(26 + 12) + \hat{k}(10)$$

$$|\vec{AB} \times \vec{AC}|^2 = (9 - \delta)^2 + (26 + 12)^2 + (10)^2$$

$$= 56^2 + 306 + 325$$

$$\text{Area of } \Delta ABC = 5\sqrt{6}$$

$$\Rightarrow \frac{1}{2} |\vec{AB} \times \vec{AC}| = 5\sqrt{6}$$

$$\begin{aligned}
 \Rightarrow |\vec{AB} \times \vec{AC}|^2 &= 600 \\
 \Rightarrow 56^2 + 306 - 275 &= 0 \\
 \Rightarrow S^2 + 66 - 55 &= 0 \\
 \Rightarrow (6 + 11)(6 - 5) &= 0 \\
 6 &= 5 \\
 \vec{CB} &= \langle 2, 3, 8 \rangle \\
 \vec{CB} \cdot \vec{CA} &= \langle 2, 4, 8 \rangle \cdot \langle 4, 3, 5 \rangle \\
 &= 8 + 12 + 40 = 60
 \end{aligned}$$

Question7

Let a , b and c be the length of sides of a triangle ABC such that $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively, then the value of $\frac{R}{r}$ is equal to:
[25-Jun-2022-Shift-1]

Options:

- A. $\frac{5}{2}$
- B. 2
- C. $\frac{3}{2}$
- D. 1

Answer: A

Solution:

$$\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9} = \lambda$$

$$a+b=7\lambda$$

$$b+c=8\lambda$$

$$c+a=9\lambda$$

$$a+b+c=12\lambda$$

$$\therefore a=4\lambda, b=3\lambda, c=5\lambda$$

$$S = \frac{4\lambda+3\lambda+5\lambda}{2} = 6\lambda$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{(6\lambda)(2\lambda)(3\lambda)(\lambda)} = 6\lambda^2$$

$$R = \frac{abc}{4\Delta} = \frac{(4\lambda)(3\lambda)(5\lambda)}{4(6\lambda^2)} = \frac{5\lambda}{2}$$

$$r = \frac{\Delta}{S} = \frac{6\lambda^2}{6\lambda} = \lambda$$

$$\frac{R}{r} = \frac{\frac{5\lambda}{2}}{\lambda} = \frac{5}{2}$$

Question8

A tower PQ stands on a horizontal ground with base Q on the ground. The point R divides the tower in two parts such that $QR = 15\text{m}$. If from a point A on the ground the angle of elevation of R is 60° and the part PR of the tower subtends an angle of 15° at A, then the height of the tower is :

[25-Jul-2022-Shift-1]

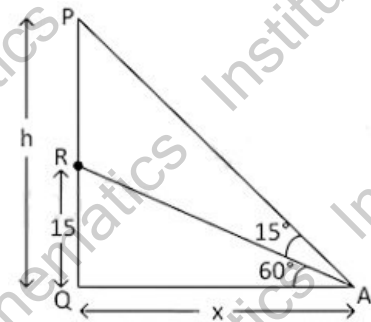
Options:

- A. $5(2\sqrt{3} + 3)\text{m}$
- B. $5(\sqrt{3} + 3)\text{m}$
- C. $10(\sqrt{3} + 1)\text{m}$
- D. $10(2\sqrt{3} + 1)\text{m}$

Answer: A

Solution:

Solution:



For $\triangle AQR$,
 $\tan 60^\circ = \frac{15}{x} \dots\dots (1)$

From $\triangle AQP$,
 $\tan 75^\circ = \frac{h}{x}$

$$\Rightarrow (2 + \sqrt{3}) = \frac{h}{x} [\because \tan 75^\circ = 2 + \sqrt{3}]$$

$$\Rightarrow h = (2 + \sqrt{3})x$$
$$= (2 + \sqrt{3}) \frac{15}{\sqrt{3}} \text{ [From (1)]}$$

$$= (2 + \sqrt{3}) \times \frac{15\sqrt{3}}{3}$$

$$= (2 + \sqrt{3}) \times 5\sqrt{3}$$

$$= 5(2\sqrt{3} + 3)\text{m}$$

Question9

Let a vertical tower AB of height $2h$ stands on a horizontal ground. Let from a point P on the ground a man can see upto height h of the tower with an angle of elevation 2α . When from P, he moves a distance d in the direction of $\vec{AP}$, he can see the top B of the tower with an angle of

elevation α . If $d = \sqrt{7}h$, then $\tan \alpha$ is equal to
[27-Jul-2022-Shift-1]

Options:

A. $\sqrt{5} - 2$

B. $\sqrt{3} - 1$

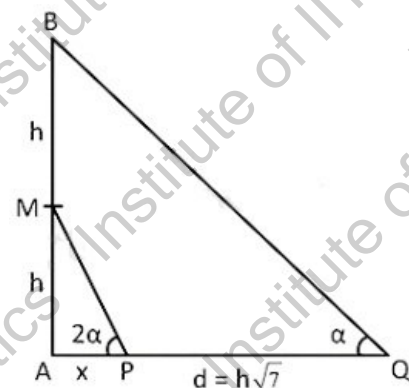
C. $\sqrt{7} - 2$

D. $\sqrt{7} - \sqrt{3}$

Answer: C

Solution:

Solution:



$\triangle APM$ gives

$$\tan 2\alpha = \frac{h}{x} \dots (i)$$

$\triangle AQB$ gives

$$\tan 2\alpha = \frac{2h}{x+d} = \frac{2h}{x+h\sqrt{7}} \dots (ii)$$

From (i) and (ii)

$$\tan 2\alpha = \frac{2 \cdot \tan 2\alpha}{1 + \sqrt{7} \cdot \tan 2\alpha}$$

Let $t = \tan \alpha$

$$\Rightarrow t = \frac{2 \cdot \frac{2t}{1-t^2}}{1 + \sqrt{7} \cdot \frac{2t}{1-t^2}}$$

$$\Rightarrow t^2 - 2\sqrt{7}t + 3 = 0$$

$$t = \sqrt{7} - 2$$

Question 10

The angle of elevation of the top P of a vertical tower PQ of height 10 from a point A on the horizontal ground is 45° . Let R be a point on AQ and from a point B, vertically above R, the angle of elevation of P is 60° . If $\angle BAQ = 30^\circ$, $AB = d$ and the area of the trapezium PQRB is α , then the ordered pair (d, α) is :
[27-Jul-2022-Shift-2]

Options:

A. $(10(\sqrt{3} - 1), 25)$

B. $\left(10(\sqrt{3} - 1), \frac{25}{2}\right)$

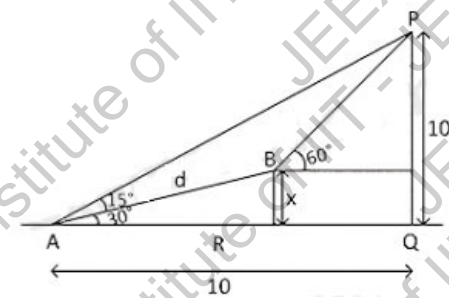
C. $(10(\sqrt{3} + 1), 25)$

D. $\left(10(\sqrt{3} + 1), \frac{25}{2}\right)$

Answer: A

Solution:

Solution:



Let $BR = x$

$$\frac{x}{d} = \frac{1}{2} \Rightarrow x = \frac{d}{2}$$

$$\frac{10 - x}{10 - x\sqrt{3}} = \sqrt{3} \Rightarrow 10 - x = 10\sqrt{3} - 3x$$

$$2x = 10(\sqrt{3} - 1)$$

$$x = 5(\sqrt{3} - 1)$$

$$d = 2x = 10(\sqrt{3} - 1)$$

$$\alpha = \frac{1}{2}(x + 10)(10 - x\sqrt{3}) = \text{Area}(PQRB)$$

$$= \frac{1}{2}(5\sqrt{3} - 5 + 10)(10 - 5\sqrt{3}(\sqrt{3} - 1))$$

$$= \frac{1}{2}(5\sqrt{3} + 5)(10 - 15 + 5\sqrt{3}) - \frac{1}{2}(75 - 25) = 25$$

Question 11

A horizontal park is in the shape of a triangle OAB with $AB = 16$. A vertical lamp post OP is erected at the point O such that $\angle PAO = \angle PBO = 15^\circ$ and $\angle PCO = 45^\circ$, where C is the midpoint of AB. Then $(OP)^2$ is equal to [28-Jul-2022-Shift-2]

Options:

A. $\frac{32}{\sqrt{3}}(\sqrt{3} - 1)$

B. $\frac{32}{\sqrt{3}}(2 - \sqrt{3})$

C. $\frac{16}{\sqrt{3}}(\sqrt{3} - 1)$

D. $\frac{16}{\sqrt{3}}(2 - \sqrt{3})$

Answer: B

Solution:

Solution:



$$OP = OA \tan 15 = OB \tan 15 \dots\dots (i)$$

$$OP = OC \tan 45 \Rightarrow OP = OC \dots\dots (ii)$$

$$OA = OB \dots\dots (iii)$$

$$OC^2 + 8^2 = OA^2$$

$$OP^2 + 64 = OP^2 \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right)^2$$

$$64 = OP^2 \left[\frac{(\sqrt{3} + 1)^2 - (\sqrt{3} - 1)^2}{(\sqrt{3} - 1)^2} \right]$$

$$= OP^2 \left(\frac{4\sqrt{3}}{(\sqrt{3} - 1)^2} \right)$$

$$OP^2 = \frac{64(\sqrt{3} - 1)^2}{4\sqrt{3}} = \frac{32}{\sqrt{3}}(2 - \sqrt{3})$$

Question 12

The angle of elevation of the top of a tower from a point A due north of it is α and from a point B at a distance of 9 units due west of A is

$\cos^{-1} \left(\frac{3}{\sqrt{13}} \right)$. If the distance of the point B from the tower is 15 units,

then $\cot \alpha$ is equal to :

[29-Jul-2022-Shift-1]

Options:

A. $\frac{6}{5}$

B. $\frac{9}{5}$

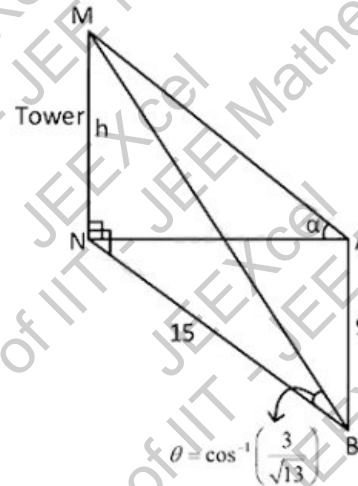
C. $\frac{4}{3}$

D. $\frac{7}{3}$

Answer: A

Solution:

Solution:



$$NA = \sqrt{15^2 - 9^2} = 12$$

$$\frac{h}{15} = \tan \theta = \frac{2}{3}$$

$$h = 10 \text{ units}$$

$$\cot \alpha = \frac{12}{10} = \frac{6}{5}$$

Question13

A man is observing, from the top of a tower, a boat speeding towards the tower from a certain point A, with uniform speed. At that point, angle of depression of the boat with the man's eye is 30° (ignore man's height). After sailing for 20s, towards the base of the tower (which is at the level of water), the boat has reached a point B, where the angle of depression is 45° . Then, the time taken (in seconds) by the boat from B to reach the base of the tower is
[25 Feb 2021 Shift 1]

Options:

- A. 10
- B. $10\sqrt{3}$
- C. $10(\sqrt{3} + 1)$
- D. $10(\sqrt{3} - 1)$

Answer: C

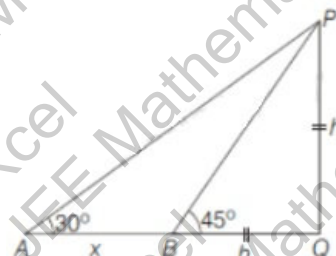
Solution:

Solution:

$$\text{In } \Delta PQB, \tan 45^\circ = \frac{PQ}{BQ}$$

$$1 = \frac{PQ}{BQ}$$

$$PQ = BQ = h \text{ (let)}$$



$$\text{In } \triangle PAQ, \tan 30^\circ = \frac{PQ}{AQ}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{x+h} \Rightarrow x+h = \sqrt{3}h$$

$$\Rightarrow x = (\sqrt{3} - 1)h$$

$$\therefore \text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\therefore \text{Speed from A to B} = \frac{AB}{20} = \frac{x}{20}$$

Also, distance from B to Q = h

$$\therefore \text{Time taken to reach Q from B} = \frac{BQ}{\text{Speed}}$$

$$= \frac{h}{x/20} = \frac{h \times 20}{x} [\because x = (\sqrt{3} - 1)h]$$

$$= \frac{h}{(\sqrt{3} - 1)h} \times 20$$

$$= \frac{(\sqrt{3} + 1) \times 20}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{(\sqrt{3} + 1) \times 20}{2}$$

$$= 10(\sqrt{3} + 1)$$

Question14

The angle of elevation of a jet plane from a point A on the ground is 60° . After a flight of 20s at the speed of 432km / h, the angle of elevation changes to 30° . If the jet plane is flying at a constant height, then its height is

[24 Feb 2021 Shift 2]

Options:

A. $1800\sqrt{3}\text{m}$

B. $3600\sqrt{3}\text{m}$

C. $2400\sqrt{3}\text{m}$

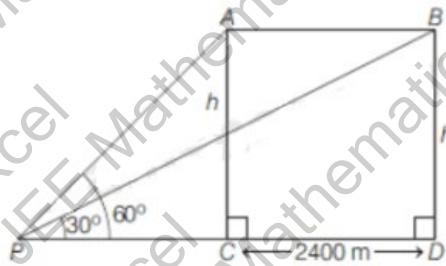
D. $1200\sqrt{3}\text{m}$

Answer: D

Solution:

Solution:

Given, angle of elevation are 60° and then 30° .



Also, in 20 sec plane covers the distance from A to B with speed 432km / h.

$$\therefore 432 \times \frac{5}{18} \text{ m / sec} = 120 \text{ m / sec}$$

$$\therefore AB = \text{distance} = S \times T = 120 \times 20 = 2400 \text{ m}$$

In $\triangle PBD$

$$\tan 30^\circ = \frac{BD}{PD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{PD}$$

$$\Rightarrow PD = h\sqrt{3}$$

In $\triangle PAC$

$$\tan 60^\circ = \frac{AC}{PC} \Rightarrow \sqrt{3} = \frac{h}{PC}$$

$$\Rightarrow PC = h / \sqrt{3}$$

$$\text{Now, } CD = PD - PC$$

$$2400 = h\sqrt{3} - \frac{h}{\sqrt{3}}$$

$$\Rightarrow 2400 = h \left(\frac{3-1}{\sqrt{3}} \right)$$

$$\therefore h = \frac{2400 \times \sqrt{3}}{2} = 1200 \times \sqrt{3}$$

$$\therefore \text{Required height} = 1200\sqrt{3} \text{ m}$$

Question 15

Two vertical poles are 150m apart and the height of one is three times that of the other. If from the middle point of the line joining their feet, an observer finds the angles of elevation of their tops to be complementary, then the height of the shorter pole (in meters) is :

24 Feb 2021 Shift 1

Options:

A. $20\sqrt{3}$

B. $25\sqrt{3}$

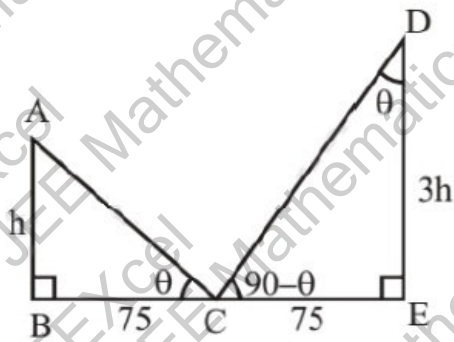
C. 30

D. 25

Answer: B

Solution:

Solution:



In right angle $\triangle ABC$ and right $\triangle CDE$

$$\tan \theta = \frac{h}{75} \text{ and } \cot(90 - \theta)$$

$$= \tan \theta = \frac{75}{3h}$$

$$\Rightarrow h^2 = \frac{(75)^2}{3}$$

$$h = 25\sqrt{3} \text{ m}$$

Question 16

Let the centroid of an equilateral triangle $\triangle ABC$ be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x + y = 3$. If R and r be the radius of circumcircle and incircle, respectively of $\triangle ABC$, then $(R + r)$ is equal to
[18 Mar 2021 Shift 2]

Options:

A. $\frac{9}{\sqrt{2}}$

B. $7\sqrt{2}$

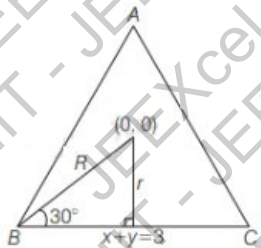
C. $2\sqrt{2}$

D. $3\sqrt{2}$

Answer: A

Solution:

Solution:



Let equation of BC is $x + y = 3$

$$\therefore r = \frac{|0 + 0 - 3|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

In equilateral triangle,

$$\sin 30^\circ = \frac{r}{R} \Rightarrow \frac{1}{2} = \frac{r}{R}$$

$$\Rightarrow R = 2r$$

$$\therefore r + R = r + 2r$$

$$= 3r = 3 \cdot \frac{3}{\sqrt{2}} = \frac{9}{\sqrt{2}}$$

Question17

A pole stands vertically inside a triangular park ABC. Let the angle of elevation of the top of the pole from each corner of the park be $\frac{\pi}{3}$. If the radius of the circumcircle of $\triangle ABC$ is 2 , then the height of the pole is equal to
[18 Mar 2021 Shift 2]

Options:

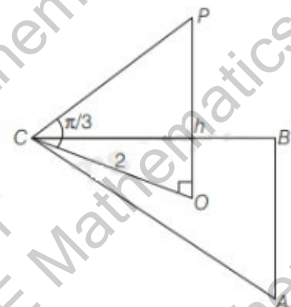
- A. $\frac{2\sqrt{3}}{3}$
- B. $2\sqrt{3}$
- C. $\sqrt{3}$
- D. $\frac{1}{\sqrt{3}}$

Answer: B

Solution:

Solution:

Let PO = h, CO = 2 = radius and $\angle PCO = \frac{\pi}{3}$



$$\tan \frac{\pi}{3} = \frac{PO}{OC} \Rightarrow \sqrt{3} = \frac{h}{2} \Rightarrow h = 2\sqrt{3}$$

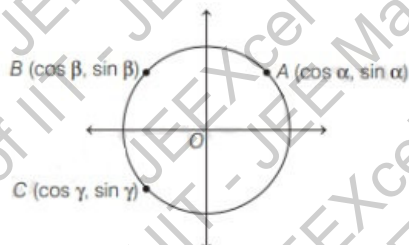
Question18

Let $\tan \alpha, \tan \beta$ and $\tan \gamma, \alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}, n \in \mathbb{N}$ be the slopes of three line segments OA, OB and OC, respectively, where O is origin. If circumcentre of $\triangle ABC$ coincides with origin and its orthocentre lies on y-axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma} \right)^2$ is equal to
[17 Mar 2021 Shift 2]

Answer: 144

Solution:

$$\text{Given, slope of OA} = \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin \alpha - 0}{\cos \alpha - 0}$$



$\therefore$ Coordinates of A must be $(\cos \alpha, \sin \alpha)$

Similarly, B = $(\cos \beta, \sin \beta)$ and C = $(\cos \gamma, \sin \gamma)$

Given, circumcentre coincides with origin and orthocentre lies on Y-axis.

So, centroid must lie on Y-axis because circumcentre, orthocentre and centroid of any triangle always lies on the same line.

$$\therefore \cos \alpha + \cos \beta + \cos \gamma = 0 \dots (i)$$

(because x-coordinate on Y-axis is zero)

$$\Rightarrow \cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma = 3 \cos \alpha \cdot \cos \beta \cdot \cos \gamma \dots (ii)$$

Now,

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = (4\cos^3 \alpha - 3\cos \alpha) + (4\cos^3 \beta - 3\cos \beta) + (4\cos^3 \gamma - 3\cos \gamma)$$

$$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 4(\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3(\cos \alpha + \cos \beta + \cos \gamma)$$

$$= 4(\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3 \times 0 \quad (\text{from Eq. (i)})$$

$$= 12 \cos \alpha \cdot \cos \beta \cdot \cos \gamma \quad (\text{from Eq. (ii)})$$

$$\therefore \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cdot \cos \beta \cdot \cos \gamma} = \frac{12 \cdot \cos \alpha \cdot \cos \beta \cdot \cos \gamma}{\cos \alpha \cdot \cos \beta \cdot \cos \gamma}$$

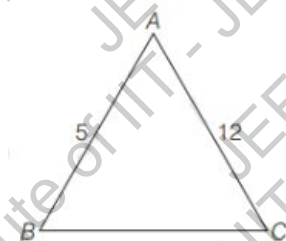
$$\Rightarrow \left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cdot \cos \beta \cdot \cos \gamma} \right)^2 = 144$$

Question 19

In $\triangle ABC$, the lengths of sides AC and AB are 12cm and 5cm, respectively. If the area of $\triangle ABC$ is 30cm^2 and R and r are respectively the radii of circumcircle and incircle of $\triangle ABC$, then the value of $2R + r$ (in cm) is equal to
[16 Mar 2021 Shift 2]

Answer: 15

Solution:

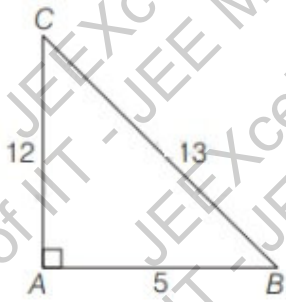


$$\text{Area of } \triangle ABC = 30\text{cm}^2$$

$$\frac{1}{2} \cdot 5 \cdot 12 \sin A = 30\text{cm}^2$$

$$\Rightarrow \sin A = 1$$

$$\Rightarrow A = 90^\circ$$



$$\text{Here, } AB^2 + AC^2 = BC^2$$

$$\Rightarrow 12^2 + 5^2 = BC^2$$

$$\Rightarrow BC = 13$$

$$\therefore R = \frac{a}{2 \sin A} = \frac{13}{2 \cdot \sin 90} = \frac{13}{2}$$

$$r = \frac{\Delta}{s} = \frac{30}{\left(\frac{12 + 5 + 13}{2} \right)} = \frac{30}{15} = 2$$

$$\therefore 2R + r = 2 \cdot \frac{13}{2} + 2 = 15$$

Question20

A spherical gas balloon of radius 16 meter subtends an angle 60° at the eye of the observer A while the angle of elevation of its center from the eye of A is 75° . Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is :
[25 Jul 2021 Shift 1]

Options:

A. $8(2 + 2\sqrt{3} + \sqrt{2})$

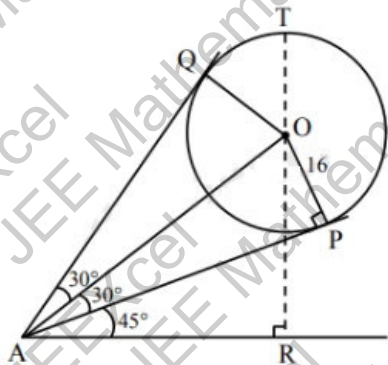
B. $8(\sqrt{6} + \sqrt{2} + 2)$

C. $8(\sqrt{2} + 2 + \sqrt{3})$

D. $8(\sqrt{6} - \sqrt{2} + 2)$

Answer: B

Solution:



O $\rightarrow$ centre of sphere

P, Q $\rightarrow$ point of contact of tangents from A

Let T be top most point of balloon & R be foot of perpendicular from O to ground.

From triangle OAP, $OA = 16 \operatorname{cosec} 30^\circ = 32$

From triangle ABO, $OR = OA \sin 75^\circ = 32 \frac{(\sqrt{3} + 1)}{2\sqrt{2}}$

So level of top most point = $OR + OT$
 $= 8(\sqrt{6} + \sqrt{2} + 2)$

Question 21

If in a triangle ABC, $AB = 5$ units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of $\triangle ABC$ is 5 units, then the area (in sq. units) of $\triangle ABC$ is :
 [20 Jul 2021 Shift 1]

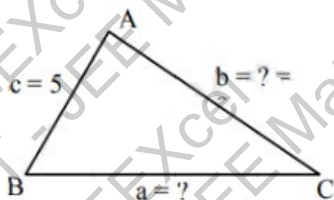
Options:

- A. $10 + 6\sqrt{2}$
- B. $8 + 2\sqrt{2}$
- C. $6 + 8\sqrt{3}$
- D. $4 + 2\sqrt{3}$

Answer: C

Solution:

Solution:



$$\text{As, } \cos B = \frac{3}{5} \Rightarrow B = 53^\circ$$

$$\text{As, } R = 5 \Rightarrow \frac{c}{\sin C} = 2R$$

$$\Rightarrow \frac{5}{10} = \sin C \Rightarrow C = 30^\circ$$

$$\text{Now, } \frac{b}{\sin B} = 2R \Rightarrow b = 2(5) \left(\frac{4}{5} \right) = 8$$

Now, by cosine formula

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\Rightarrow \frac{3}{5} = \frac{a^2 + 25 - 64}{2(5)a}$$

$$\Rightarrow a^2 - 6a - 39 = 0$$

$$\therefore a = \frac{6 \pm \sqrt{192}}{2} = \frac{6 \pm 8\sqrt{3}}{2}$$

$$\Rightarrow 3 + 4\sqrt{3} \text{ (Reject } a = 3 - 4\sqrt{3} \text{)}$$

$$\text{Now, } \Delta = \frac{abc}{4R} = \frac{(3 + 4\sqrt{3})(8)(5)}{4(5)} = 2(3 + 4\sqrt{3})$$

$$\Rightarrow \Delta = (6 + 8\sqrt{3})$$

$\Rightarrow$ Option (3) is correct.

Question 22

Let in a right angled triangle, the smallest angle be θ . If a triangle formed by taking the reciprocal of its sides is also a right angled triangle, then $\sin \theta$ is equal to:
[20 Jul 2021 Shift 2]

Options:

A. $\frac{\sqrt{5} + 1}{4}$

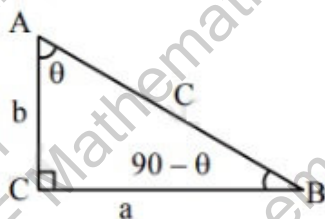
B. $\frac{\sqrt{5} - 1}{2}$

C. $\frac{\sqrt{2} - 1}{2}$

D. $\frac{\sqrt{5} - 1}{4}$

Answer: B

Solution:



$$\angle A = \theta$$

$$\angle B = 90^\circ - \theta$$

a = smallest side

$$c^2 = a^2 + b^2$$

$$\frac{1}{a^2} = \frac{1}{b^2} + \frac{1}{c^2}$$

$$\frac{b^2 c^2}{a^2} = b^2 + c^2$$

$$\text{Use } a = 2R \sin A = 2R \sin \theta$$

$$b = 2R \sin B = 2R \sin(90^\circ - \theta) = 2R \cos \theta$$

$$c = 2R \sin C = 2 \sin 90^\circ = 2R$$

$$\frac{4R^2 \cos^2 \theta}{4R^2 \sin^2 \theta} = 4R^2 \cos^2 \theta + 4R^2$$

$$\cos^2 \theta = \sin^2 \theta \cos^2 \theta + \sin^2 \theta$$

$$1 - \sin^2 \theta = \sin^2 \theta (1 - \sin^2 \theta) + \sin^2 \theta$$

$$\sin^2 \theta = \frac{3 - \sqrt{5}}{2}$$

$$\Rightarrow \sin \theta = \frac{\sqrt{5} - 1}{2}$$

Question 23

A vertical pole fixed to the horizontal ground is divided in the ratio 3 : 7 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a point on the ground 18m away from the base of the pole, then the height of the pole (in m) is [31 Aug 2021 Shift 1]

Options:

A. $12\sqrt{15}$

B. $12\sqrt{10}$

C. $8\sqrt{10}$

D. $6\sqrt{10}$

Answer: B

Solution:

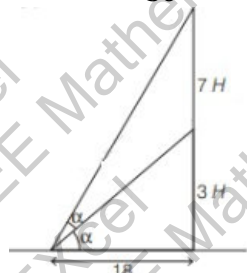
Solution:

Let height of pole = $10h$

$$\tan \alpha = \frac{3h}{18}$$

$$\tan 2\alpha = \frac{10h}{18} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\frac{5h}{9} = \frac{2 \cdot \left(\frac{h}{6}\right)}{1 - \frac{h^2}{36}}$$



$$1 - \frac{h^2}{36} = \frac{3}{5}$$

$$\frac{h^2}{36} = \frac{2}{5}$$

$$h^2 = \frac{72}{5}$$

$$h = \sqrt{\frac{72}{5}}$$

$$10h = 12\sqrt{10}$$

Question 24

Let $\frac{\sin A}{\sin B} = \frac{\sin(A - C)}{\sin(C - B)}$, where A, B and C are angles of a $\triangle ABC$. If the lengths of the sides opposite these angles are a, b and c respectively, then
[27 Aug 2021 Shift 1]

Options:

- A. $b^2 - a^2 = a^2 + c^2$
- B. b^2, c^2 and a^2 are in AP
- C. c^2, a^2 and b^2 are in AP
- D. a^2, b^2 and c^2 are in AP

Answer: B

Solution:

Solution:

$$\begin{aligned}\text{Given, } \frac{\sin A}{\sin B} &= \frac{\sin(A - C)}{\sin(C - B)} \\ \Rightarrow \frac{\sin(B + C)}{\sin(A + C)} &= \frac{\sin(A - C)}{\sin(C - B)} \quad [\because A + B + C = \pi] \\ \Rightarrow \sin(C + B) \cdot \sin(C - B) &= \sin(A + C) \cdot \sin(A - C) \\ \Rightarrow \sin^2 C - \sin^2 B &= \sin^2 A - \sin^2 C \\ \Rightarrow 2\sin^2 C &= \sin^2 A + \sin^2 B \\ \Rightarrow 2(2R \sin C)^2 &= (2R \sin A)^2 + (2R \sin B)^2 \\ \Rightarrow 2c^2 &= a^2 + b^2 \quad \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \right] \\ \Rightarrow a^2, c^2, b^2 &\text{ are in AP.} \\ \Rightarrow b^2, c^2, a^2 &\text{ are in AP}\end{aligned}$$

Question 25

Two poles, AB of length a m and CD of length a + b (b $\neq$ a) metres are erected at the same horizontal level with bases at B and D.

If $BD = x$ and $\tan \angle ACB = \frac{1}{2}$, then

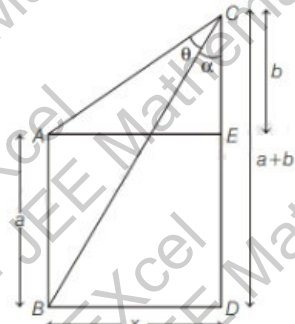
[27 Aug 2021 Shift 2]

Options:

- A. $x^2 + 2(a + 2b)x - b(a + b) = 0$
- B. $x^2 + 2(a + 2b)x + a(a + b) = 0$
- C. $x^2 - 2ax + b(a + b) = 0$
- D. $x^2 - 2ax + a(a + b) = 0$

Answer: C

Solution:



In $\triangle ACE$

$$\tan(\theta + \alpha) = \frac{x}{b}$$

In $\triangle ACD$

$$\tan \alpha = \frac{x}{a+b} \text{ and } \tan \theta = \frac{1}{2}$$

Then,

$$\tan(\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \cdot \tan \alpha}$$

$$\Rightarrow \frac{\frac{1}{2} + \frac{x}{a+b}}{1 - \frac{1}{2} \cdot \frac{x}{a+b}} = \frac{x}{b}$$

$$\Rightarrow \frac{a+b+2x}{2(a+b)-x} = \frac{x}{b}$$

$$\Rightarrow ab + b^2 + 2bx = 2ax + 2bx - x^2$$

$$\Rightarrow x^2 - 2ax + ab + b^2 = 0$$

$$\Rightarrow x^2 - 2ax + b(a+b) = 0$$

Question 26

A ray of light coming from the point $(2, 2\sqrt{3})$ is incident at an angle 30° on the line $x = 1$ at the point A. The ray gets reflected on the line $x = 1$ and meets x -axis at the point B. Then, the line AB passes through the point:

[Sep. 06, 2020 (I)]

Options:

A. $\left(3, -\frac{1}{\sqrt{3}}\right)$

B. $\left(4, -\frac{\sqrt{3}}{2}\right)$

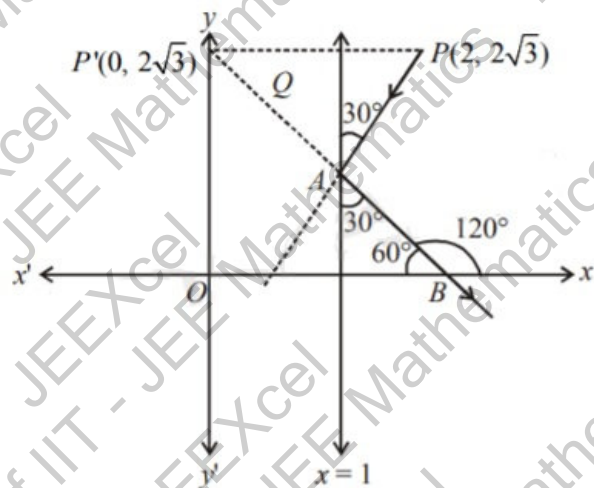
C. $(3, -\sqrt{3})$

D. $(4, -\sqrt{3})$

Answer: C

Solution:

Solution:



Slope of AB = $\tan 120^\circ = -\sqrt{3}$
 therefore Equation of line AB (i.e. BP')
 $y - 2\sqrt{3} = -\sqrt{3}(x - 0)$
 $\Rightarrow \sqrt{3}x + y = 2\sqrt{3}$
 $\therefore$ Point $(3, -\sqrt{3})$ lies on line AB.

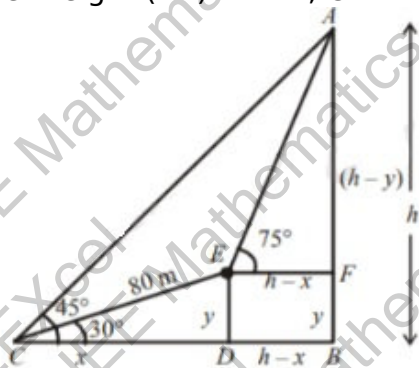
Question 27

The angle of elevation of the top of a hill from a point on the horizontal plane passing through the foot of the hill is found to be 45° . After walking a distance of 80 meters towards the top, up a slope inclined at an angle of 30° to the horizontal plane, the angle of elevation of the top of the hill becomes 75° . Then the height of the hill (in meters) is _____.
 [Sep. 06, 2020 (I)]

Answer: 80

Solution:

Let height (AB) = h m, CD = x m and ED = y m



In rt. $\triangle CDE$,

$$\sin 30^\circ = \frac{y}{80} \Rightarrow y = 40$$

$$\cos 30^\circ = \frac{x}{80} \Rightarrow x = 40\sqrt{3}$$

Now, in $\triangle AEF$,

$$\tan 75^\circ = \frac{h - y}{h - x}$$

$$\Rightarrow (2 + \sqrt{3}) = \frac{h - 40}{h - 40\sqrt{3}}$$

$$\begin{aligned}\Rightarrow (2 + \sqrt{3})(h - 40\sqrt{3}) &= h - 40 \\ \Rightarrow 2h - 80\sqrt{3} + \sqrt{3}h - 120 &= h - 40 \\ \Rightarrow h + \sqrt{3}h &= 80 + 80\sqrt{3} \\ \Rightarrow (\sqrt{3} + 1)h &= 80(\sqrt{3} + 1) \\ \therefore h &= 80\text{m}\end{aligned}$$

Question28

The angle of elevation of the summit of a mountain from a point on the ground is 45° . After climbing up on km towards the summit at an inclination of 30° from the ground, the angle of elevation of the summit is found to be 60° . Then the height (in km) of the summit from the ground is:

[Sep. 06, 2020 (II)]

Options:

A. $\frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

B. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

C. $\frac{1}{\sqrt{3} - 1}$

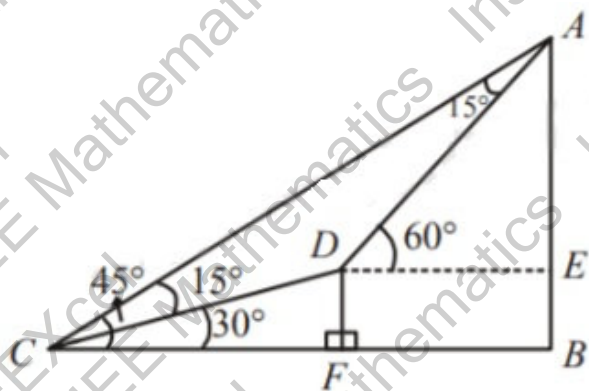
D. $\frac{1}{\sqrt{3} + 1}$

Answer: C

Solution:

$$\because \angle DCA = \angle DAC = 30^\circ$$

$$\therefore AD = DC = 1\text{km}$$



In $\triangle DEA$,

$$\frac{AE}{AD} = \sin 60^\circ \Rightarrow AE = \frac{\sqrt{3}}{2} \text{ km}$$

$$\text{In } \triangle CDF, \sin 30^\circ = \frac{DF}{CD} \Rightarrow DF = \frac{1}{2} \text{ km}$$

$$\therefore EB = DF = \frac{1}{2} \text{ km}$$

$$\therefore \text{Height of mountain} = AE + EB$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{1}{2} \right) = \left(\frac{\sqrt{3} + 1}{2} \right) \text{ km}$$

$$= \frac{1}{\sqrt{3} - 1} \text{ km}$$

Question 29

Two vertical poles $AB = 15$ m and $CD = 10$ m are standing apart on a horizontal ground with points A and C on the ground. If P is the point of intersection of BC and AD , then the height of P (in m) above the line AC is :

[Sep. 04, 2020 (I)]

Options:

A. $\frac{20}{3}$

B. 5

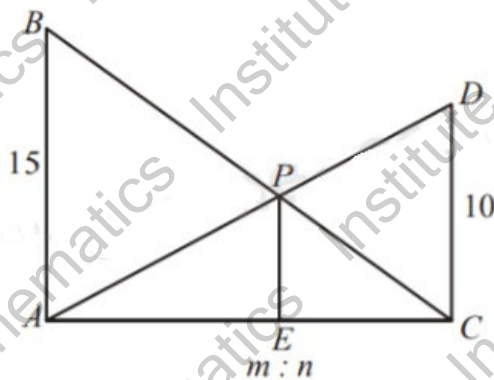
C. $\frac{10}{3}$

D. 6

Answer: D

Solution:

Solution:



Let $PE \perp AC$ and $\frac{AE}{EC} = \frac{m}{n}$

$\because \triangle AEP \sim \triangle ACD, \frac{m}{PE} = \frac{m+n}{10}$

$\Rightarrow PE = \frac{10m}{m+n} \dots (i)$

$\because \triangle CEP \sim \triangle CAB, \frac{n}{PE} = \frac{m+n}{15}$

$\Rightarrow PE = \frac{15n}{m+n} \dots (ii)$

From (i) and (ii),

$10m = 15n \Rightarrow m = \frac{3}{2}n$

So, $PE = 6$